Thailand Statistician
July 2025; 23(3): 553-572
http://statassoc.or.th
Contributed paper

On the Estimation of the Reliability Characteristics of a Weighted Generalized Positive Exponential Family of Distributions

Aditi Chaturvedi [a]* and Surinder Kumar [b]
[a] Sharda School of Business Studies, Sharda University, Greater Noida, India
[b] Department of Statistics, Babasaheb Bhimrao Ambedkar University,
Lucknow, Uttar Pradesh, India
*Corresponding author; e-mail: aditi.chaturvedi@sharda.ac.in; caditic@gmail.com

Received: 1 March 2022
Revised: 20 January 2023
Accepted: 12 February 2023

Abstract

A weighted generalization of positive exponential family of distributions is taken into consideration and its properties are studied. Considering two measures of reliability, namely $R(t) = P(X > t)$ and $P = P(X > Y)$, their Uniformly minimum variance unbiased (UMVU) estimators, Maximum Likelihood (ML) estimators and Method of Moment (MM) estimators are developed and the performance of the estimators are investigated using Monte Carlo Simulation. We investigate two empirical data sets to illustrate the proposed approach.

Keywords: Monte Carlo simulation, MLE, point estimation, UMVUE, weighted exponential.

1. Introduction

The concept of weighted distributions was introduced by Fisher (1934) and first applied by Rao (1965) to model statistical data which standard distributions could not formulate. Moreover, weighted distribution theory gives a unified approach to dealing with model specification and data interpretation problems. Weighted distributions frequently occur in studies related to reliability, survival analysis, analysis of family data, biomedicine, ecology and several other areas.

Suppose X is a non-negative continuous random variable (r.v.) with probability density function (pdf) $f(x)$. The pdf of the weighted r.v. X_w is given by

$$f_w(x) = \frac{w(x)\, f(x)}{\mu_w}, \quad x > 0, \tag{1}$$

where $w(x)$ is a non-negative weight function and $\mu_w = E[W(X)] < \infty$.

For the weight function $w(x) = x^c$ in equation (1), the resultant distribution is named size biased distribution. For $c = 1$, the weight function depends on the length of units of interest and the resulting distribution is called length biased distribution while, for $c = 2$, the resulting distribution is called area biased distribution. Gupta and Kundu (2009) introduced a new weighted exponential model which has the pdf whose shape is very close to the shape of Weibull, gamma or generalized exponential distributions and hence can be used as their alternative. Das and Roy (2011a), Das and Roy (2011b) considered length-biased weighted generalized Rayleigh distribution and length-biased weighted Weibull distribution. They explored various properties of these distributions and demonstrated their relationship with several well known distributions. Kilany (2016) considered weighted

Lomax distribution and studied its properties. Fallah and Kazemi (2020) revisited the generalized weighted exponential distribution. They developed some new distributional results for this distribution and provided its closed form expressions and related characteristics. **?** introduced weighted Half-Logistic distribution and discussed in detail its properties and applications. Alahmadi et al. (2022) have introduced a new version of weighted Weibull distribution and showed its application to model COVID-19 data. Yazgan et al. (2022) have presented a fuzzy stress-strength reliability model for weighted exponential distribution.

Kumar and Chaturvedi (2020) generalized the positive exponential family proposed by Liang (2008), which covers as many as ten distributions to be particular cases. They explored the properties and different methods of estimation of the parameters and the reliability characteristics associated with this family. Chaturvedi et al. (2021a) have developed sequential and two-stage procedures for the scale parameter of this generalized positive exponential family of distributions. The main goal of the present paper is to provide an extension of the generalized positive exponential family of distributions. The size biased distribution is proposed to increase the flexibility of modelling data. The advantage of this size biased positive exponential family of distributions is that for different values of β, the size biased form of the distributions described by Kumar and Chaturvedi (2020) are special cases of this family of distributions. The present investigation aims to study some structural properties of the proposed weighted generalization of the positive exponential family of distributions.

The rest of the paper is organized as follows: In Section 2, the weighted generalization of the positive exponential family of distributions is proposed and the associated properties are investigated. In Section 3, we derive UMVU estimators, ML estimators and MM estimators of the q^{th} power of the parameter θ of the proposed weighted family of distributions, when other parameters are known. We also derive UMVU estimators and ML estimators of the reliability functions. In section 4, we derive ML estimators when all the parameters are unknown. Section 5 of our paper comprises of an extensive simulation study followed by real data illustrations in Section 6. We end with a brief set of conclusions in Section 7.

2. The Weighted Generalized Positive Exponential Family of Distributions and Its Properties

A r.v. X is said to follow generalized positive exponential family of distributions if its pdf and cdf are respectively given by

$$f\left(x;\alpha,\beta,\nu,\theta\right)=\alpha\left(\frac{\beta}{\theta}\right)^{\nu}\frac{1}{\Gamma\nu}x^{\alpha\nu-1}\exp\left(\frac{-\beta x^{\alpha}}{\theta}\right);x>0,\alpha,\beta,\nu,\theta>0 \tag{2}$$

and

$$F(x)=\frac{\gamma\left(\nu,\frac{\beta x^{\alpha}}{\theta}\right)}{\Gamma\nu}, \tag{3}$$

where $\gamma(x,a)=\int_0^x t^{a-1}e^{-t}dt$ is the lower incomplete gamma function.

We construct a weighted family of distributions by taking $w(x)=x^c$, and hence size biased weighted generalized positive exponential family of distributions is given by

$$g\left(x;\alpha,\beta,\nu,\theta,c\right)=\frac{\alpha}{\Gamma\left(\nu+\frac{c}{\alpha}\right)}\left(\frac{\beta}{\theta}\right)^{\nu+\frac{c}{\alpha}}x^{\alpha\nu+c-1}\exp\left(\frac{-\beta x^{\alpha}}{\theta}\right); \quad x>0,\alpha,\beta,\nu,\theta,c>0. \tag{4}$$

We denote it by $WGPEFD(c,\alpha,\beta,\nu,\theta)$. The pdfs of length biased and area biased generalized positive exponential family of distributions (LGPEFD and AGPEFD) distributions can be, respectively, obtained by substituting $c=1$ and $c=2$ in (4). The corresponding cdf and hazard function are respectively given by

$$G\left(x;\alpha,\beta,\nu,\theta,c\right)=\frac{\gamma\left(\nu+\frac{c}{\alpha},\frac{\beta x^{\alpha}}{\theta}\right)}{\Gamma\left(\nu+\frac{c}{\alpha}\right)} \tag{5}$$

and

$$h(x)=\frac{\alpha\left(\frac{\beta}{\theta}\right)^{\nu+\frac{c}{\alpha}}x^{\alpha\nu+c-1}\exp\left(\frac{-\beta x^{\alpha}}{\theta}\right)}{\Gamma\left(\nu+\frac{c}{\alpha}\right)-\gamma\left(\nu+\frac{c}{\alpha},\frac{\beta x^{\alpha}}{\theta}\right)}. \tag{6}$$

Figures 1 and 2 provide possible shapes of probability density functions and hazard functions of this family of distributions for different values of α, β, ν, c and θ.

For different values of β, this family covers the following distributions as special cases:

1. For $\alpha=\nu=\beta=1$, we get one parameter size biased exponential distribution.
2. For $\alpha=\beta=1$, it gives size biased gamma distribution. Further, for integral values of α, it gives size biased Erlang distribution.
3. For $\beta=1$, it leads to size biased generalized gamma distribution.
4. For $\beta=\nu=1$, it turns out to be size biased Weibull distribution.
5. For $\nu=\frac{1}{2},\beta=1,\alpha=2$, it is known as size biased half normal distribution.
6. For $\nu=\frac{m}{2}$, $\alpha=2$, $\beta=\frac{1}{2},m>0$ we get size biased chi distribution and for m = 3 we get size biased Maxwell distribution (Sharma et al., 2017).
7. For $\alpha=2$,$\nu=1$,$\beta=1$, we get a size biased Rayleigh distribution.
8. For $\alpha=2$,$\beta=1$,$\nu=k+1$; $k\geq 0$ we get a size biased generalized Rayleigh distribution of Voda (1978).
9. For $\nu=\beta$ and $\alpha=2$, $\nu>o$, $\beta>0$ we get the size biased Nakagami distribution (Mudsir and Ahmad, 2018).

The various distributional properties and reliability characteristics related to this family of distributions are stated below:

1. **Moments**
   The rth raw moment (about the origin) of this family of distributions is given by

$$\mu_r^{'}=\left(\frac{\theta}{\beta}\right)^{r/\alpha}\frac{1}{\Gamma\left(\nu+\frac{c}{\alpha}\right)}\Gamma\left(\nu+\frac{c}{\alpha}+\frac{r}{\alpha}\right).$$

   In particular, mean and variance of $WGPEFD(c,\alpha,\beta,\gamma,\nu,\theta)$ are respectively given by

$$\mu=\left(\frac{\theta}{\beta}\right)^{1/\alpha}\frac{\Gamma\left(\nu+\frac{c}{\alpha}+\frac{1}{\alpha}\right)}{\Gamma\left(\nu+\frac{c}{\alpha}\right)} \tag{7}$$

   and

$$\sigma^2=\left(\frac{\theta}{\beta}\right)^{2/\alpha}\frac{1}{\Gamma\left(\nu+\frac{c}{\alpha}\right)}\left[\Gamma\left(\nu+\frac{c}{\alpha}+\frac{2}{\alpha}\right)-\frac{\left[\left(\Gamma\left(\nu+\frac{c}{\alpha}+\frac{1}{\alpha}\right)\right)^2\right.}{\Gamma\left(\nu+\frac{c}{\alpha}\right)}\right]. \tag{8}$$

2. **Skewness and Kurtosis**
   The coefficients of skewness (β_1) and kurtosis (β_2) can be obtained as

$$\beta_1=\frac{\left[a_3a_0^2-3a_2a_1a_0+2a_1^3\right]^2}{\left[a_2a_0-a_1^2\right]^3}$$

and

$$\beta_2 = \frac{a_0^3 a_4 - 4a_0^2 a_1 a_3 + 6a_0 a_2 a_1^2 - 3a_1^4}{[a_0 a_2 - a_1^2]^2},$$

respectively, where $a_r = \Gamma\left(\nu + \frac{c}{\alpha} + \frac{r}{\alpha}\right)$. From Figure 3, we observe that WGPEFD is positively skewed as $\beta_1 > 0$ for different values of parameters. Figure 4 clearly indicates that this family exhibits the shapes higher than normal curve as β_2 is larger than 3 for the given c and different values of θ and hence this family is leptokurtic.

3. **Mode**
   Mode of the distribution is given by

$$X_{mode} = \left(\frac{\alpha\nu + c - 1}{\alpha}\left(\frac{\theta}{\beta}\right)\right)^{1/\alpha}.$$

The pdf of this family is uni-modal for given α, β, ν, c and θ.

4. **Median**
   Median is the solution of the following equation:

$$F(Md) = 0.5$$

$$\Rightarrow \frac{\gamma\left(\nu + \frac{c}{\alpha}, \frac{\beta(Md)^\alpha}{\theta}\right)}{\Gamma\left(\nu + \frac{c}{\alpha}\right)} - 0.5 = 0$$

.

5. **Quantiles**
   The q^{th} quantile x_q can be obtained by solving the equation

$$\begin{aligned} q =& F_X(x_q;\theta) \\ \Longrightarrow q =& \frac{\gamma\left(\nu + \frac{c}{\alpha}, \frac{\beta x_q^\alpha}{\theta}\right)}{\Gamma(\nu + \frac{c}{\alpha})} \\ \Longrightarrow x_q =& \left(\frac{\theta}{\beta}\mathcal{G}^{-1}\left(\nu + \frac{c}{\alpha}, q\right)\right)^{\frac{1}{\alpha}}, \end{aligned}$$

where $\mathcal{G}^{-1}\left(\nu + \frac{c}{\alpha}, q\right)$ is an inverse gamma regularized function and can be approximated by using the following series expansion

$$\mathcal{G}^{-1}(a,z) = (-(z-1)\Gamma(a+1))^{1/a} + \frac{\left[(-(z-1)\Gamma(a+1))^{1/a}\right]^2}{a+1} + \frac{(3a+5)\left[(-(z-1)\Gamma(a+1))^{1/a}\right]^3}{2(a+1)^2(a+2)} + \mathcal{O}((z-1)^{4/a}),$$

where $\mathcal{O}(\cdot)$ represents higher order terms.

6. **Moment Generating Function and Characteristic Function**
   The moment generating function of X is given by

$$M_X(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!}\left(\frac{\theta}{\beta}\right)^{j/\alpha} \frac{\Gamma\left(\nu + \frac{c}{\alpha} + \frac{j}{\alpha}\right)}{\Gamma\left(\nu + \frac{c}{\alpha}\right)}.$$

Consequently, the characteristic function $\phi_X(t) = E(e^{tX})$ is given by

$$\sum_{j=0}^{\infty} \frac{(it)^j}{j!} \left(\frac{\theta}{\beta}\right)^{j/\alpha} \frac{\Gamma\left(\nu + \frac{c}{\alpha} + \frac{j}{\alpha}\right)}{\Gamma\left(\nu + \frac{c}{\alpha}\right)}.$$

7. **Conditional Moments and Conditional Moment Generating Function**
Let X be a random variable following the WGPEFD, then the conditional moment $E(X^r|X > t)$ and the conditional moment generating function $E(e^{tX}|X > x_\circ)$ are respectively given by

$$E(X^r|X > t) = \frac{\left(\frac{\theta}{\beta}\right)^{r/\alpha} \left[\Gamma\left(\nu + \frac{c}{\alpha} + \frac{r}{\alpha}\right) - \gamma\left(\nu + \frac{c}{\alpha}, \frac{\beta t^\alpha}{\theta}\right)\right]}{\Gamma\left(\nu + \frac{c}{\alpha}\right) - \gamma\left(\nu + \frac{c}{\alpha}, \frac{\beta t^\alpha}{\theta}\right)}$$

and

$$E(e^{tX}|X > x_\circ) = \frac{\sum_{i=0}^{\infty} \frac{t^i}{i!} \left(\frac{\theta}{\beta}\right)^{i/\alpha} \Gamma\left(\nu + \frac{c}{\alpha} + \frac{i}{\alpha}\right)}{\Gamma\left(\nu + \frac{c}{\alpha}\right) - \gamma\left(\nu + \frac{c}{\alpha}, \frac{\beta x_\circ^\alpha}{\theta}\right)}.$$

8. **Stochastic Ordering**
A r.v. X is said to be stochastically greater than Y, i.e., $Y \leq_{st} X$, if $F_Y(t) \leq F_X(t)$ for all t. Further, X is said to be greater than Y in the

(a) hazard rate order, $Y \leq_{hr} X$, if $h_Y(t) \geq h_X(t)$ for all t.

(b) mean residual life order, $Y \leq_{mrl} X$ if $m_Y(t) \geq m_X(t)$ for all t.

(c) likelihood ratio order, $Y \leq_{lr} X$ if $\frac{f_X(t)}{f_Y(t)}$ decreases in t.

Shaked and Shanthikumar (1994) gave a result regarding stochastic ordering which shows that the existence of likelihood ratio ordering implies the existence of all the orderings mentioned above.
Let $X \sim WGPEFD(\alpha_1, \beta_1, \nu_1, \theta_1, c_1)$ and $Y \sim WGPEFD(\alpha_2, \beta_2, \nu_2, \theta_2, c_2)$. Then, the likelihood ratio is given by

$$\frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1}{\alpha_2} \left(\frac{\beta_1^{\nu_1 + \frac{c_1}{\alpha_1}}}{\beta_2^{\nu_2 + \frac{c_2}{\alpha_2}}}\right) \left(\frac{\theta_2^{\nu_2 + \frac{c_2}{\alpha_2}}}{\theta_1^{\nu_1 + \frac{c_1}{\alpha_1}}}\right) \left(\frac{\Gamma\left(\nu_2 + \frac{c_2}{\alpha_2}\right)}{\Gamma\left(\nu_1 + \frac{c_1}{\alpha_1}\right)}\right) x^{\alpha_1\nu_1 - \alpha_2\nu_2 + c_1 - c_2} \times exp\left[-\left(\frac{\beta_1\theta_2 x^{\alpha_1} - \beta_2\theta_1 x^{\alpha_2}}{\theta_1\theta_2}\right)\right]$$

$$\implies \frac{d}{dx}\frac{f_X(x)}{f_Y(x)} = \frac{f_X(x)}{f_Y(x)} \left[\frac{\alpha_1\nu_1 - \alpha_2\nu_2 + c_1 - c_2}{x} - \frac{\alpha_1\beta_1\theta_2 x^{\alpha_1 - 1} - \alpha_2\beta_2\theta_1 x^{\alpha_2 - 1}}{\theta_1\theta_2}\right]. \quad (9)$$

From (9), we can observe that $\frac{d}{dx}\frac{f_X(x)}{f_Y(x)}$ is decreasing in x, if $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$, $\nu_1 < \nu_2$, $\theta_2 < \theta_1$ and $c_1 < c_2, \forall x, 0 < \alpha_1, \alpha_2 < 1$.

Hence, $Y \leq_{lr} X$ when $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$, $\nu_1 < \nu_2$, $\theta_2 < \theta_1$, $c_1 < c_2$ and $0 < \alpha_1, \alpha_2 < 1$ and hence,

$$(Y \leq_{lr} X) \implies (Y \leq_{hr} X) \implies (Y \leq_{mrl} X)$$
$$\Downarrow$$
$$(Y \leq_{st} X)$$

9. **Mean Residual Life Function**
   The Mean Residual Life function is given by

$$\mu_(t) = \frac{\left[\int_t^\infty 1 - \frac{\gamma\left(\nu+\frac{c}{\alpha}, \frac{\beta u^\alpha}{\theta}\right)}{\Gamma(\nu)}\right] du}{\left[1 - \frac{\gamma\left(\nu, \frac{\beta t^\alpha}{\theta}\right)}{\Gamma(\nu+\frac{c}{\alpha})}\right]}.$$

10. **Mean Time to System Failure**
    Mean time to system failure of this family of distributions is given by

$$MTSF = \frac{\left(\frac{\theta}{\beta}\right)^{\left(\frac{1}{\alpha}\right)}}{\Gamma\left(\nu+\frac{c}{\alpha}\right)} \Gamma\left(\nu + \frac{c}{\alpha} + \frac{1}{\alpha}\right)$$

All the properties described above can be derived for LGPEFD and AGPEFD by taking $c = 1$ and $c = 2$ respectively.

3. UMVU and ML Estimators of Powers of Parameter θ, $R(t)$ and P

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the WGPEFD. Then, assuming α, β, ν and c to be known, the likelihood function of the parameter θ given the sample observations $\underline{x} = (x_1, x_2, \ldots, x_n)$ is given by:

$$L(\theta \mid \underline{x}) = \left(\frac{\alpha}{\Gamma(\nu+\frac{c}{\alpha})}\right)^n \left(\frac{\beta}{\theta}\right)^{n\left(\nu+\frac{c}{\alpha}\right)} e^{-\frac{\beta}{\theta}\sum\limits_{i=1}^{n} x_i^\alpha} \prod_{i=1}^{n} x_i^{\alpha\nu+c-1}. \tag{10}$$

The following theorem provides uniformly minimum variance unbiased (UMVU) estimator of powers of θ.

Theorem 1 *For $q \in (-\infty, \infty)$, the UMVU estimator of θ^q is given by:*

$$\widetilde{\theta^q} = \begin{cases} \left\{\frac{\Gamma\left(n\left(\nu+\frac{c}{\alpha}\right)\right)}{\Gamma(n\left(\nu+\frac{c}{\alpha}\right)+q)}\right\} S^q; & n\left(\nu+\frac{c}{\alpha}\right)+q > 0 \\ 0; & \text{otherwise.} \end{cases}$$

where $S = \beta \sum\limits_{i=1}^{n} X_i^\alpha$.

Proof: It follows from (10) and factorization theorem (Rohatgi and Saleh, 2012) that S is sufficient statistic for θ and the pdf of S is

$$f_s(s \mid \theta) = \frac{s^{n\left(\nu+\frac{c}{\alpha}\right)-1}}{\Gamma\left(n\left(\nu+\frac{c}{\alpha}\right)\right)\theta^{n\left(\nu+\frac{c}{\alpha}\right)}} \exp\left(-\frac{s}{\theta}\right); \quad \nu,\ \alpha,\ \beta,\ \theta,\ c > 0,\ s \geq 0. \tag{11}$$

From (11), since the distribution of S belongs to exponential family, it is also complete. Now, it follows from (11) that

$$E[S^q] = \left\{\frac{\Gamma\left(n\left(\nu+\frac{c}{\alpha}\right)+q\right)}{\Gamma\left(n\left(\nu+\frac{c}{\alpha}\right)\right)}\right\}\theta^q, \tag{12}$$

and the theorem follows. □

In the following theorem, we obtain UMVU estimator of the sampled pdf at a specified point x.

Theorem 2 *The UMVU estimator of the sampled pdf at a specified point x is*

$$\widetilde{g}(x;\alpha,\beta,\nu,\theta,c)=\begin{cases}\frac{\alpha}{\beta\left(\nu+\frac{c}{\alpha},(n-1)\left(\nu+\frac{c}{\alpha}\right)\right)}\left(\frac{\beta}{S}\right)^{\nu+\frac{c}{\alpha}}x^{\alpha\nu+c-1}\left[1-\frac{\beta x^{\alpha}}{S}\right]^{(n-1)\left(\nu+\frac{c}{\alpha}\right)-1}; & \beta x^{\alpha}<S\\ 0; & \text{otherwise.}\end{cases}$$

Proof: We can write the pdf given in (4) as

$$\begin{aligned}g(x;\alpha,\beta,\nu,\theta,c)&=\frac{\alpha}{\Gamma\left(\nu+\frac{c}{\alpha}\right)}\left(\frac{\beta}{\theta}\right)^{\nu+\frac{c}{\alpha}}x^{\alpha\nu+c-1}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left(\frac{\beta x^{\alpha}}{\theta}\right)^{i}\\&=\frac{\alpha\beta^{\nu+\frac{c}{\alpha}}x^{\alpha\nu+c-1}}{\Gamma\left(\nu+\frac{c}{\alpha}\right)}\sum_{i=0}^{\infty}\frac{(-1)^i\left(\beta x^{\alpha}\right)^i}{i!}\theta^{-\left(\nu+\frac{c}{\alpha}+i\right)}.\end{aligned}$$

Applying Theorem 1,

$$\begin{aligned}\widetilde{g}(x;\alpha,\beta,\nu,\theta,c)&=\frac{\alpha}{\Gamma\left(\nu+\frac{c}{\alpha}\right)}\beta^{\nu+\frac{c}{\alpha}}\,x^{\alpha\nu+c-1}\sum_{i=0}^{\infty}\frac{(-1)^i\left(\beta x^{\alpha}\right)^i}{i!}(\widetilde{\theta})^{\left(-(\nu+\frac{c}{\alpha}+i)\right)}\\&=\frac{\alpha\left(\frac{\beta}{S}\right)^{\nu+\frac{c}{\alpha}}x^{(\alpha\nu+c-1)}}{\beta\left(\nu+\frac{c}{\alpha},(n-1)\left(\nu+\frac{c}{\alpha}\right)\right)}\sum_{i=0}^{(n-1)\left(\nu+\frac{c}{\alpha}\right)-1}(-1)^i\binom{(n-1)\left(\nu+\frac{c}{\alpha}\right)-1}{i}\times\\&\quad\left(\frac{\beta x^{\alpha}}{S}\right)^{i},\end{aligned}$$

and the result follows. □

The following theorem provides UMVU estimator of the reliability function $R(t)$.

Theorem 3 *The UMVU estimator of $R(t)$ is*

$$\widetilde{R}(t)=\begin{cases}1-I_{\frac{\beta t^{\alpha}}{S}}\left(\nu+\frac{c}{\alpha},(n-1)\left(\nu+\frac{c}{\alpha}\right)\right); & \beta t^{\alpha}<S\\ 0 & ;\text{otherwise,}\end{cases}$$

where $I_x(p,q)=\frac{1}{\beta(p,q)}\int_0^x y^{p-1}(1-y)^{q-1}dy;\ 0\le y\le 1,\ x<1,\ p,q>0$ is the incomplete beta function.

Proof: On applying Theorem 2, the UMVU estimator of $R(t)$ is given by

$$\begin{aligned}\widetilde{R}(t)_{II}&=\int_t^{\infty}\widetilde{g}(x;\alpha,\beta,\nu,\theta,c)dx\\&=\frac{\alpha}{\beta\left(\nu+\frac{c}{\alpha},(n-1)\left(\nu+\frac{c}{\alpha}\right)\right)}\left(\frac{\beta}{S}\right)^{\nu+\frac{c}{\alpha}}\times\\&\quad\int_t^{\infty}x^{\alpha\nu+c-1}\left[1-\frac{\beta x^{\alpha}}{S}\right]^{(n-1)\left(\nu+\frac{c}{\alpha}\right)-1}dx,\end{aligned}$$

and the result follows by substituting $\frac{\beta x^{\alpha}}{S}=z$.

Let X and Y be two independent random variables with respective pdf:

$$g\left(x;\alpha_1,\beta_1,\nu_1,\theta_1,c_1\right)=\frac{\alpha_1}{\Gamma\left(\nu_1+\frac{c_1}{\alpha_1}\right)}\left(\frac{\beta_1}{\theta_1}\right)^{\nu_1+\frac{c_1}{\alpha_1}}x^{\alpha-1\nu_1+c_1-1}\exp\left(\frac{-\beta_1x^{\alpha_1}}{\theta_1}\right);$$

$$x>0,\alpha_1,\beta_1,\nu_1,\theta_1,c_1>0$$

and

$$g\left(y;\alpha_2,\beta_2,\nu_2,\theta_2,c_2\right)=\frac{\alpha_2}{\Gamma\left(\nu_2+\frac{c_2}{\alpha_2}\right)}\left(\frac{\beta_2}{\theta_2}\right)^{\nu_2+\frac{c_2}{\alpha_2}}y^{\alpha_2\nu_2+c_2-1}\exp\left(\frac{-\beta_2y^{\alpha_2}}{\theta_2}\right);$$

$$y>0,\alpha_2,\beta_2,\nu_2,\theta_2,c_2>0.\ \square$$

Now the UMVU estimator of P is given in the following theorem.

Theorem 4 *The UMVU estimator of P is*

$$\tilde{P}=\begin{cases}\int_{z=0}^{1}\frac{1}{\beta\left(\nu_1+\frac{c_1}{\alpha_1},(n-1)\left(\nu_1+\frac{c_1}{\alpha_1}\right)\right)}z^{\nu_1+\frac{c_1}{\alpha_1}-1}\left(1-z\right)^{(n-1)\left(\nu_1+\frac{c_1}{\alpha_1}\right)-1}\times\\ I_{\left\{\frac{\beta_2\left(\frac{Sz}{\beta_1}\right)^{\left(\frac{\alpha_2}{\alpha_1}\right)}}{T}\right\}}\left(\nu_2+\frac{c_2}{\alpha_2},(m-1)\left(\nu_2+\frac{c_2}{\alpha_2}\right)\right);\text{for}\left(\frac{S}{\beta_1}\right)^{1/\alpha_1}\le\left(\frac{T}{\beta_2}\right)^{1/\alpha_2}\\ 1-\frac{1}{\beta\left(\nu_2+\frac{c_2}{\alpha_2},(m-1)\left(\nu_2+\frac{c_2}{\alpha_2}\right)\right)}\int_{z=0}^{1}z^{\nu_2+\frac{c_2}{\alpha_2}-1}\left(1-z\right)^{(m-1)\left(\nu_2+\frac{c_2}{\alpha_2}\right)-1}\times\\ I_{\left\{\frac{\beta_1\left(\frac{Tz}{\beta_2}\right)^{\left(\frac{\alpha_1}{\alpha_2}\right)}}{S}\right\}}\left(\nu_1+\frac{c_1}{\alpha_1},(n-1)\left(\nu_1+\frac{c_1}{\alpha_1}\right)\right);\text{for}\left(\frac{S}{\beta_1}\right)^{1/\alpha_1}>\left(\frac{T}{\beta_2}\right)^{1/\alpha_2}.\end{cases}$$

Proof: Let $X_1,X_2,\ldots,X_n$ be a random sample of size n from $g(x;\alpha_1,\beta_1,\nu_1,\theta_1,c_1)$ and $Y_1,Y_2,\ldots,Y_m$ be a random sample of size m from $g(y;\alpha_2,\beta_2,\nu_2,\theta_2,c_2)$. Further, let $S=\sum_{i=1}^{n}\beta_1X_i^{\alpha_1}$ and $T=\sum_{i=1}^{m}\beta_2Y_i^{\alpha_2}$. It follows from Theorem 2 that

$$\widetilde{g}(x;\alpha_1,\beta_1,\nu_1,\theta_1,c_1)=\frac{\alpha_1}{\beta\left(\nu_1+\frac{c_1}{\alpha_1},(n-1)\left(\nu_1+\frac{c_1}{\alpha_1}\right)\right)}\left(\frac{\beta_1}{S}\right)^{\nu_1+\frac{c_1}{\alpha_1}}x^{\alpha_1\nu_1+c_1-1}\times \left[1-\frac{\beta_1y^{\alpha_1}}{S}\right]^{(n-1)\left(\nu_1+\frac{c_1}{\alpha_1}\right)-1};\beta_1x^{\alpha_1}<S \tag{13}$$

and

$$\widetilde{g}(y;\alpha_2,\beta_2,\nu_2,\theta_2,c_2)=\frac{\alpha_2}{\beta\left(\nu_2+\frac{c_2}{\alpha_2},(m-1)\left(\nu_2+\frac{c_2}{\alpha_2}\right)\right)}\left(\frac{\beta_2}{T}\right)^{\nu_2+\frac{c_2}{\alpha_2}}y^{\alpha_2\nu_2+c_2-1}\times \left[1-\frac{\beta_2y^{\alpha_2}}{T}\right]^{(m-1)\left(\nu_2+\frac{c_2}{\alpha_2}\right)-1};\beta_2y^{\alpha_2}<T. \tag{14}$$

The UMVU estimator of P can be obtained as

$$\widetilde{P}=\int_{x=0}^{\infty}\int_{y=0}^{x}\widetilde{g}(x;\alpha_1,\beta_1,\nu_1,\theta_1,c_1)\widetilde{g}(y;\alpha_2,\beta_2,\nu_2,\theta_2,c_2)\,dx\,dy$$

$$=\int_{x=0}^{\min\left(\left(\frac{S}{\beta_1}\right)^{\frac{1}{\alpha_1}},\left(\frac{T}{\beta_2}\right)^{\frac{1}{\alpha_2}}\right)}\frac{\alpha_1x^{\alpha_1\nu_1+c_1-1}}{\beta\left(\nu_1+\frac{c_1}{\alpha_1},(n-1)\left(\nu_1+\frac{c_1}{\alpha_1}\right)\right)}\left(\frac{\beta_1}{S}\right)^{\nu_1+\frac{c_1}{\alpha_1}}\times \left[1-\frac{x^{\alpha_1}}{S}\right]^{(n-1)\left(\nu_1+\frac{c_1}{\alpha_1}\right)-1}I_{\frac{\beta_2x^{\alpha_2}}{T}}\left(\nu_2+\frac{c_2}{\alpha_2},(m-1)\left(\nu_2+\frac{c_2}{\alpha_2}\right)\right)dx.$$

When $\left(\frac{S}{\beta_1}\right)^{\frac{1}{\alpha_1}} \le \left(\frac{T}{\beta_2}\right)^{\frac{1}{\alpha_2}}$, the UMVU estimator of P is given by

$$\widetilde{P} = \int_{x=0}^{\left(\frac{S}{\beta_1}\right)^{\frac{1}{\alpha_1}}} \frac{\alpha_1 x^{\alpha_1\nu_1+c_1-1}}{\beta\left(\nu_1+\frac{c_1}{\alpha_1},(n-1)\left(\nu_1+\frac{c_1}{\alpha_1}\right)\right)} \left(\frac{\beta_1}{S}\right)^{\nu_1+\frac{c_1}{\alpha_1}} \left[1-\frac{\beta_1 x^{\alpha_1}}{S}\right]^{(n-1)\left(\nu_1+\frac{c_1}{\alpha_1}\right)-1} \times$$
$$I_{\frac{\beta_2 x^{\alpha_2}}{T}}\left(\nu_2+\frac{c_2}{\alpha_2},(m-1)\left(\nu_2+\frac{c_2}{\alpha_2}\right)\right) dx, \tag{15}$$

and the first assertion follows by substituting $\frac{\beta_1 x^{\alpha_1}}{S} = z$.

When $\left(\frac{S}{\beta_1}\right)^{1/\alpha_1} > \left(\frac{T}{\beta_2}\right)^{1/\alpha_2}$, the UMVU estimator of P is given by

$$\begin{aligned}\widetilde{P} &= \int_{y=0}^{\infty}\int_{x=y}^{\infty} \widetilde{g}(x;\alpha_1,\beta_1,\nu_1,\theta_1,c_1)\widetilde{g}(y;\alpha_2,\beta_2,\nu_2,\theta_2,c_2)dx\,dy \\ &= \int_{y=0}^{\infty} \widetilde{R}(y;\alpha_1,\beta_1,\nu_1,\theta_1,c_1)\widetilde{g}(y;\alpha_2,\beta_2,\nu_2,\theta_2,c_2)dy\end{aligned}$$

$$= \frac{\alpha_2}{\beta\left(\nu_2+\frac{c_2}{\alpha_2},(m-1)\left(\nu_2+\frac{c_2}{\alpha_2}\right)\right)} \left(\frac{\beta_2}{T}\right)^{\nu_2+\frac{c_2}{\alpha_2}} y^{\alpha_2\nu_2+c_2-1} \times$$
$$\left[1-\frac{\beta_2 y^{\alpha_2}}{T}\right]^{(m-1)\left(\nu_2+\frac{c_2}{\alpha_2}\right)-1} \left[1 - I_{\frac{\beta_1 y^{\alpha_1}}{S}}\left(\nu_1+\frac{c_1}{\alpha_1},(n-1)\left(\nu_1+\frac{c_1}{\alpha_1}\right)\right)\right] dy, \tag{16}$$

and the second assertion follows by substituting $\frac{\beta_2 y^{\alpha_2}}{T} = z$.

It can be easily seen from (10) that the maximum likelihood (ML) estimator of θ^q is given by

$$\widehat{\theta^q} = \left(\frac{S}{n\left(\nu+\frac{c}{\alpha}\right)}\right)^q, \tag{17}$$

where, $S = \beta \Sigma x_i{}^{\alpha}$. □

Now we provide ML estimator of $R(t)$ in the following theorem.

Theorem 5 *The ML estimator of $R(t)$ is given by*

$$\widehat{R}(t) = 1 - \frac{\gamma\left(\nu+\frac{c}{\alpha}, \frac{n\beta t^{\alpha}}{S}\left(\nu+\frac{c}{\alpha}\right)\right)}{\Gamma\left(\nu+\frac{c}{\alpha}\right)},$$

where $\gamma(a,r) = \int_0^r y^{a-1}e^{-y}dy$ is the lower incomplete gamma function.

The ML estimator of P is given in the following theorem.

Theorem 6 *The ML estimator of P is given by*

$$\widehat{P} = 1 - \frac{1}{\Gamma\left(\nu_1+\frac{c_1}{\alpha_1}\right)\Gamma\left(\nu_2+\frac{c_2}{\alpha_2}\right)} \int_{z=0}^{\infty} z^{\nu_2+\frac{c_2}{\alpha_2}-1}e^{-z} \times$$
$$\gamma\left(\nu_1+\frac{c_1}{\alpha_1}, \frac{n\beta_1\left(\nu_1+\frac{c_1}{\alpha_1}\right)\left(\frac{zT}{m\beta_2}\left(\nu_2+\frac{c_2}{\alpha_2}\right)^{-1}\right)^{\frac{\alpha_1}{\alpha_2}}}{S}\right) dz.$$

Next, we derive the mathod of moment (MM) estimator of θ.

Theorem 7 *The MM estimator of θ^q is given by*

$$\widehat{\theta^q_M} = \left(\left(\frac{\Gamma\left(\nu+\frac{c}{\alpha}\right)}{\Gamma\left(\nu+\frac{c}{\alpha}+\frac{1}{\alpha}\right)}\right)^{\alpha}\overline{X}^{\alpha}\beta\right)^{q}.$$

The proofs of Theorems 5, 6 and 7 can be easily derived on the similiar lines as derived in Kumar and Chaturvedi (2020).

4. ML Estimators When All the Parameters are Unknown

Now we discuss the case when all the parameters α, ν, θ and c are unknown. The log-likelihood function of the parameters α, ν, θ and c given the sample observations $\underline{x}$ and different values of β is

$$l(\alpha,\nu,\theta,c \mid \underline{x},\beta) = n\log(\alpha) - n\log\left(\Gamma\left(\nu+\frac{c}{\alpha}\right)\right) + n\nu\log(\beta) + \frac{nc}{\alpha}\log(\beta) - n\nu\log(\theta) - \frac{nc}{\alpha}\log(\theta) - \frac{\beta}{\theta}\sum_{i=1}^{n}x_i^{\alpha} + (\alpha\nu+c-1)\sum_{i=1}^{n}\log(x_i).$$

The ML estimators of α, ν and c are given by the simultaneous solution of the following four equations.

$$\frac{\partial l}{\partial\alpha} = \frac{n}{\alpha} + \frac{n}{\Gamma\left(\nu+\frac{c}{\alpha}\right)}\frac{\partial\Gamma\left(\nu+\frac{c}{\alpha}\right)}{\partial\alpha}\frac{c}{\alpha^2} - \frac{nc}{\alpha^2}log(\beta) + \frac{nc}{\alpha^2}log(\theta) - \frac{\beta}{\theta}\sum_{i=1}^{n}x_i^{\alpha}\log(x_i) + \nu\sum_{i=1}^{n}\log(x_i) = 0 \tag{18}$$

$$\frac{\partial l}{\partial\nu} = \frac{-n}{\Gamma\left(\nu+\frac{c}{\alpha}\right)}\frac{\partial\Gamma\left(\nu+\frac{c}{\alpha}\right)}{\partial\nu} + nlog(\beta) - n\log(\theta) + n\log(\beta) + \alpha\sum_{i=1}^{n}\log(x_i) = 0 \tag{19}$$

$$\frac{\partial l}{\partial c} = \frac{-n}{\Gamma\left(\nu+\frac{c}{\alpha}\right)}\frac{\partial\Gamma\left(\nu+\frac{c}{\alpha}\right)}{\partial c}\left(\frac{1}{\alpha}\right) + \frac{n}{\alpha}log(\beta) - \frac{n}{\alpha}log(\theta) + \sum_{i=1}^{n}\log(x_i) = 0 \tag{20}$$

$$\frac{\partial l}{\partial\theta} = -\frac{n\nu}{\theta} - \frac{nc}{\alpha\theta} + \frac{\beta\sum_{i=1}^{n}x_i^{\alpha}}{\theta^2} = 0 \tag{21}$$

Since, these non-linear equations don't have a closed form solution, we apply Newton Raphson algorithm to compute ML estimators of α, ν, c and θ. Further, using the invariance property of ML estimators, the ML estimator of $R(t)$ is given as

$$\widehat{R}(t) = 1 - \frac{\gamma\left(\widehat{\nu}+\frac{\widehat{c}}{\widehat{\alpha}},\frac{\beta x^{\widehat{\alpha}}}{\widehat{\theta}}\right)}{\Gamma\left(\widehat{\nu}+\frac{\widehat{c}}{\widehat{\alpha}}\right)}, \tag{22}$$

where $\widehat{\alpha}$, $\widehat{\nu}$, $\widehat{\theta}$ and $\widehat{c}$ are the ML estimators of α, ν, θ and c respectively, and the ML estimator of P is given as

$$\widehat{P} = 1 - \frac{1}{\Gamma\left(\widehat{\nu}_1+\frac{\widehat{c}_1}{\widehat{\alpha}_1}\right)\Gamma\left(\widehat{\nu}_2+\frac{\widehat{c}_2}{\widehat{\alpha}_2}\right)}\int_{z=0}^{\infty} z^{\widehat{\nu}_2+\frac{\widehat{c}_2}{\widehat{\alpha}_2}-1}e^{-z}\times \gamma\left(\widehat{\nu}_1+\frac{\widehat{c}_1}{\widehat{\alpha}_1},\frac{\beta_1}{\widehat{\theta}_1}\left(\frac{\widehat{\theta}_2 z}{\beta_2}\right)^{\frac{\widehat{\alpha}_1}{\widehat{\alpha}_2}}\right)dz. \tag{23}$$

5. Simulation Studies

In order to compare the performance of $\widetilde{\theta^q}$, $\widehat{\theta^q_M}$ and $\widehat{\theta^q}$ for different sample sizes, we conduct a Monte Carlo simulation study. For $\alpha = 0.5$, $\beta = 2, \nu = 3, c = 0.8$, we generate 10,000 samples each of size n from WGPEFD and repeat this procedure for several values of θ. Figure 5 shows the mean square error (MSE) of the UMVU estimator, MM estimator and ML estimator of θ^q. From these figures, we note that for smaller sample sizes and for $q = 2$, the ML estimator performs the best and the MM estimator performs the worst. The performance of UMVU estimator is in between the two. As the sample size increases, the three curves come close to each other.

Along the similar lines, we perform the simulation studies to compare the performance of $\widetilde{R}(t)$ and $\widehat{R}(t)$ for different sample sizes. For $t = 10$ and $\alpha = 0.5, \beta = 2, \nu = 1.5, c = 0.8$, we generate 10,000 samples each of size n from the WGPEFD and repeat this procedure for several values of θ. Figure 6 shows the MSE of the UMVU estimator and ML estimator of $R(t)$. From these figures, we note that the MSE of the ML estimator of $R(t)$ is always less than that of the UMVU estimator and hence ML estimator of $R(t)$ performs better than UMVU estimator of $R(t)$. However, for large sample sizes these estimators of $R(t)$ are almost equally efficient.

Now, we compare the performance of $\widetilde{P}$ and $\widehat{P}$ for different sample sizes. By Monte Carlo simulation, for $\alpha_1 = 0.4$, $\beta_1 = 0.6$, $\nu_1 = 4$, $c_1 = 0.5$ and $\alpha_2 = 0.1$, $\beta_2 = 0.3$, $\nu_2 = 1$, $c_2 = 0.5$, we generate 10,000 samples each of size n and m from WGPEFD and repeat this procedure for several values of θ_1 and $\theta_2 = 0.8$. Figure 7 shows the MSE of the UMVU estimator and ML estimator of P. From these figures, we note that the MSE of the UMVU estimator of P is always greater than that of the ML estimator, however, for large sample sizes these estimators of P are almost equally efficient.

5.1. ML estimation of the reliability functions when all parameters are unknown

We consider the special case of weighted generalized gamma distribution (WGGD) which is given by taking $\beta = 1$ in (4). In order to compute the ML estimates of the reliability functions, when all the parameters are unknown, we have first generated 1000 random samples of size $n = 50$ from the WGGD (say, X population or random strength X) with $\alpha_1 = 0.2$, $\nu_1 = 2$, $\theta_1 = 2.5$ and $c_1 = 2$. We generated ML estimators of these parameters for these 1000 samples and obtained the mean of estimates of them. The ML estimators of α_1, ν_1, θ_1 and c_1 comes out to be $\widehat{\alpha}_1 = 0.3311$, $\widehat{\nu}_1 = 1.6521$, $\widehat{\theta}_1 = 2.3548$ and $\widehat{c}_1 = 1.5833$ respectively. Also for $t = 50$, actual $R(t) = 0.9871$ and $\widehat{R(t)} = 0.9159$.

Now, to obtain the estimates of P, when all the parameters are unknown, we have generated 1000 random samples of size $m = 60$ from the WGGD (say, Y population or random stress Y) with $\alpha_2 = 0.18$, $\nu_2 = 1.6$, $\theta_2 = 2.6$ and $c_2 = 1.8$. We generated ML estimators of these parameters for these 1000 samples and obtained mean of estimates of them. The ML estimators of α_2, ν_2, θ_2 and c_2 are obtained as $\widehat{\alpha}_2 = 0.1752$, $\widehat{\nu}_2 = 1.5798$, $\widehat{\theta}_2 = 2.5697$ and $\widehat{c}_2 = 1.7456$ respectively. Here actual value of $P = 0.9941$ and the MLE of P comes out to be $\widehat{P} = 0.9946$.

The ML estimators obtained here are close to the true parameter values, though we observe that all the parameters are slightly underestimated except P, which is slightly overestimated. These findings verify the validity of our theoretical results.

6. Real Life Data Examples

This section deals with examples of real data to establish the superiority of Weighted Gamma distribution (WGD) over Gamma and Weighted Generalized Gamma distribution (WGGD) which are special cases of WGPEFD and to illustrate the proposed estimation methods.

Data set I (representing Population X) was reported by Stablein et al. (1981) [see also Chaturvedi et al. (2021b)]. It corresponds to the survival times in days from a clinical trial on a locally advanced, non-resectable gastric carcinoma, involving 90 patients randomized to either chemotherapy alone or a combination of chemotherapy and radiation. The plot of empirical and theoretical cdfs of Weighted Gamma distribution given in Figure 8 shows that it fits well to this data.

For comparing WGD with Gamma and WGGD, we use the concept of Akaike Information Criterion (AIC), Kolmogorov Smirnov (KS) distance and corresponding p-value. The best model is the one that has the least values of AIC and comparatively high p-value. For establishing the superiority of WGD, the calculated values of AIC, KS distance and p-values are reported in Table 1.

Since AIC is the least and p-value is reasonably good for WGD, we choose it to fit the data. Let the population $X \sim WG(x;\nu_1,\theta_1,c_1)$. We obtain ML estimators of the parameters ν_1, θ_1 and c_1 and then assuming ν_1 and c_1 to be known, we obtain UMVU estimator and moment estimator of θ_1. We also obtain UMVU estimator and ML estimator of reliability function $R_X(t)$ at $t = 10$. All the estimated values are listed in Table 2.

Data set II (representing Population Y), reported by Efron (1988) [see also Shanker et al. (2016)], represent the survival times of a group of patients suffering from Head and Neck cancer disease and treated using a combination of radiotherapy and chemotherapy (RT+CT). The plot of empirical and theoretical cdfs of Weighted Gamma distribution given in Figure 9 shows that it fits well to this data. For comparison purposes, the calculated values of AIC, KS distance and p-values are reported in Table 3.

Since, AIC is least and p-value is reasonably good for WGD we choose it to fit the data. Let the population $Y \sim WG(y;\nu_2,\theta_2,c_2)$. We obtain ML estimator of the prameters ν_2, θ_2 and c_2 and then assuming ν_2 and c_2 to be known, we obtain UMVU estimator and moment estimator of θ_2. We also obtain UMVU estimator and ML estimator of reliability function $R_Y(t)$ at $t = 10$. All the estimated values are listed in Table 4.

Now, for the above two data sets, we obtain estimators of $P = P(X > Y)$. The ML estimator and UMVU estimator of P are obtained as 0.7374 and 0.7225, respectively.

7. Concluding Remarks

In this article, we have considered the weighted family of the generalization of positive exponential family of distributions developed by Kumar and Chaturvedi (2020). We have considered size biased family of distributions by taking weight $w(x) = x^c$. UMVU, ML and MM estimators are developed for the powers of parameters, $R(t)$ and P. All the estimates can be obtained for length biased and area biased generalization of positive exponential family of distributions by taking $c = 1$ and $c = 2$, respectively. Efficiency comparison of the three methods of estimation through Monte Carlo Simulation studies is done. Real life data sets are studied to show the superiority of Weighted Gamma distribution, which is special case of this family of distributions and to illustrate the proposed estimation methods.

Acknowledgements

The authors would like to thank the Editor and anonymous referees for their comments and suggestions on an earlier version of this manuscript leading to an improved presentation.

References

Alahmadi AA, Alqawba M, Almutiry W, Shawki AW, Alrajhi S, Al-Marzouki S, Elgarhy M. A new version of weighted Weibull distribution: modelling to COVID-19 data. Discrete Dyn Nat Soc. 2022; Article ID 3994361, 12 Pages.

Alizadeh M, Cordeiro GM, Nascimento ADC, Lima MCS, Ortega EMM. Odd-Burr generalized family of distributions with some applications. J Stat Comput Simul. 2017; 87:367-389.

Bourguignon M, Silva RB, Cordeiro GM. The Weibull-G family of probability distributions. J Data Sci. 2014; 12:53-68.

Chaturvedi A, Malhotra A. Estimation of $P(X > Y)$ for the positive exponential family of distributions. Statistica. 2018; 78(2):149-167.

Chaturvedi A, Bapat SR, Joshi N. Two-stage and sequential procedures for estimation of powers of parameter of a family of distributions. Seq Anal. 2021a; 40(2):170-197.

Chaturvedi A, Chattopadhyay S, Bapat SR, Joshi N. Sequential point estimation procedures for the parameter of a family of distributions. Commun Stat Simulat .2021b; 50(9):2678-2704.

Cordeiro GM, Castro M. A new family of generalized distributions. J Stat Comput Simul. 2011; 81(7):883 - 898.

Cordeiro GM, Ortega EMM, Popovic BV, Pescim RR. The Lomax generator of distributions: Properties, minification process and regression model. Appl Math Comput. 2014; 247:465-486.

Das KK, Roy DB. Applicability of length biased weighted generalized Rayleigh distribution. Adv Appl Sci Res. 2011a; 2(4):320-327.

Das KK, Roy DB. On some length-biased weighted Weibull distribution. Adv Appl Sci Res. 2011b; 2(5):465-475.

Efron B. Logistic regression survival analysis and the Kaplan-Meier curve. J Amer Stat Assoc. 1988; 83(402):414-425.

Eugene N, Lee C, Famoye F. Beta-normal distribution and it's applications. Commun Stat Theory. 2002; 31(4): 497-512.

Fallah A, Kazemi R. Statistical inference for the generalized weighted exponential distribution. Commun Stat Simulat. 2020; 51(8): 4296 - 4313.

Fisher RA. The effects of methods of ascertainment upon the estimation of frequencies. Ann Eugen. 1934; 6:13-25.

Gupta RD, Kundu D. A new class of weighted exponential distributions. Statistics. 2009; 43:621 - 634.

Hashempour M, Alizadeh M. A new weighted half-logistic distribution: properties, applications and different method of estimations. Stat Optim Inf Comput. 2023; 11(3): 554-569.

Kilany NM. Weighted Lomax distribution. SpringerPlus 2016; 5:1862.

Kumar S, Chaturvedi A. On a generalization of the positive exponential family of distributions and the estimation of reliability characteristics. Statistica 2020; 80(1): 57-77.

Liang T. Empirical Bayes estimation of reliability in a positive exponential family. Commun Stat Theory. 2008; 37(13): 2052-2065.

Mudsir S, Ahmad SP. Classical and Bayesian estimation of scale parameter in weighted Nakagami distribution using real life data sets. Pak J Stat. 2018; 34(5): 397-423.

Rao CR. On discrete distributions arising out of methods of ascertainment. In: Patil GP, editor. Classical and Contagious Discrete Distribution. Pergamon Press and Statistical Publishing Society, Calcutta. 1965; 320-332.

Rohatgi VK, Saleh AKMdE. An introduction to probability and statistics. Wiley, New York.2012.

Shanker R, Fesshaye H, Selvaraj S. On modeling of lifetime data using one parameter akash, lindley and exponential distributions. Biom Biostat Int J. 2016; 3(2): 5662.

Shaked M, Shanthikumar J. Stochastic orders and their applications. Academic Press, Boston.1994.

Sharma VK, Dey S, Singh SK, Manzoor U. On length and area-biased Maxwell distributions. Commun Stat Simulat. 2017; 47(5): 1506-1528.

Stablein DM, Carter JrWH, Novak JW. Analysis of survival data with non proportional hazard functions. Control Clin Trials. 1981; 2(2): 149 - 159.

Tahir MH, Cordeiro GM, Alzaatreh A, Mansoor M, Zubair M. The logistic-X family of distributions and its applications. Commun Stat Theory. 2016; 45(24): 7326-7349.

Voda VG. Concerning an application of the Weibull distribution to reliability aspects in metrology. Metrologia Aplicat $\widetilde{a}$. 1978; 25: 123-124.

Yazgan E, Gurler S, Esemen M, Sevinc B. Fuzzy stress-strength reliability for weighted exponential distribution. Qual Reliab Eng Int. 2022; 38: 550-559.

Table 1 Comparison of different distributions based on AIC, KS-Distance and P values for data set I

CRITERIA	AIC	KS-Distance	KS P-value
WGD	111.5418	0.1034	0.6837
Gamma	396.3332	0.1132	0.6811
WGGD	401.7463	0.0716	0.8664

Table 2 Estimates Based on Data Set I

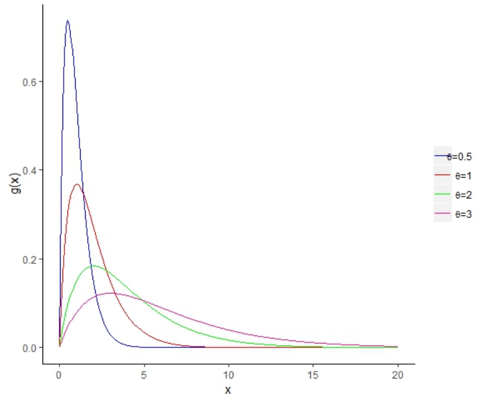
$\widehat{\nu_1}$	$\widehat{c_1}$	$\widehat{\theta_1}$	$\widetilde{\theta_1}$	$\widehat{\theta}_{iMM}$	$\widehat{R}_X(t)$	$\widetilde{R}_X(t)$
0.5701	0.5701	500.1502	465.6973	465.6972	0.9876	0.9853

Table 3 Comparison of different distributions based on AIC, KS-Distance and P values for data set II

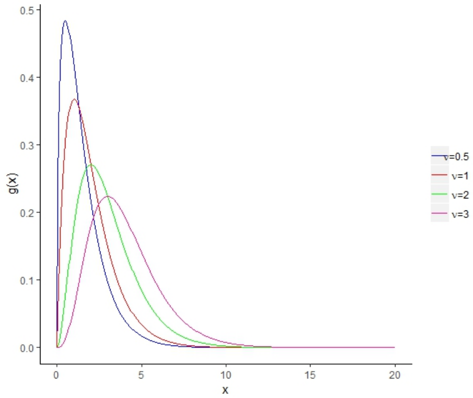
CRITERIA	AIC	KS-Distance	KS P-value
WGD	144.0705	0.1473	0.668
Gamma	566.1502	0.1573	0.268
WGGD	566.4392	0.09923	0.7419

Table 4 Estimates Based on Data Set II

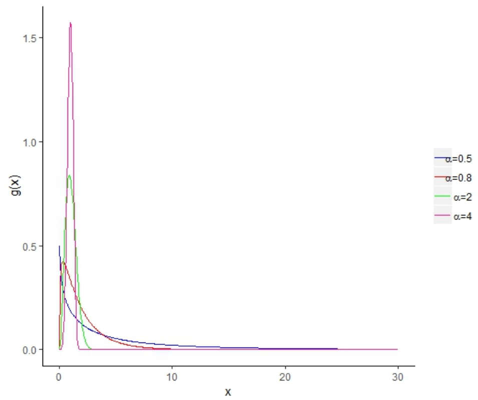
$\widehat{\nu_2}$	$\widehat{c_2}$	$\widehat{\theta_2}$	$\widetilde{\theta_2}$	$\widehat{\theta}_{2MM}$	$\widehat{R}_Y(t)$	$\widetilde{R}_Y(t)$
0.5118	0.5118	218.3114	218.3116	218.3116	0.9583	0.9597



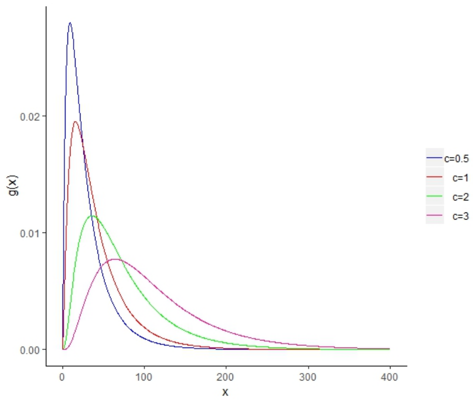
(a) $\alpha = \beta = \nu = c = 1$ and $\theta = 0.5, 1, 2, 3$



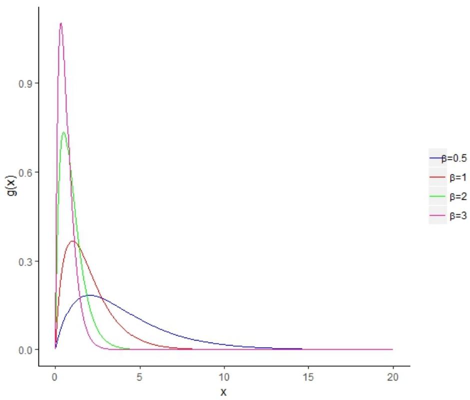
(b) $\alpha = \beta = \theta = c = 1$ and $\nu = 0.5, 1, 2, 3$



(c) $\beta = \nu = \theta = 1, c = 0.5$ and $\alpha = 0.5, 0.8, 2, 4$

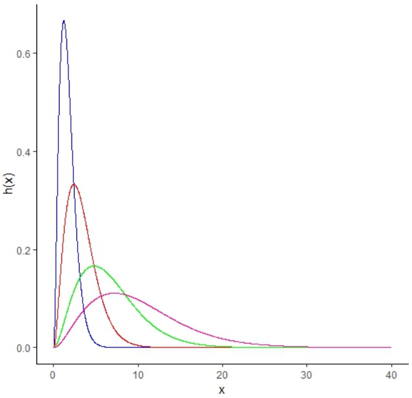


(d) $\alpha = \theta = \nu = \beta = 1$ and $c = 0.5, 1, 2, 3$

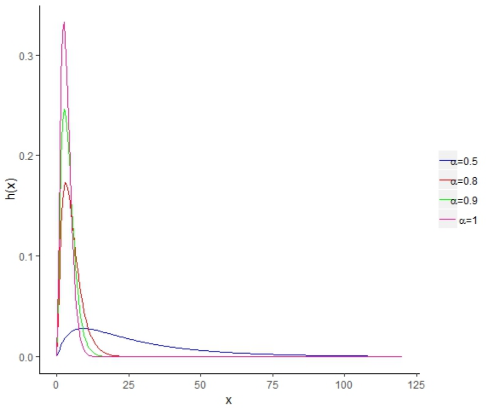


(e) $\alpha = \theta = \nu = c = 1$ and $\beta = 0.5, 1, 2, 3$

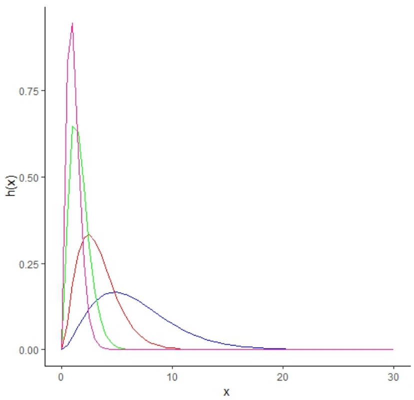
Figure 1 Probability density function plots for different values of parameters



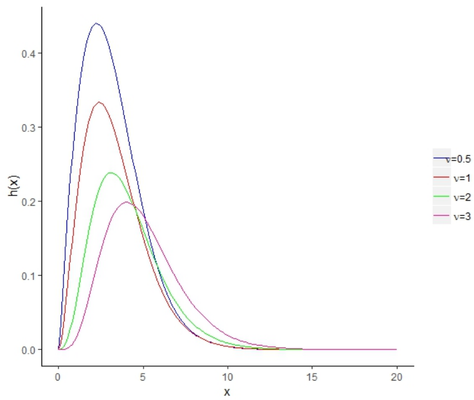
(a) $\alpha = \beta = \nu = 1, c = 2$ and $\theta = 0.5, 1, 2, 3$



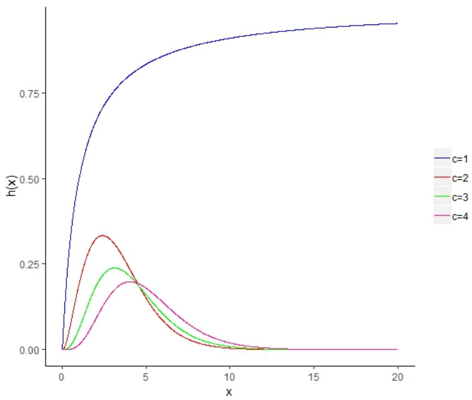
(b) $\nu = \beta = \theta = 1, c = 2$ and $\alpha = 0.5, 0.8, 0.9, 1$



(c) $\alpha = \nu = \theta = 1, c = 2$ and $\beta = 0.5, 1, 2, 3$

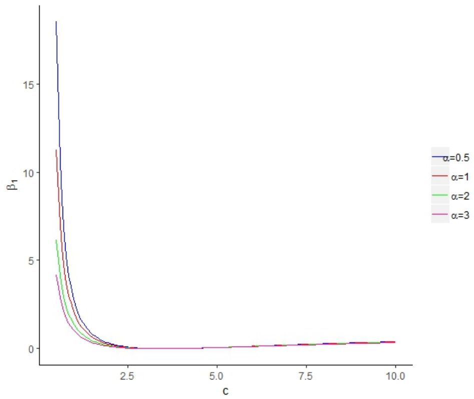


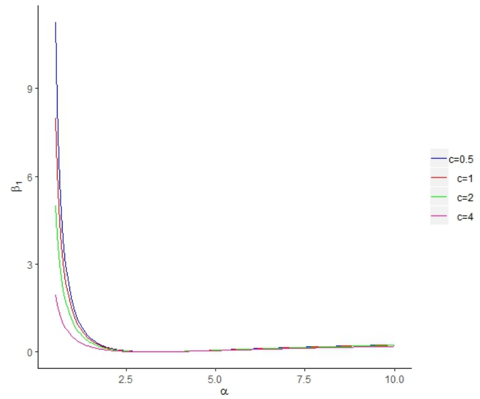
(d) $\alpha = \theta = \beta = 1, c = 2$ and $\nu = 0.5, 1, 2, 3$



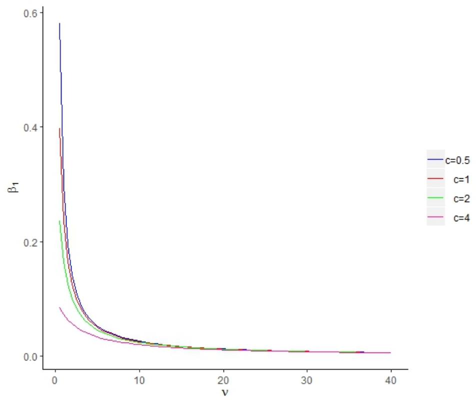
(e) $\alpha = \theta = \nu = \beta = 1$ and $c = 1, 2, 3, 4$

Figure 2 Hazard rate function plots for different values of parameters



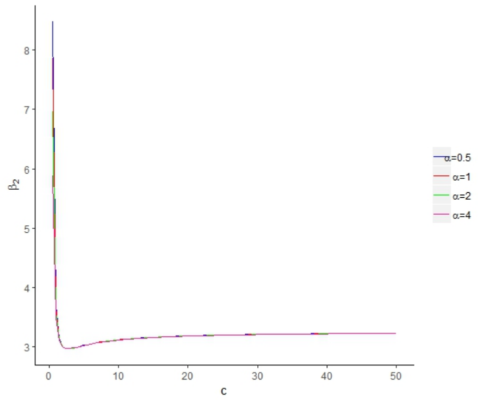


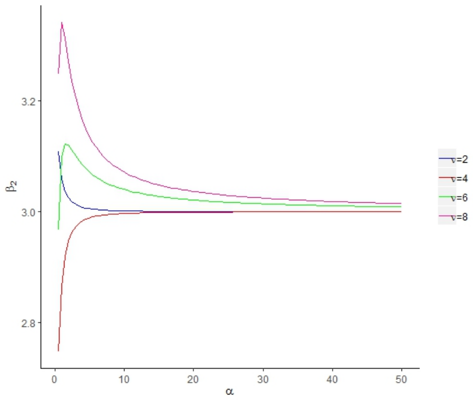
(a) Skewness plot along c for different values of α **(b)** Skewness plot along α for different values of c



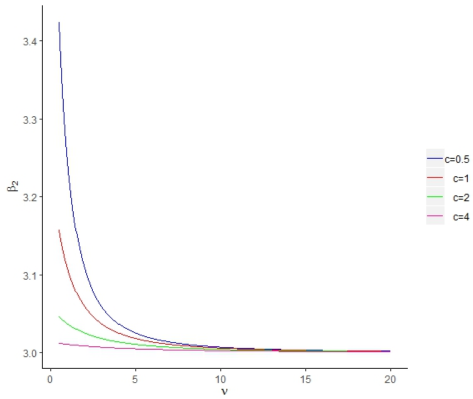
(c) Skewness plot along ν for different values of c

Figure 3 Skewness plots for different values of parameters



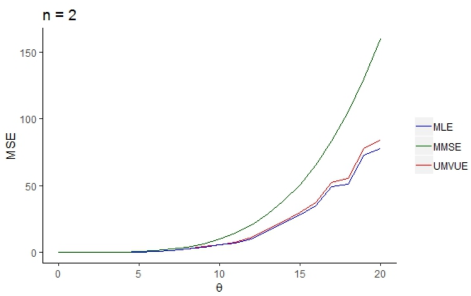


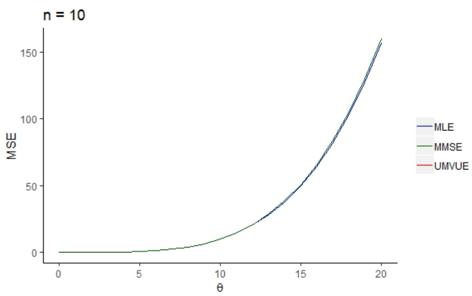
(a) Kurtosis plot along c for different values of α **(b)** Kurtosis plot along α for different values of ν



(c) Kurtosis plot along ν for different values of c

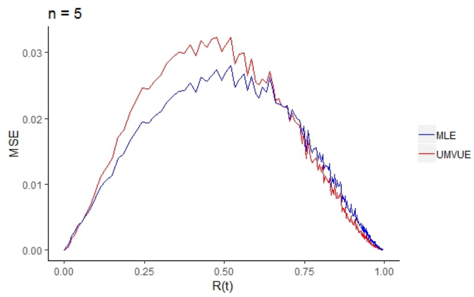
Figure 4 Kurtosis plots for different values of parameters



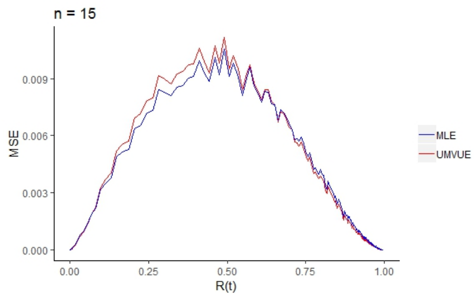


(a) MSE of the UMVU, ML and MM Estimator of θ^q for n=2 **(b)** MSE of the UMVU, ML and MM Estimator of θ^q for n=10

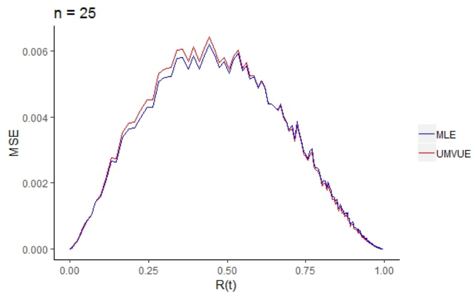
Figure 5 MSE of the UMVU, ML and MM Estimator of θ^q for different sample sizes



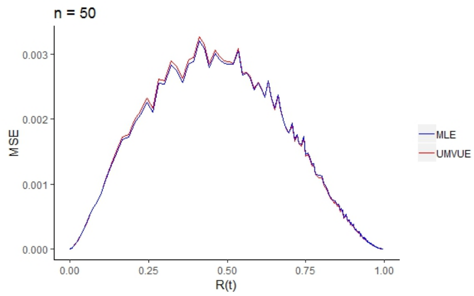
(a) MSE of the UMVU and ML Estimator of $R(t)$ for n=5



(b) MSE of the UMVU and ML Estimator of $R(t)$ for n=15

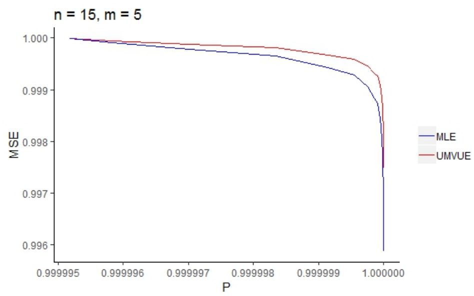


(c) MSE of the UMVU and ML Estimator of $R(t)$ for n=25

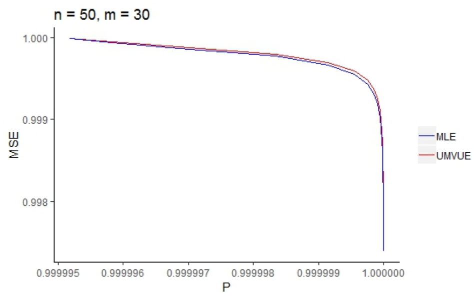


(d) MSE of the UMVU and ML Estimator of $R(t)$ for n=50

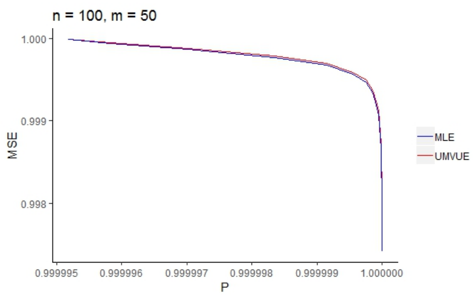
Figure 6 MSE of the UMVU and ML Estimator of $R(t)$ for different sample sizes



(a) MSE of the UMVU and ML Estimator of P for n=15 and m=5



(b) MSE of the UMVU and ML Estimator of P for n=50 and m=30



(c) MSE of the UMVU and ML Estimator of P for n=100 and m=50

Figure 7 MSE of the UMVU and ML Estimator of P for different sample sizes

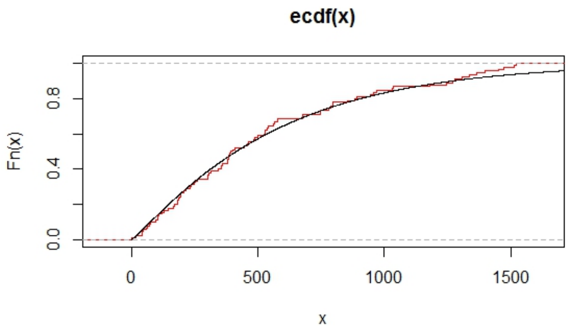


Figure 8 The empirical and theoretical cdf of WGD(ν_1, θ_1, c_1) model

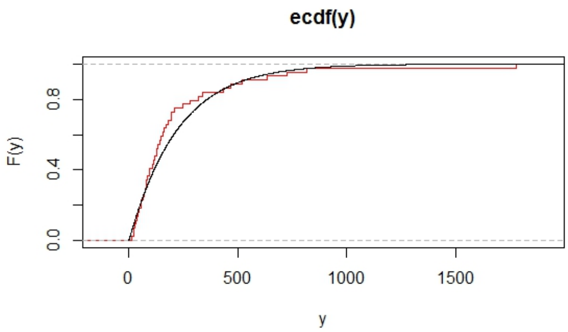


Figure 9 The empirical and theoretical cdf of WGD(ν_2, θ_2, c_2) model