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Log Type Estimators Using Multi-Auxiliary Information Under Ranked Set Sampling

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Abstract

It is well known that the relevant utilization of auxiliary information associated with the auxiliary variable helps to enhance the efficiency of the estimates. Therefore, we introduce some log type estimators based on multi-auxiliary information under ranked set sampling. The mean square error (MSE) of the suggested estimators is derived to the first order approximation. The efficiency conditions are obtained by comparing the MSE of the suggested estimators with the MSE of the contemporary estimators. Further, numerical and simulation studies are conducted over real and artificially generated populations to support the theoretical results. The empirical results show that the suggested estimators perform better than the usual mean estimator, classical ratio estimator, Abu-Dayyeh et al. (2009) estimator, Mehta and Mandowara (2014) estimator, Khan and Shabbir (2016) estimator and Khan et al. (2019) estimator.

Keywords: Mean square error, auxiliary information, ranked set sampling.

1. Introduction

McIntyre (1952) introduced the idea of ranked set sampling (RSS) as a cost-independent alternative to simple random sampling but did not furnish any mathematical formula. The necessary mathematical formulation to the theory of RSS was provided by Takahasi and Wakimoto (1968). McIntyre (1952) and Takahasi and Wakimoto (1968) developed RSS with perfect ranking of units. Dell and Clutter (1972) demonstrated in the case of perfect and imperfect ranking of units that the mean under RSS is an unbiased estimator of the population mean. Muttlak and McDonald (1990) investigated RSS when units are selected with size biased probability regarding the concomitant variable, whereas Muttlak and McDonald (1992) introduced an efficient line intercept method under RSS. Muttlak (1995) employed RSS to estimate the parameters in simple linear regression. Samawi and Muttlak (1996) showed that ranking of the denominator variable in the ratio estimator improves the efficiency. Singh et al. (2014) suggested a general family of estimators of population mean under RSS. Bhushan and Kumar (2020) suggested some log type class of estimators of population mean under RSS, whereas Bhushan et al. (2021) developed some novel classes of estimators for the estimation of population mean under RSS. Kumar and Dudeja (2021) introduced shadowed type 2 fuzzy-based Markov model to predict shortest path with optimized waiting time. Moreover, an extensive work is also done by Bhushan and Kumar (2021, 2022a, 2022b, 2022c) to estimate the population mean under RSS which may be of reader's interest. When the surveys are rather extensive, the efficiency of the estimators may be improved by considering the information on more than one auxiliary variable. Abu-Dayyeh et al. (2009) introduced a ratio estimator of population mean based on auxiliary information in RSS. Mehta and Mandowara (2014) opined an improved ratio estimator consisting two auxiliary variables in RSS. Khan and Sabbir (2016, 2017) envisaged improved ratio type estimators utilizing two auxiliary variables under RSS. Recently, some efficient estimators of population mean are suggested by Khan et al. (2019) using two auxiliary variables. This article proposes some log type estimators of population mean based on bivariate auxiliary information in RSS.

The article is designed in a few sections: In Section 2, a concise review of conventional estimators is considered along with their properties. The proposed estimators are given in Section 3 along with their properties. The conditions of efficiency for the proposed estimators are obtained in Section 4. The conditions of efficiency are enhanced by numerical and simulation studies in Section 5 and Section 6, respectively. The results of numerical and simulation studies are discussed in Section 7. Finally, the conclusion is presented in Section 8.

2. Conventional Estimators

In ranked set sampling, m independent random sets, each of size m, are randomly drawn from the population with equal probability and without replacement. The members of each random set are ranked concerning the variable of choice. Then, the first smallest unit is quantified from the ranked set and the remaining units of the set are discarded. The second smallest unit is quantified from the second ranked set and the remaining units of the set are discarded. In this way, this procedure is continued until the unit with the largest rank is quantified from the m^{th} set and the remaining units of the set are discarded. This whole process is referred to as a cycle and the repetition of such cycles r times provides $n = mr$ ranked set samples.
The procedure of selecting n ranked set samples is defined in the following steps:
Step 1: Randomly draw m^2 trivariate samples from the parent population.
Step 2: Allocate randomly drawn m^2 units into m sets, each of size m units.
Step 3: Each set is ranked with respect to the variable of choice.
Step 4: Measure the unit with rank i from the set i $(i = 1, 2, ..., m)$ for actual measurement.
Step 5: Repeat steps 1 to 4 for r cycles until the desired samples of size $n = mr$ units are obtained.
Let the ranking be performed on the auxiliary variable z_1 while the ranking of study variable y and auxiliary variable z_2 are measured with errors in ranking. Thus, $(z_{1(i)}, z_{2[i]}, y_{[i]})$ denote i^{th} order statistic in the i^{th} sample for variable z_1 and the i^{th} judgment ordering in the i^{th} sample for the auxiliary variable z_2 and study variable y, respectively. The parentheses () and [] associated, respectively, with z_1 and z_2, y show the perfect and imperfect ranking of units.
Further, this section considers existing conventional estimators under RSS using bivariate auxiliary information.

(i) The conventional mean estimator under RSS is expressed by

$$\bar{y}_m = \bar{y}_{[n]} \tag{1}$$

where $\bar{y}_{[n]} = \sum_{i=1}^{m} y_{[i]}/mr$ is the ranked set sample mean of the study variable y.

(ii) The conventional ratio estimator using bivariate auxiliary information under RSS is defined as

$$\bar{y}_r = \bar{y}_{[n]} \left(\frac{\bar{Z}_1}{\bar{z}_{1(n)}}\right)\left(\frac{\bar{Z}_2}{\bar{z}_{2[n]}}\right) \tag{2}$$

where $\bar{z}_{1(n)} = \sum_{i=1}^{m} z_{1(i)}/mr$ and $\bar{z}_{2[n]} = \sum_{i=1}^{m} z_{2[i]}/mr$ are the ranked set sample means of auxiliary variables z_1 and z_2, respectively.

(iii) Abu-Dayyeh et al. (2009) envisaged a class of ratio type estimators using bivariate auxiliary information under RSS as

$$\bar{y}_w = \bar{y}_{[n]}\left[w_1\left(\frac{\bar{Z}_1}{\bar{z}_{1(n)}}\right)^{a_1} + w_2\left(\frac{\bar{Z}_2}{\bar{z}_{2[n]}}\right)^{a_2}\right] \tag{3}$$

where $a_1,\ a_2,\ w_1$ and w_2 are suitably chosen constants and $w_1 + w_2 = 1$.

(iv) On the lines of Olkin (1958), Mehta and Mandowara (2014) suggested ratio type estimator under RSS as

$$\bar{y}_{mm} = \bar{y}_{[n]}\left[w_1\left(\frac{\bar{Z}_1}{\bar{z}_{1(n)}}\right) + w_2\left(\frac{\bar{Z}_2}{\bar{z}_{2[n]}}\right)\right] \tag{4}$$

where w_1 and w_2 are duly opted scalars.

(v) Khan and Shabbir (2016) developed a class of ratio in exponential ratio type estimators based on bivariate auxiliary information under RSS as

$$\bar{y}_{k_1} = \bar{y}_{[n]}\left(\frac{\bar{Z}_1}{\bar{z}_{1(n)}}\right)^{\eta_1}\left(\frac{\bar{Z}_2}{\bar{z}_{2[n]}}\right)^{\eta_2}\left[k_1\exp\left(\frac{\bar{Z}_1 - \bar{z}_{1(n)}}{\bar{Z}_1 + \bar{z}_{1(n)}}\right) + k_2\exp\left(\frac{\bar{Z}_2 - \bar{z}_{2[n]}}{\bar{Z}_2 + \bar{z}_{2[n]}}\right)\right] \tag{5}$$

where η_1 and η_2 are unknown constants and k_1 and k_2 are the weights such that $k_1 + k_2 = 1$.

(vi) Khan et al. (2019) suggested a class of difference in exponential ratio type estimators using bivariate auxiliary information under RSS as

$$\bar{y}_{k_2} = [\bar{y}_{[n]} + \Phi_1(\bar{Z}_1 - \bar{z}_{1(n)}) + \Phi_2(\bar{Z}_2 - \bar{z}_{2[n]})]\left[k_1\exp\left(\frac{\bar{Z}_1 - \bar{z}_{1(n)}}{\bar{Z}_1 + \bar{z}_{1(n)}}\right) + k_2\exp\left(\frac{\bar{Z}_2 - \bar{z}_{2[n]}}{\bar{Z}_2 + \bar{z}_{2[n]}}\right)\right] \tag{6}$$

where Φ_1 and Φ_2 are scalars.

The MSE of the above estimators are expressed in the Appendix A for ready reference.

3. Suggested Estimators

Motivated by the work of Bhushan et al. (2020a, b), we have extended the work of Bhushan and Kumar (2020) using bivariate and multi-auxiliary information under RSS.

3.1. Suggested estimators using bivariate auxiliary information

The proposed estimators using bivariate auxiliary information under RSS are given by

$$T_{b_1} = \bar{y}_{[n]}\left[1 + \log\left(\frac{\bar{z}_{1(n)}}{\bar{Z}_1}\right)\right]^{\alpha_1}\left[1 + \log\left(\frac{\bar{z}_{2[n]}}{\bar{Z}_2}\right)\right]^{\beta_1} \tag{7}$$

$$T_{b_2} = \bar{y}_{[n]}\left[1 + \alpha_2\log\left(\frac{\bar{z}_{1(n)}}{\bar{Z}_1}\right)\right]\left[1 + \beta_2\log\left(\frac{\bar{z}_{2[n]}}{\bar{Z}_2}\right)\right] \tag{8}$$

where α_j and $\beta_j,\ j = 1, 2$ are duly opted scalars to minimize the MSE.

3.2. Suggested estimators using multi-auxiliary information

In the sample survey, sometimes the information is available on multiple auxiliary variables. Let the information be available on p multi-auxiliary variables $z_1, z_2, ..., z_p$, then the proposed estimators based on multi-auxiliary information are defined as

$$T_{b_3} = \bar{y}_{[n]} \prod_{j=1}^{p} \left[1 + \log\left(\frac{\bar{z}_{j\,(n)}}{\bar{Z}_j}\right)\right]^{\alpha_j} \tag{9}$$

$$T_{b_4} = \bar{y}_{[n]} \prod_{j=1}^{p} \left[1 + \beta_j \log\left(\frac{\bar{z}_{j\,(n)}}{\bar{Z}_j}\right)\right] \tag{10}$$

where α_j and $\beta_j,\ j = 1, 2, ..., p$ are duly opted scalars.

Theorem 1 *The MSE and minimum MSE of the proposed class of estimators $T_{b_i},\ i = 1, 2$ are given by*

$$MSE(T_{b_i}) = \bar{Y}^2 \left[\Delta_0 + \alpha_i^2\Delta_1 + \beta_i^2\Delta_2 + 2\alpha_i\Delta_{01} + 2\beta_i\Delta_{02} + 2\alpha_i\beta_i\Delta_{12}\right],\ \ i = 1, 2 \tag{11}$$

$$MSE(T_{b_3}) = \bar{Y}^2 \left[\Delta_0 + \sum_{j=1}^{p} \alpha_j^2\Delta_j + 2\sum_{j=1}^{p} \alpha_j\Delta_{0j} + 2\sum\sum_{i>j}^{p} \alpha_i\alpha_j\Delta_{ij}\right] \tag{12}$$

$$MSE(T_{b_4}) = \bar{Y}^2 \left[\Delta_0 + \sum_{j=1}^{p} \beta_j^2\Delta_j + 2\sum_{j=1}^{p} \beta_j\Delta_{0j} + 2\sum\sum_{i>j}^{p} \beta_i\beta_j\Delta_{ij}\right] \tag{13}$$

$$minMSE(T_{b_i}) = \bar{Y}^2 \left[\ \Delta_0 - \frac{(\Delta_2\Delta_{01}^2 + \Delta_1\Delta_{02}^2 - 2\Delta_{01}\Delta_{02}\Delta_{12})}{(\Delta_1\Delta_2 - \Delta_{12}^2)}\ \right],\ \ i = 1, 2 \tag{14}$$

$$minMSE(T_{b_i}) = \bar{Y}^2 \left[\ \Delta_0 - \frac{(\sum_{j=1}^{p}\Delta_j\Delta_{0p-j+1}^2 - 2\sum\sum_{i>j}^{p}\Delta_{0i}\Delta_{0j}\Delta_{ij})}{(\prod_{j=1}^{p}\Delta_j - \sum\sum_{i>j}^{p}\Delta_{ij}^2)}\ \right],\ \ i = 3, 4 \tag{15}$$

Proof. To find the MSE, we assume the following notations as
$\bar{y}_{[n]} = \bar{Y}(1+\epsilon_0),\ \bar{z}_{1(n)} = \bar{Z}_1(1+\epsilon_1),\ \bar{z}_{2[n]} = \bar{Z}_2(1+\epsilon_2)$ such that $E(\epsilon_t) = 0,\ t = 0, 1, 2,\ E(Z_{1_{(i)}}) = \mu_{z_{1(i)}},\ E(Z_{2_{[i]}}) = \mu_{z_{2[i]}},\ E(Y_{[i]}) = \mu_{y_{[i]}},\ E({\epsilon_0}^2) = (\gamma C_y^2 - W_{y_{[i]}}^2) = \Delta_0,\ E({\epsilon_1}^2) = (\gamma C_{z_1}^2 - W_{z_{1(i)}}^2) = \Delta_1,\ E({\epsilon_2}^2) = (\gamma C_{z_2}^2 - W_{z_{2[i]}}^2) = \Delta_2,\ E(\epsilon_0, \epsilon_1) = (\gamma\rho_{z_1y}C_{z_1}C_y - W_{z_1y_{[i]}}) = \Delta_{01},\ E(\epsilon_0, \epsilon_2) = (\gamma\rho_{z_2y}C_{z_2}C_y - W_{z_2y_{[i]}}) = \Delta_{02}$ and $E(\epsilon_1, \epsilon_2) = (\gamma\rho_{z_1z_2}C_{z_1}C_{z_2} - W_{z_1z_{2[i]}}) = \Delta_{12}$.
Where $\gamma = 1/mr,\ C_{z_1} = S_{z_1}/\bar{Z}_1,\ C_{z_2} = S_{z_2}/\bar{Z}_2,\ C_y = S_y/\bar{Y},\ W_{z_{1(i)}}^2 = \sum_{i=1}^{m}\tau_{z_{1(i)}}^2/m^2r\bar{Z}_1^2,\ W_{z_{2[i]}}^2 = \sum_{i=1}^{m}\tau_{z_{2[i]}}^2/m^2r\bar{Z}_2^2,\ W_{y_{[i]}}^2 = \sum_{i=1}^{m}\tau_{y_{[i]}}^2/m^2r\bar{Y}^2,\ W_{z_1y_{[i]}} = \sum_{i=1}^{m}\tau_{z_1y_{[i]}}/m^2r\bar{Z}_1\bar{Y},\ \ W_{z_2y_{[i]}} = \sum_{i=1}^{m}\tau_{z_2y_{[i]}}/m^2r\bar{Z}_2\bar{Y}$,
$W_{z_1z_{2[i]}} = \sum_{i=1}^{m}\tau_{z_1z_{2[i]}}/m^2r\bar{Z}_1\bar{Z}_2,\ \ \tau_{z_{1(i)}} = (\mu_{z_{1(i)}} - \bar{Z}_1),\ \ \tau_{z_{2[i]}} = (\mu_{z_{2[i]}} - \bar{Z}_2)\ \ \tau_{y_{[i]}} = (\mu_{y_{[i]}} - \bar{Y}),\ \tau_{z_1y_{[i]}} = (\mu_{z_{1(i)}} - \bar{Z}_1)(\mu_{y_{[i]}} - \bar{Y}),\ \tau_{z_2y_{[i]}} = (\mu_{z_{2[i]}} - \bar{Z}_2)(\mu_{y_{[i]}} - \bar{Y})$ and $\tau_{z_1z_{2[i]}} = (\mu_{z_{1(i)}} - \bar{Z}_1)(\mu_{z_{2[i]}} - \bar{Z}_2)$.

Using the above notations, the estimators $T_{b_i},\ i = 1, 2$ can be expressed as

$$T_{b_1} - \bar{Y} = \bar{Y}\left[\begin{array}{l} \epsilon_0 + \alpha_1\epsilon_1 + \beta_1\epsilon_2 + \left(\frac{\alpha_1^2}{2} - \alpha_1\right)\epsilon_1^2 + \left(\frac{\beta_1^2}{2} - \beta_1\right)\epsilon_2^2 + \alpha_1\epsilon_0\epsilon_1 \\ +\beta_1\epsilon_0\epsilon_2 + \alpha_1\beta_1\epsilon_1\epsilon_2 \end{array}\right] \tag{16}$$

$$T_{b_2} - \bar{Y} = \bar{Y}\left[\ \epsilon_0 + \alpha_2\epsilon_1 + \beta_2\epsilon_2 - \tfrac{\alpha_2}{2}\epsilon_1^2 - \tfrac{\beta_2}{2}\epsilon_2^2 + \alpha_2\epsilon_0\epsilon_1 + \beta_2\epsilon_0\epsilon_2 + \alpha_2\beta_2\epsilon_1\epsilon_2\ \right]. \tag{17}$$

Squaring and taking expectation both sides of (16) and (17), we get the MSE of the estimators $T_{b_i},\ i = 1, 2$ up to the first order of approximation as

$$MSE(T_{b_i}) = \bar{Y}^2 \left[\Delta_0 + \alpha_i^2\Delta_1 + \beta_i^2\Delta_2 + 2\alpha_i\Delta_{01} + 2\beta_i\Delta_{02} + 2\alpha_i\beta_i\Delta_{12}\right]. \tag{18}$$

Minimizing the $MSE(T_{b_i})$ regarding α_i and β_i, we get

$$\alpha_{i(opt)} = \frac{(\Delta_{02}\Delta_{12} - \Delta_2\Delta_{01})}{(\Delta_1\Delta_2 - \Delta_{12}^2)} \tag{19}$$

$$\beta_{i(opt)} = \frac{(\Delta_{01}\Delta_{12} - \Delta_1\Delta_{02})}{(\Delta_1\Delta_2 - \Delta_{12}^2)}. \tag{20}$$

Putting $\alpha_{i(opt)}$ and $\beta_{i(opt)}$ in the $MSE(T_{b_i})$, we get the minimum MSE up to the first order of approximation as

$$minMSE(T_{b_i}) = \bar{Y}^2 \left[\Delta_0 - \frac{(\Delta_2\Delta_{01}^2 + \Delta_1\Delta_{02}^2 - 2\Delta_{01}\Delta_{02}\Delta_{12})}{(\Delta_1\Delta_2 - \Delta_{12}^2)} \right] \tag{21}$$

Similarly, the properties of estimators $T_{b_i},\ i = 3, 4$ can be obtained. □

4. Efficiency Conditions

This section presents the efficiency conditions by comparing the minimum MSE of the proposed estimators from (14) with the minimum MSE of the conventional estimators from (A.28), (A.29), (A.31), (A.35), (A.36) and (A.37) as

$$MSE(\bar{y}_m) > MSE(T_{b_i})$$

$$\frac{(\Delta_2\Delta_{01}^2 + \Delta_1\Delta_{02}^2 - 2\Delta_{01}\Delta_{02}\Delta_{12})}{(\Delta_1\Delta_2 - \Delta_{12}^2)} > -1 \tag{22}$$

$$MSE(\bar{y}_r) > MSE(T_{b_i})$$

$$\frac{(\Delta_2\Delta_{01}^2 + \Delta_1\Delta_{02}^2 - 2\Delta_{01}\Delta_{02}\Delta_{12})}{(\Delta_1\Delta_2 - \Delta_{12}^2)} > 2\Delta_{01} + 2\Delta_{02} - 2\Delta_{12} - \Delta_1 - \Delta_2 \tag{23}$$

$$MSE(\bar{y}_w) > MSE(T_{b_i})$$

$$a_2^2\Delta_2 > 2a_2\Delta_{02} \tag{24}$$

$$MSE(\bar{y}_{MM}) > MSE(T_{b_i})$$

$$\frac{(\Delta_2\Delta_{01}^2 + \Delta_1\Delta_{02}^2 - 2\Delta_{01}\Delta_{02}\Delta_{12})}{(\Delta_1\Delta_2 - \Delta_{12}^2)} > \frac{(\Delta_{02} - \Delta_2 - \Delta_{01} + \Delta_{12})^2}{(\Delta_1 + \Delta_2 - 2\Delta_{12})} - \Delta_2 - 2\Delta_{02} \tag{25}$$

$$MSE(\bar{y}_{k_1}) > MSE(T_{b_i})$$

$$\frac{(\Delta_2\Delta_{01}^2 + \Delta_1\Delta_{02}^2 - 2\Delta_{01}\Delta_{02}\Delta_{12})}{(\Delta_1\Delta_2 - \Delta_{12}^2)} > - \left[\begin{array}{l} +\frac{1}{4}(k_1 + 2\eta_1)^2\Delta_1 + \frac{1}{4}(k_2 + 2\eta_2)^2\Delta_2 \\ -(k_1 + 2\eta_1)\Delta_{01} - (k_2 + 2\eta_2)\Delta_{02} \\ +\frac{1}{2}(k_1 + 2\eta_1)(k_2 + 2\eta_2)\Delta_{12} \end{array} \right] \tag{26}$$

$$MSE(\bar{y}_{k_2}) > MSE(T_{b_i})$$

$$\frac{(\Delta_2\Delta_{01}^2 + \Delta_1\Delta_{02}^2 - 2\Delta_{01}\Delta_{02}\Delta_{12})}{(\Delta_1\Delta_2 - \Delta_{12}^2)} > -\frac{1}{\bar{Y}^2} \left[\begin{array}{l} -\frac{1}{4}(k_1\bar{Y} + 2\Phi_1\bar{Z}_1)^2\Delta_1 - \frac{1}{4}(\bar{Y}k_2 + 2\Phi_2\bar{Z}_2)^2\Delta_2 \\ +\bar{Y}(\bar{Y}k_1 + 2\Phi_1\bar{Z}_1)\Delta_{01} + \bar{Y}(\bar{Y}k_2 + 2\Phi_2\bar{Z}_2)\Delta_{02} \\ -\frac{1}{2}(k_2\bar{Y} - 2\Phi_1\bar{Z}_1)(\bar{Y}k_2 + 2\Phi_2\bar{Z}_2)\Delta_{12} \end{array} \right] \tag{27}$$

If the above conditions are well satisfied, then the suggested estimators become superior than the existing estimators.

5. Numerical Study

To support the efficiency conditions presented in the previous section, we perform a numerical study using some real populations that are given here under.

(1) **Source:** Singh (2003, pp. 1115), y=Season average price (in \$) per pound in 1996, z_1=Season average price (in \$) per pound in 1995, z_2=Season average price (in \$) per pound in 1994, N=36, $\bar{Y}$=0.2032, $\bar{Z}_1$=0.1856, $\bar{Z}_2$=0.1708, S_y=0.0803, S_{z_1}=0.0752, S_{z_2}=0.0634, ρ_{z_1y}=0.8775, ρ_{z_2y}=0.8577 and $\rho_{z_1z_2}$=0.8780.

(2) **Source:** Sarndal et al. (2003, pp. 652-659), y=Total number of seats in municipal council of Sweden in 1982, z_1=Number of conservative seats in municipal council of Sweden in 1982, z_2=Number of social-democratic seats in municipal council of Sweden in 1982, N=284, $\bar{Y}$=47.5, $\bar{Z}_1$=9.05, $\bar{Z}_2$=22.11, S_y=11.06, S_{z_1}=4.95, S_{z_2}=7.34, ρ_{z_1y}=0.65, ρ_{z_2y}=0.75 and $\rho_{z_1z_2}$=0.20.

(3) **Source:** Singh (2003, pp. 1116), y=Number of fish caught throughout the year 1995, z_1=Number of fish caught throughout the year 1994, z_2=Number of fish caught throughout the year 1993, N=69, $\bar{Y}$=4514.89, $\bar{Z}_1$=4954.43, $\bar{Z}_2$=4591.07, $S_y = 6099.14$, $S_{z_1} = 7058.98$, $S_{z_2} = 6315.21$, $\rho_{z_1y} = 0.9601$, $\rho_{z_2y} = 0.9564$ and $\rho_{z_1z_2} = 0.9729$.

From these populations, we select a ranked set sample of size $n = 12$ using RSS such that the set size $m = 3$ and the number of cycles $r = 4$. The percent relative efficiency (PRE) of various estimators T (T=$\bar{y}_r$, $\bar{y}_w$, $\bar{y}_{mm}$, $\bar{y}_{k_i}, i = 1, 2$ and $T_{b_i}, i = 1, 2$) w.r.t. mean per unit estimator $\bar{y}_m$ is calculated utilizing the following expression.

$$PRE = \frac{MSE(\bar{y}_m)}{MSE(T)} \times 100.$$

The results of the numerical study for these populations are reported in Table 1 by PRE.

Table 1 PRE of different estimators for real populations

Estimators	Population 1	Population 2	Population 3
$\bar{y}_m$	100	100	100
$\bar{y}_r$	81.6331	22.4555	78.3802
$\bar{y}_w$	252.3034	272.8118	851.6827
$\bar{y}_{mm}$	28.6098	20.5741	25.3191
$\bar{y}_{k_1}$	108.9130	282.0447	44.2769
$\bar{y}_{k_2}$	90.8850	319.1730	17.2778
$T_{b_i},\ i = 1, 2$	512.4137	559.7327	1428.252

6. Simulation Study

To enhance the efficiency conditions, we have carried out a simulation study based on artificially generated symmetric and asymmetric populations. The description of the populations is given below.

(1) We generate a trivariate normal population of size N=1000 with parameters $\bar{Y} = 10, \bar{Z}_1 = 15, \bar{Z}_2 = 20, \sigma_y = 15, \sigma_{z_1} = 20, \sigma_{z_2} = 25$ with correlation coefficients ρ_{z_1y}=0.89, ρ_{z_2y}=0.79 and $\rho_{z_1z_2}$=0.69.

(2) The triplet (y, z_1, z_2) is generated of size N=500 such that $x_1 \sim$ Weibull(0.5, 1), $e \sim$ N(0, 1), z_2=$1.5z_1^{0.5} + e$ and $y = 8z_1 + 7z_2 + e$, where $\rho_{z_1y} > \rho_{z_2y}$.

The RSS procedure is considered to draw 6 ranked set samples each with set size $m = 3$ and number of cycle $r = 4$ $i.e.$ $n = 12$ from each population. Using 15,000 iterations, the gain in PRE of different estimators T w.r.t. mean per unit estimator $\bar{y}_m$ is obtained as

$$PRE = \frac{MSE(\bar{y}_m)}{MSE(T)} \times 100 = \frac{\frac{1}{15{,}000}\sum_{i=1}^{15{,}000}(\bar{y}_m - \bar{Y})^2}{\frac{1}{15{,}000}\sum_{i=1}^{15{,}000}(T - \bar{Y})^2} \times 100.$$

The simulation outcomes are displayed in terms of PRE from 2 to 3 which show the superiority of the suggested estimators $T_{b_i},\ i=1,2$ in comparison to the usual mean estimator $\bar{y}_m$, classical ratio estimator $\bar{y}_r$, ratio type estimator $\bar{y}_a$, $\bar{y}_w$ envisaged by Abu-Dayyeh et al. (2009), Mehta and Mandowara (2014) estimator $\bar{y}_{mm}$, Khan and Shabbir (2016) estimator $\bar{y}_{k_1}$ and Khan et al. (2019) estimator $\bar{y}_{k_2}$.

Table 2 PRE of different estimators for artificially generated normal population

Samples Estimators	1	2	3	4	5	6
$\bar{y}_m$	100	100	100	100	100	100
$\bar{y}_r$	69.4632	103.5917	118.2773	58.6357	87.9153	82.5761
$\bar{y}_w$	112.2593	130.5996	134.4317	113.1973	101.4816	110.2743
$\bar{y}_{mm}$	48.9815	131.9844	100.6957	56.9013	84.3605	73.5406
$\bar{y}_{k_1}$	102.8539	135.0452	120.2090	116.9718	101.2254	106.1422
$\bar{y}_{k_2}$	107.3609	122.1814	122.8091	117.5623	101.2495	106.2039
$T_{b_i},\ i=1,2$	113.0088	135.4848	134.795	118.2017	101.5006	111.9497
$W^2_{y_{[i]}}$	0.018550	0.001211	0.032321	0.000000	0.067096	0.000304
$W^2_{z_1(i)}$	0.090040	0.029912	0.005574	0.024037	0.043274	0.045211
$W^2_{z_2[i]}$	0.017093	0.000000	0.003656	0.000462	0.000313	0.000390
$W^2_{z_1y[i]}$	-0.040868	0.006020	0.013422	0.000328	-0.053884	-0.003710
$W^2_{z_1y[i]}$	0.017807	-0.000006	0.010870	0.000005	-0.004587	-0.000344
$W^2_{z_1z_2[i]}$	-0.039231	-0.000308	0.004514	0.00333	0.003683	0.004199

Table 3 PRE of different estimators for artificially generated Weibull population

Samples Estimators	1	2	3	4	5	6
$\bar{y}_m$	100	100	100	100	100	100
$\bar{y}_r$	80.7379	39.2487	42.3787	41.6355	42.1923	46.8327
$\bar{y}_w$	163.8211	105.4807	105.3451	121.0608	112.7139	121.1342
$\bar{y}_{mm}$	27.4983	63.6670	59.4895	42.3909	55.6484	120.2273
$\bar{y}_{k_1}$	155.6881	106.8107	103.9006	118.3606	110.8535	117.3293
$\bar{y}_{k_2}$	160.2798	107.0126	102.6795	119.4986	113.0195	117.2040
$T_{b_i},\ i=1,2$	163.8268	107.2256	105.9513	121.7621	113.3722	129.7583
$W^2_{y_{[i]}}$	0.004433	0.000006	0.047772	0.008802	0.001311	0.006879
$W^2_{z_1(i)}$	0.058527	0.047702	0.021969	0.085072	0.000195	0.008661
$W^2_{z_2[i]}$	0.002116	0.004293	0.004699	0.016382	0.004851	0.002903
$W^2_{z_1y[i]}$	-0.016107	-0.001753	0.032396	-0.027364	-0.000505	-0.007719
$W^2_{z_1y[i]}$	0.003063	-0.000525	0.014982	0.012008	0.002522	0.004469
$W^2_{z_1z_2[i]}$	-0.011131	0.014311	0.010160	-0.037332	-0.000972	-0.005014

7. Discussion of Results

Table 1 based on the results of the numerical study in terms of PRE for the real populations show the ascendancy of the proposed class of estimators over the usual mean estimator, classical ratio estimators, Aby-Dayyeh et al. (2009) estimator, Mehta and Mandowara (2014) estimator, Khan and Shabbir (2016) estimator and Khan et al. (2019) estimator. Furthermore, the numerical study is generalized by the simulation study using artificially generated normal and Weibull populations and the results are reported in Tables 2 and 3, respectively. From Table 2, the PRE of the proposed estimators dominate the PRE of the existing estimators in samples 1-6. The similar inclination in the PRE values can be observed from the simulation results of 3.

8. Conclusion

In this article, we have proposed some log type estimators based on bivariate auxiliary information under RSS along side the expressions of their bias and MSE. The theoretical comparison is made with respect to the usual mean estimator $\bar{y}_m$, classical ratio estimator $\bar{y}_r$, ratio type estimator $\bar{y}_w$ envisaged by Abu-Dayyeh et al. (2009), Mehta and Mandowara (2014) estimator $\bar{y}_{mm}$, Khan and Shabbir (2016) estimator $\bar{y}_{k_1}$ and Khan et al. (2019) estimator $\bar{y}_{k_2}$ and the efficiency conditions are obtained. The theoretical results are illustrated with a numerical study based on some real populations and a simulation study based on some artificially generated symmetric and asymmetric populations. It has been observed from the results of numerical and simulation studies disclosed from 1 to 3 that the PRE of the suggested estimators repress the PRE of the existing estimators. Hence, the suggested estimators may be considered by survey practitioners in practical use as an efficient alternative of the existing estimators.

Furthermore, the proposed estimators can be developed for the estimation of population mean using stratified ranked set sampling.

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Appendix A

The MSE of the conventional mean estimator is expressed by

$$MSE(\bar{y}_m) = \bar{Y}^2\Delta_0. \tag{A.28}$$

The MSE of the conventional ratio estimator $\bar{y}_r$ is expressed by

$$MSE(\bar{y}_r) = \bar{Y}^2\left[\ \Delta_0 + \Delta_1 + \Delta_2 - 2\Delta_{01} - 2\Delta_{02} + 2\Delta_{12}\ \right]. \tag{A.29}$$

The MSE of the estimator $\bar{y}_w$ is expressed by

$$MSE(\bar{y}_w) = \bar{Y}^2\left[\ \Delta_0 + w_1^2a_1^2\Delta_1 + w_2^2a_2^2\Delta_2 - 2w_1a_1\Delta_{01} - 2w_2a_2\Delta_{02} + 2w_1w_2a_1a_2\Delta_{12}\ \right]. \tag{A.30}$$

The optimum values of scalars are expressed by

$$\begin{aligned} a_{1(opt)} &= \frac{(\Delta_2\Delta_{01} - \Delta_{02}\Delta_{12})}{(\Delta_1\Delta_2 - \Delta_{12}^2)} \\ a_{2(opt)} &= \frac{(\Delta_1\Delta_{02} - \Delta_{01}\Delta_{12})}{(\Delta_1\Delta_2 - \Delta_{12}^2)} \\ w_{1(opt)} &= \frac{(a_2^2\Delta_2 + a_1\Delta_{01} + a_2\Delta_{02} - a_1a_2\Delta_{12})}{(a_1^2\Delta_1 + a_2^2\Delta_2 - 2a_1a_2\Delta_{12})} \\ w_{2(opt)} &= 1 - w_{1(opt)}. \end{aligned}$$

The minimum MSE at optimum values of scalars is expressed by

$$minMSE(\bar{y}_w) = \bar{Y}^2\left[\Delta_0 + a_2^2\Delta_2 - 2a_2\Delta_{02} - \frac{(\Delta_2\Delta_{01}^2 + \Delta_1\Delta_{02}^2 - 2\Delta_{01}\Delta_{02}\Delta_{12})}{(\Delta_1\Delta_2 - \Delta_{12}^2)}\right]. \tag{A.31}$$

The MSE of the estimator $\bar{y}_{mm}$ is expressed by

$$MSE(\bar{y}_{mm}) = \bar{Y}^2 \left[\ \Delta_0 + w_1^2\Delta_1 + w_2^2\Delta_2 + 2w_1\Delta_{01} + 2w_2\Delta_{02} + 2w_1w_2\Delta_{12} \ \right]. \tag{A.32}$$

The optimum value of w_1 and w_2 are obtained by minimizing (A.32) w.r.t. w_1 and w_2 as

$$w_{1(opt)} = \left[\frac{\Delta_2 + \Delta_{01} + \Delta_{02} - \Delta_{12}}{\Delta_1 + \Delta_2 - 2\Delta_{12}}\right] \tag{A.33}$$

$$w_{2(opt)} = 1 - w_{1(opt)}. \tag{A.34}$$

The minimum MSE at optimum values of scalars is expressed by

$$minMSE(\bar{y}_{mm}) = \bar{Y}^2 \left[\Delta_0 + \Delta_2 - 2\Delta_{02} - \frac{(\Delta_2\Delta_{01}^2 + \Delta_1\Delta_{02}^2 - 2\Delta_{01}\Delta_{02}\Delta_{12})}{(\Delta_1\Delta_2 - \Delta_{12}^2)}\right]. \tag{A.35}$$

The minimum MSE of estimator $\bar{y}_{k_1}$ is expressed by

$$minMSE(\bar{y}_{k_1}) = \bar{Y}^2 \left[\begin{array}{l} \Delta_0 + \frac{1}{4}(k_1 + 2\eta_1)^2\Delta_1 + \frac{1}{4}(k_2 + 2\eta_2)^2\Delta_2 - (k_1 + 2\eta_1)\Delta_{01} \\ -(k_2 + 2\eta_2)\Delta_{02} + \frac{1}{2}(k_1 + 2\eta_1)(k_2 + 2\eta_2)\Delta_{12} \end{array} \right] \tag{A.36}$$

where

$$\eta_{1(opt)} = \frac{(\Delta_{01}\Delta_2 - \Delta_{02}\Delta_{01})}{(\Delta_1\Delta_2 - \Delta_{12}^2)} - \frac{k_1}{2}$$

$$\eta_{2(opt)} = \frac{(\Delta_{02}\Delta_1 - \Delta_{01}\Delta_{12})}{(\Delta_1\Delta_2 - \Delta_{12}^2)} - \frac{(k_2)}{2}.$$

The minimum MSE of estimator $\bar{y}_{k_2}$ is expressed by

$$minMSE(\bar{y}_{k_2}) = \left[\begin{array}{l} \bar{Y}^2\Delta_0 + \frac{1}{4}(k_1\bar{Y} + 2\Phi_1\bar{Z}_1)^2\Delta_1 + \frac{1}{4}(\bar{Y}k_2 + 2\Phi_2\bar{Z}_2)^2\Delta_2 \\ -\bar{Y}(\bar{Y}k_1 + 2\Phi_1\bar{Z}_1)\Delta_{01} - \bar{Y}(\bar{Y}k_2 + 2\Phi_2\bar{Z}_2)\Delta_{02} \\ +\frac{1}{2}(k_1\bar{Y} - 2\Phi_1\bar{Z}_1)(\bar{Y}k_2 + 2\Phi_2\bar{Z}_2)\Delta_{12} \end{array} \right] \tag{A.37}$$

where

$$\Phi_{1(opt)} = \frac{\bar{Y}}{\bar{Z}_1}\eta_{1(opt)}$$

$$\Phi_{2(opt)} = \frac{\bar{Y}}{\bar{Z}_2}\eta_{2(opt)}.$$