# A New Stereographic T-X Family Probability Distribution

**Durgada Vara Lakshmi Prasanna [a]\*, Rachakonda Subba Rao [b] and Gunturu Venkata Sita Rama Deekshitulu [c]**

[a] Research Scholar, J.N.T.U.K, Kakinada, Assistant Professor, Department of Mathematics, Aditya University, Surampalem, Andhra Pradesh, India

[b] Department of Engineering Mathematics & Humanities, S.R.K.R Engineering College, Bhimavaram, Andhra Pradesh, India

[c] Department of Mathematics, J.N.T.U.K, Kakinada, Andhra Pradesh, India

*Corresponding author; e-mail: lakshmisreee.21@gmail.com

## Abstract

A new circular Transformed Transformer family distribution-Stereographic Weibull Rayleigh Distribution (SWRD), is generated by adopting the methodology of inverse stereographic projection. The graphs of probability density function(pdf) and cumulative probability distribution functions(cdf) are plotted, both in linear and circular forms. A few characteristics like trigonometric moments, central trigonometric moments, mean, variance, circular standard deviation, skewness, kurtosis of SWRD are evaluated. Also reliability, hazard function and reverse hazard function of SWRD are studied. A real data set is considered to study the Goodness of fit for the proposed model. The suitable graphs are also presented for the assumed distribution.

## 1. Introduction

T-X family or transformed transformer family is a new family of probability distributions in which two existing probability distributions are combined to obtain more flexible models for fitting the real data. Alzaatreh et al. (2013) developed this new family of distributions using quantile function. Subba Rao et al. (2016) worked on acceptance sampling based on life tests using a T-X family distribution, Pareto Rayleigh. Ahmad et al. (2017) developed a mixed model, Weibull-Rayleigh distribution. El-Morshedy et al. (2021) proposed a new generalization of the odd Weibull-G family by consolidating two notable families of distributions. Choudary et al. (2021) extended the modified Weibull distribution with an additional shape parameter, in order to provide more flexibility to its density and the hazard rate function.

From the existing probability distributions defined on the real line, number of discrete and continuous circular probability models were generated by Fisher (1996), Jammalamadaka and Sengupta (2001), Dattatreya Rao et al. (2007), Toshihiro et al. (2010), Girija et al. (2013), Radhika et al. (2013), Phani et al. (2014). In all the above models they adopted the methodologies wrapping linear distribution around unit circle, stereographic projections, offset distributions, raising sun function etc. Minh

and Farnum (2003) introduced a transformation to convert points on the real line to unit circle in the complex plane.

In this paper the basic definitions which are required to study the new model are given in Section 1. A stereographic weibull rayleigh distribution is defined and its graphs for pdf and cdf are plotted in Section 2. The characteristic function and trigonometric moments of SWRD are presented in Section 3. Section 4 deals with reliability function, hazard function and reverse hazard function. In Section 5, the Goodness of fit of the stereographic Weibull Rayleigh distribution is verified for a real data set and conclusions and further scope of the work are mentioned in Section 6.

### 1.1. Basic definitions
**T-X family**

The cumulative probability distribution function of T-X family for a random variable $X$ is

$$J(x) = \int_a^{W(S(x))} m(t)dt = M\{W(S(x))\},$$

and the corresponding probability density function is given by

$$j(x) = \frac{d}{dx} W(S(x))m\{W(S(x))\},$$

where $M(t)$ and $m(t)$ are the probability distribution function and density function of a continuous random variable $T > 0$ defined on $[0, \infty)$. $S(x)$ is the cdf of a random variable $X$.

**Inverse Stereographic Projection**

Inverse Stereographic Projection (ISP) is a one-to-one mapping defined as

$$T(\theta) = y = k + l\frac{sin\theta}{1 + cos\theta} = k + ltan\left(\frac{\theta}{2}\right),$$

where $y \in (-\infty, \infty), k \in R, l > 0$ and $\theta \in [-\pi, \pi)$ . Then

$$T^{-1}(y) = \theta = 2tan^{-1}\left(\frac{y - k}{l}\right),$$

is a random point on the unit circle.

**Theorem 1** *$Y$ is random variable defined on the interval $(-\infty, \infty)$. The probability distribution function and density function of $Y$ are denoted by $G(y)$ and $g(y)$ respectively . Let the probability distribution function and probability density functions of this random point be $H(\theta)$ and $h(\theta)$ respectively. Then $H(\theta)$ and $h(\theta)$ can be expressed in terms of $G(y)$ and $g(y)$ as*
for $l > 0$,

(i) $H(\theta) = G\left(k + ltan\left(\frac{\theta}{2}\right)\right),$
$$= \frac{l}{2}sec^2\left(\frac{\theta}{2}\right)g\left(k + ltan\left(\frac{\theta}{2}\right)\right).$$

(ii) $h(\theta) = l\left[\frac{1+tan\left(\frac{\theta}{2}\right)}{2}\right]g\left(k + ltan\left(\frac{\theta}{2}\right)\right).$

### 2. Stereographic Weibull Rayleigh Distribution
The pdf of a Weibull Rayleigh distribution is

$$g(y) = \frac{\alpha}{\lambda}\frac{y}{p^2}\left(\frac{y^2}{2\lambda p^2}\right)^{n-1} e^{-\left(\frac{y^2}{2\lambda p^2}\right)^\alpha}, \alpha, \lambda > 0, p > 0,$$

where $\alpha$ and $\lambda$ are shape parameter and scale parameters, respectively.

It's probability distribution function is given by

$$G(y) = 1 - e^{-\left(\frac{y^2}{2\lambda p^2}\right)^\alpha}, \alpha, \lambda > 0, p > 0.$$

By applying Theorem 1 on Weibull Rayleigh distribution, a new circular sterographic T-X family probability model-Stereographic Weibull Rayleigh Distribution (SWRD) is developed, its pdf and cdf are given by

$$h(\theta) = \frac{\alpha}{2\lambda p^2} tan\left(\frac{\theta}{2}\right) sec^2\frac{\theta}{2}\left(\frac{1}{2\lambda p^2}tan^2\left(\frac{\theta}{2}\right)\right)^{\alpha-1} e^{-\left(\frac{1}{2\lambda p^2}tan^2\left(\frac{\theta}{2}\right)\right)^\alpha},$$

where $\alpha$ and $\lambda$ are shape and scale parameters, respectively.

$$H(\theta) = 1 - e^{-\left(\frac{tan^2\left(\frac{\theta}{2}\right)}{2\lambda p^2}\right)^\alpha}, \alpha > 0, \lambda > 0, p > 0, 0 \le \theta \le \pi.$$

The graphical representations of pdf of stereographic Weibull Rayleigh distribution are shown below by fixing shape parameter and varying scale parameter and vice versa both in linear and circular form.
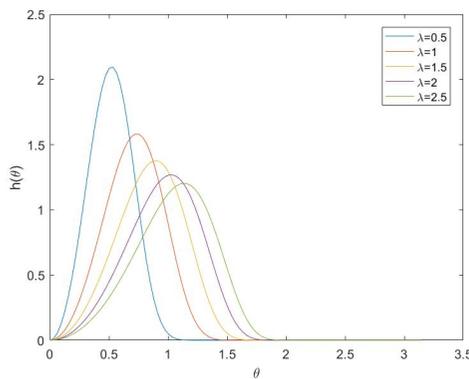


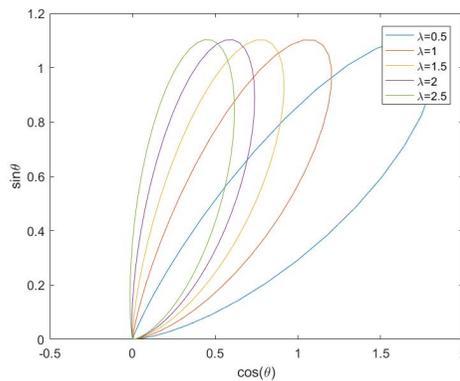**Figure 1** Graph of pdf of stereographic Weibull Rayleigh model for $\alpha$=1.5 and $p$=1.5 (Linear)



**Figure 2** Graph of pdf of stereographic Weibull Rayleigh model for $\alpha$=1.5 and $p$=1.5 (Circular)
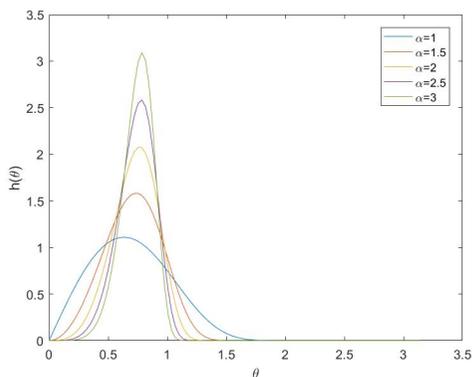
**Figure 3** Graph of pdf of stereographic Weibull Rayleigh model for $\lambda$=1 and $p$=1.5 (Linear)
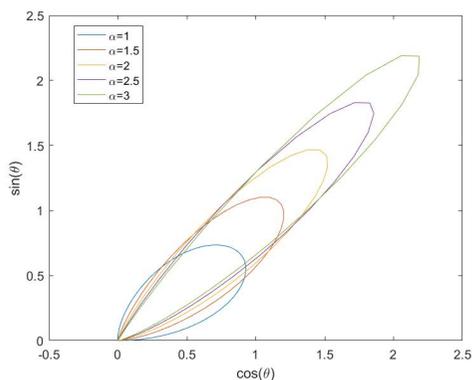


**Figure 4** Graph of pdf of stereographic Weibull Rayleigh model for $\lambda$=1 and $p$=1.5 (Circular)
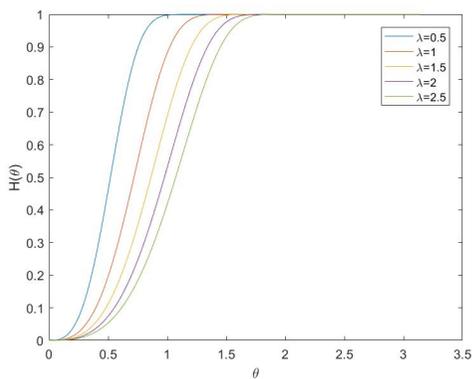


**Figure 5** Graph of CDF of stereographic Weibull Rayleigh model for $\alpha$=1.5 and $p$=1.5 (Linear)
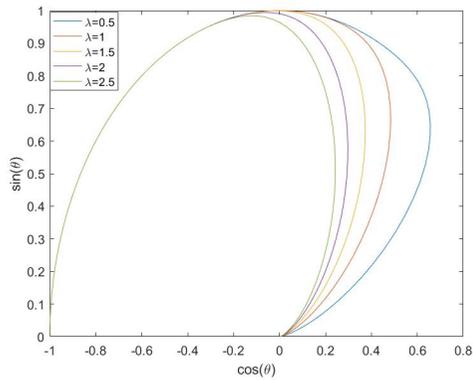
**Figure 6** Graph of CDF of stereographic Weibull Rayleigh model for $\alpha$=1.5 and $p$=1.5 (Circular)
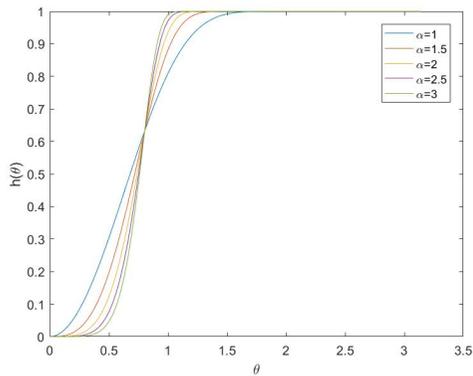


**Figure 7** Graph of CDF of stereographic Weibull Rayleigh model for $\lambda$=1 and $p$=1.5 (Linear)
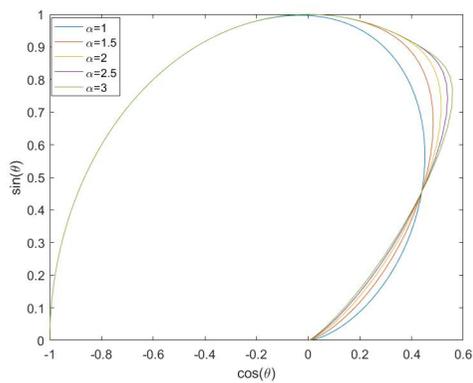


**Figure 8** Graph of CDF of stereographic Weibull Rayleigh model for $\lambda$=1 and $p$=1.5 (Circular)

The graph of pdf is positively skewed and tends to be more and more peaked for each value of shape parameter ($\alpha$) by keeping scale parameter ($\lambda$) fixed. It is also noticed that the peakedness of the curve decreases as the scale parameter value increases by fixing the shape parameter. Also in both the cases, curve tends to be symmetric.

The graphical representations of CDF of stereographic Weibull Rayleigh distribution are shown below by fixing shape parameter and varying scale parameter and vice versa both in linear and circular form.

## 3. Characteristics of SWRD

### Characteristic function of stereographic Weibull Rayleigh distribution

The characteristic function of SWRD is given by

$$\phi_{X_s}(q) = \int\limits_{-\pi}^{\pi} e^{iq\theta} h(\theta) d\theta,$$

$$= \int\limits_{-\pi}^{\pi} (cosq\theta + isinq\theta) h(\theta) d\theta,$$

$$= \int\limits_{-\pi}^{\pi} (cosq\theta h(\theta) + isinq\theta h(\theta)) d\theta,$$

$$= \int\limits_{-\pi}^{\pi} cosq\theta h(\theta) d\theta + i \int\limits_{-\pi}^{\pi} sinq\theta h(\theta) d\theta.$$

### Trigonometric moments

The trigonometric moments of the probability distribution are given by

$$\{\phi_q : q = \pm 1, \pm 2, \pm 3, \dots\}, \text{ where } \phi_q = \alpha_q + i\beta_q.$$

$\alpha_q = E(cosq\theta)$, represents $q^{th}$ order cosine moments and $\beta_q = E(sinq\theta)$ represents $q^{th}$ order sine moments of the random angle $\theta$.

The sine moments of the stereographic Weibull Rayleigh distribution are given by

$$\beta_q = E(sinq\theta),$$

$$= \int\limits_{-\pi}^{\pi} sinq\theta h(\theta) d\theta,$$

$$= \int\limits_{-\pi}^{\pi} sinq\theta \frac{\alpha}{2\lambda p^2} tan\left(\frac{\theta}{2}\right) sec^2\frac{\theta}{2} \left(\frac{1}{2\lambda p^2} tan^2\left(\frac{\theta}{2}\right)\right)^{\alpha-1} e^{-\left(\frac{1}{2\lambda p^2} tan^2\left(\frac{\theta}{2}\right)\right)^{\alpha}} d\theta.$$

The cosine moments of the stereographic Weibull Rayleigh distribution are given by

$$\alpha_q = E(cosq\theta),$$

$$= \int\limits_{-\pi}^{\pi} cosq\theta h(\theta) d\theta,$$

$$= \int\limits_{-\pi}^{\pi} cosq\theta \frac{\alpha}{2\lambda p^2} tan\left(\frac{\theta}{2}\right) sec^2\frac{\theta}{2} \left(\frac{1}{2\lambda p^2} tan^2\left(\frac{\theta}{2}\right)\right)^{\alpha-1} e^{-\left(\frac{1}{2\lambda p^2} tan^2\left(\frac{\theta}{2}\right)\right)^{\alpha}} d\theta.$$

The cosine and sine (trigonometric) moments $\alpha_q$ and $\beta_q$, which are computed using MATLAB, are shown in the given Table 1.

The characteristics of circular distribution are evaluated using the expressions given in Mardia, Jupp (2000).

**Table 1** Characteristics of stereographic Weibull Rayleigh distribution for $\alpha = 1.5$ and $p = 1.5$

| Stereographic Weibull Rayleigh Distribution | | $\lambda = 0.5$ | $\lambda = 1$ | $\lambda = 1.5$ | $\lambda = 2$ | $\lambda = 2.5$ |
|---|---|---|---|---|---|---|
| Mean | $\mu$ | 0.5229 | 0.7198 | 0.8520 | 0.9566 | 1.0556 |
| Trigonometric Moments | $\alpha_1$ | 0.85 | 0.73 | 0.63 | 0.55 | 0.47 |
| | $\alpha_2$ | 0.48 | 0.13 | -0.12 | -0.29 | -0.42 |
| | $\beta_1$ | 0.49 | 0.64 | 0.72 | 0.78 | 0.70 |
| | $\beta_2$ | 0.81 | 0.88 | 0.85 | 0.78 | 0.70 |
| Resultant length | $\rho$ | 0.9811 | 0.9708 | 0.9567 | 0.9544 | 0.9538 |
| | | 0.9415 | 0.8895 | 0.8584 | 0.8322 | 0.8163 |
| Variance | $V_0$ | 0.0189 | 0.0292 | 0.0433 | 0.0456 | -0.0556 |
| Circular Standard deviation | $\sigma_0$ | 0.1953 | 0.2435 | 0.2975 | 0.3055 | 0.329i |
| Central Trigonometric Moments | $\alpha_1^*$ | 0.9811 | 0.9708 | 0.9567 | 0.9544 | 0.9538 |
| | $\alpha_2^*$ | 0.9415 | 0.8894 | -0.8584 | -0.8322 | -0.8163 |
| | $\beta_1^*$ | 0 | 0 | 0 | 0 | 0 |
| | $\beta_2^*$ | -0.0094 | -0.0138 | -0.0061 | -0.0113 | -0.000006 |
| Skewness | $\gamma_1^0$ | -3.6177 | -2.7657 | -0.677 | -1.1605 | |
| Kurtosis | $\gamma_2^0$ | 41.9462 | 1.3875 | -904.655 | -799.237 | -665.7087 |

**Table 2** Characteristics of stereographic Weibull Rayleigh distribution for $\lambda = 1$ and $p = 1.5$

| Stereographic Weibull Rayleigh Distribution | | $\alpha = 1$ | $\alpha = 1.5$ | $\alpha = 2$ | $\alpha = 2.5$ | $\alpha = 3$ |
|---|---|---|---|---|---|---|
| Mean | $\mu$ | 0.696 | 0.7197 | 0.727 | 0.743 | 0.743 |
| Trignometric Moments | $\alpha_1$ | 0.73 | 0.73 | 0.73 | 0.73 | 0.73 |
| | $\alpha_2$ | 0.15 | 0.13 | 0.11 | 0.09 | 0.07 |
| | $\beta_1$ | 0.61 | 0.64 | 0.65 | 0.67 | 0.67 |
| | $\beta_2$ | 0.79 | 0.88 | 0.93 | 0.95 | 0.96 |
| Resultant length | $\rho$ | 0.9513 | 0.9708 | 0.9774 | 0.9908 | 0.9908 |
| | | 0.8041 | 0.8895 | 0.9364 | 0.9542 | 0.9625 |
| Variance | $V_0$ | 0.0487 | 0.0292 | 0.0226 | 0.0092 | 0.0092 |
| Circular Standard deviation | $\sigma_0$ | 0.3160 | 0.2435 | 0.2138 | 0.1360 | 0.1360 |
| Central Trignometric Moments | $\alpha_1^*$ | 0.9513 | 0.9708 | 0.9774 | 0.9908 | 0.9908 |
| | $\alpha_2^*$ | 0.8041 | 0.8894 | 0.9364 | 0.9542 | 0.9624 |
| | $\beta_1^*$ | 0 | 0 | 0 | 0 | 0 |
| | $\beta_2^*$ | -0.00723 | -0.0137 | -0.00094 | -0.0095 | -0.0115 |
| Skewness | $\gamma_1^0$ | -0.6727 | -2.7456 | -0.2767 | -10.766 | 13.032 |
| Kurtosis | $\gamma_2^0$ | -6.2714 | 1.3875 | 46.5607 | -112.296 | -15.415 |

## 4. Reliability and Hazard Function of SWARD

**Reliability**

The probability that the system will not fail for $a$ time $t$ is known as reliability of the system. The reliability function of SWRD is

$$R(\theta) = 1 - H(\theta),$$

$$= 1 - \left(1 - e^{-\left(\frac{tan^2\left(\frac{\theta}{2}\right)}{2\lambda p^2}\right)^{\alpha}}\right),$$

$$R(\theta) = e^{-\left(\frac{tan^2\left(\frac{\theta}{2}\right)}{2\lambda p^2}\right)^{\alpha}}.$$

**Hazard rate**

The hazard rate of SWRD is given by

$$s(\theta) = \frac{h(\theta)}{R(\theta)},$$

$$= \frac{\frac{\alpha}{2\lambda p^2} tan\left(\frac{\theta}{2}\right) sec^2 \frac{\theta}{2} \left(\frac{1}{2\lambda p^2} tan^2\left(\frac{\theta}{2}\right)\right)^{\alpha-1} e^{-\left(\left(\frac{1}{2\lambda p^2} tan^2\left(\frac{\theta}{2}\right)\right)^{\alpha}\right)}}{e^{-\left(\frac{tan^2\left(\frac{\theta}{2}\right)}{2\lambda p^2}\right)^{\alpha}}}.$$

$$\text{Hazard rate } = \frac{\alpha}{2\lambda p^2} tan\left(\frac{\theta}{2}\right) sec^2 \frac{\theta}{2} \left(\frac{1}{2\lambda p^2} tan^2\left(\frac{\theta}{2}\right)\right)^{\alpha-1}.$$

**Reverse hazard rate**

Reversed hazard rate of a random life is the ratio of density function and its probability distribution function. The reverse hazard function is given by

$$Q(\theta) = \frac{h(\theta)}{H(\theta)},$$

$$= \frac{\frac{\alpha}{2\lambda p^2} tan\left(\frac{\theta}{2}\right) sec^2 \frac{\theta}{2} \left(\frac{1}{2\lambda p^2} tan^2\left(\frac{\theta}{2}\right)\right)^{\alpha-1} e^{-\left(\left(\frac{1}{2\lambda p^2} tan^2\left(\frac{\theta}{2}\right)\right)^{\alpha}\right)}}{1 - e^{-\left(\frac{tan^2\left(\frac{\theta}{2}\right)}{2\lambda p^2}\right)^{\alpha}}}.$$

## 5. Application

To verify the goodness of fit for the proposed model, a real data set(in radians) of cross-bed azimuths of paleocurrents measured in the Belford Anticline(New South Wales) is considered from Fisher (1996):

| | | | | |
|---|---|---|---|---|
| 4.9567 | 5.4280 | 5.8294 | 5.5851 | 5.1313 |
| 4.2586 | 4.2412 | 4.2237 | 4.5902 | 4.0841 |
| 4.6600 | 5.4978 | 5.7421 | 4.1015 | 4.2062 |
| 5.5676 | 5.3756 | 4.2761 | 4.5902 | 4.8520 |
| 4.8695. | | | | |

The linear, circular, linear histogram and rose diagrams of above dataset are shown below.
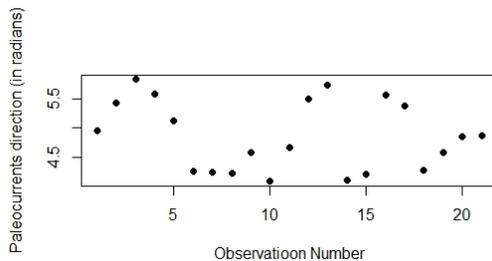
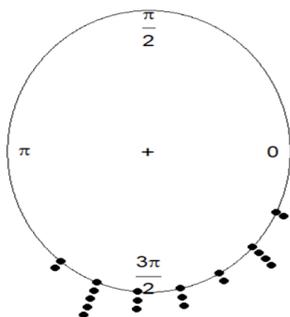**Figure 9** Linear plot of paleocurrents directions measured in radians



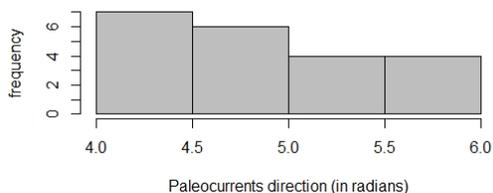**Figure 10** Circular plot of paleocurrents directions measured in radians



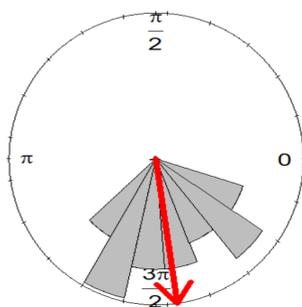**Figure 11** Linear histogram of paleocurrents measured in radians

**Figure 12** Rose diagram of paleocurrents measured in radians

In the rose diagram, red arrow represents mean of circular data. It shows that paleocurrent indicators are unidirectional. So the direction of water flow is unimodal in the geologic past of the Belford Anticline(New South Wales).

The estimates $\theta_1, \theta_2, \theta_3, ...., \theta_{21}$ are obtained by substituting given data set in cdf of stereographic weibull rayleigh distribution which are further used to carry out the goodness of fit tests.The test statistics of Kuiper test, Watson $U^2$ test are tabulated.

**Table 3** Test statistic values for sample size($n$)=21

| Tests | Test Statistic |
|---|---|
| Kuiper test | 4.7904 |
| Watson $U^2$ test | 0.04 |
| Range Test | 0.4171 |

From the data available in Deva (2012), the cut off points for sample size 21 at various levels of significance are tabulated below.

**Table 4** Cut off points for sample size($n$)=21

| Test | Level of Significance | | |
|---|---|---|---|
| | 1% | 5% | 10% |
| Kuipers Test | 0.6990 - 2.1200 | 0.7893 - 1.8521 | 0.8288 - 1.7453 |
| Watsons $U^2$ - Test | 0.0150 - 0.3048 | 0.0203 - 0.2132 | 0.0242 - 0.1814 |
| Range Test | 3.9980 - 5.6940 | 4.4727 - 5.6122 | 4.6404 - 5.5687 |

It is noticed that the value of Watson $U^2$ test lies between the cutoff points at all level of significances. The proposed model is good fit for the given data.

## 6. Conclusion

In this paper a new stereographic T-X family distribution (SWRD) is developed by adopting the technique inverse stereographic projection. Various characteristics of SWRD are tabulated by keeping shape parameter fixed and varying the scale parameter and vice versa. The results are given in Table 1 and Table 2. Also reliability, hazard rate and inverse hazard rate of the distribution are discussed keeping the further research in view. The proposed SWRD is verified for the goodness of fit to a real data set considered from Fisher (1996).

## Acknowledgements

## References

Ahmad A, Ahmad SP, Ahmed A. Characterization and Estimation of Weibull-Rayleigh distribution with Applications to life time data. Appl Math Inf Sci Lett. 2017; 5(2): 71-79.

Ayman A, Carl L, Felix F. A new method for generating families of continuous distributions. Metron. 2013; 71(1): 63-79.

Choudhary N, Abhishek T, Bhupendra S. A Flexible Bathtub-Shaped Failure Time Model: Properties and Associated Inference. Statistica. 2021; 81(1): 65-92.

Dattatreya Rao AV, Ramabhadra Sarma I, Girija SVS. On wrapped version of some life testing models. Commun Stat Theory. 2007; 36(11): 2027-2035.

Raaj D. Some contributions to circular statistics. PhD [dissertation], Andhra Pradesh: Acharya Nagarjuna University; 2012.

El-Morshedy M, Alshammari FS, Tyagi A, Elbatal I, Hamed YS, Eliwa MS. Bayesian and frequentist inferences on a type I half-logistic odd Weibull generator with applications in engineering. Entropy. 2021; 23(4): 446.

Fisher NI. Statistical analysis of circular data. Cambridge: Cambridge University Press; 1995.

Girija SVS, Dattatreya Rao AV, Phani Y. On stereographic lognormal distribution. Int J Advances Appl Sci. 2013; 2(3): 125-132.

Jammalamadaka S R, Sengupta A. Topics in circular statistics. Singapore: World scientific; 2001.

Mardia KV, Jupp PE. Statistics of directional data. England: John Wiley & Sons; 2000.

Minh DL, Farnum NR. Using bilinear transformations to induce probability distributions. Commun Stat Theor M. 2003; 32(1) :1-9.

Phani Y, Radhika AJV, Girija SVS, Dattatreya Rao AV. New circular models induced by modified inverse stereographic projection on arc tan exponential-type distribution. Int J Math Arch. 2014; 5(4): 1-6.

Radhika AJV, Phani Y, Girija SVS, Dattatreya Rao AV. On Stereographic Semicircular Gamma Model. Proceedings of the National Conference on Recent Trends in Mathematical ComputingNCRTMC. 2013.

Subba Rao R, Parsad G, Kantam RRI. Acceptance sampling based on life tests: Pareto-Rayleigh model. Int J Adv Res Sci Eng Technol. 2016; 5(8): 427-436.

Toshihiro A, Shimizu K, Pewsey A. Symmetric unimodal models for directional data motivated by inverse stereographic projection. J Jpn Stat Soc. 2010; 40(1): 045-061.