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Generalized Inverse Xgamma Distribution: Properties, Estimation and Its Applications To Survival Data

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Abstract

This article introduces a new form of IXGD called the generalized inverse Xgamma distribution. The proposed model exhibits the pattern of an inverted bathtub type hazard rate and it belongs to the family of positively skewed models. The explicit expressions of some distributional properties, such as, moments, inverse moments, conditional moments, mean deviation, quantile function etc. are derived. To estimate the unknown model parameters as well as survival characteristics, viz., survival function and hazard rate function, we used different estimation procedures, namely, method of maximum likelihood estimation, ordinary and weighted least squares estimation, Cramer-von-Mises estimation and maximum product of spacings estimation. Also, the Bayesian estimation of the same is studied with respect to the squared error loss function. The asymptotic confidence intervals and the Bayes credible intervals of the parameters are computed. Monte Carlo simulations are performed to compare the performances of the proposed methods of estimation in terms of average mean squared errors for the point estimates, average widths and coverage probabilities for interval estimates. Finally, the potential and practical applicability of the proposed model is illustrated through two real life examples.

Keywords: Bayesian estimation, classical methods of estimation, inverse xgamma distribution, moments, reliability curve, order statistics.

1. Introduction

It is impossible to analyse the reliability and survival characteristics and other properties of any lifetime product without the support of distributions. The lifespan of any item or product must follow a particular distribution shape. When modelling monotonic hazard rate functions, exponential, gamma, Weibull and log-normal distributions may be the initial choices. But these distributions have several limitations. These distributions exhibit only monotonically increasing, decreasing or constant Hazard rate functions (HRF)s. However, the most realistic HRF is bathtub-shaped. Even, this occurs in most of the real-life situations. For instance, such shapes of HRF occur when the population is divided into several sub-populations having early failures, wear out failures, and more or less constant failures. Therefore, a perfect bathtub consists of two change points and a constant part enclosed within the change points. The counter part of bathtub HRF is also very interesting for analysing real-life data through several inverted family of distributions. In the literature, there are many lifetime distributions are available from inverse families of distributions, viz., Inverse exponential distribution (IED) [see, Lin et al. (1989)], Inverse Weibull distribution (IWD) [see, Kundu and Howlader (2010)], Inverse Rayleigh distribution (IRD) [see, Voda (1972)], Inverse Lindley distribution (ILD) [see, Sharma et al. (2014)], Inverse xgamma distribution (IXGD) [see, Yadav et al. (2018)] distributions and many more. Also the generalization of inverse family of distributions are available in the literature, such as, Generalized inverted exponential distribution (GIED) [see, Abouammoh et al. (2009)], Generalized inverted gamma distribution (GIGD) [see, Mead (2015)], Generalized inverted Lindley distribution (GILD) [see Sharma et al. (2015)], Exponentiated generalized inverse Weibull distribution (EGIWD) [see, Elbatal et al. (2014)] APTXGD [see, Shukla et al. (2022)], FEXGD [see, Tripathi et al. (2022)], EXGD [see, Yadav et al. (2021)], new class of Xgamma [see, Demirci Bicer (2019)] and many more. Recently, IXGD, the inverted version of Xgamma distribution (XGD) [see, Sen et al. (2016)] is introduced by Yadav et al. (2018). They have also discussed several statistical properties of IXGD and showed the superiority of IXGD among the one parameter inverted family of distributions. If X followed IXGD with scale parameter θ, then the Probability density function (PDF) and Cumulative distribution function (CDF) of IXGD are given as;

$$f(x \mid \theta) = \frac{\theta^2}{x^2(1+\theta)}\left(1+\frac{\theta}{2x^2}\right)e^{-\theta/x} \;\; ; x>0,\ \theta>0, \tag{1}$$

$$F(x \mid \theta) = \left[1+\frac{\theta^2}{2x^2(\theta+1)}+\frac{\theta}{x(\theta+1)}\right]e^{-\theta/x} \;\; ; x>0,\ \theta>0. \tag{2}$$

The main aim of this article is: introduce a generalized version of IXGD by using the power transformation, named as Generalized inverse Xgamma distribution (GIXGD). The estimation of the parameters and associated survival characteristics have been derived using different methods of classical estimation. Further, Bayes estimation of the same has been discussed using informative/non-informative priors under squared error loss function. The ACI and Highest posterior density (HPD) credible interval of the parameters are also computed. After through exploration of the literature, we found that no work has been done in the direction to introduced GIXGD. Our aim is to fill up this gap through this present study and also considered two data sets of survival time of guinea pigs with different doses of tubercle bacilli and survival time of 44 patients suffering from head and neck cancer disease for the illustration of application of proposed model over the some other well known models.

The rest of the article is organized as follows. In Section 2, we introduced the PDF and the CDF of GIXGD and also derived the expression of survival and hazard rate functions. Different statistical properties, viz., moments, inverse moments, conditional moments, harmonic mean, mean deviation, quantile function, Bonferroni and Lorenz curves and a procedure to generate random numbers from GIXGD are discussed in Section 3. In Section 4, estimation of parameters, survival and hazard rate functions by using Maximum likelihood estimator (MLE), Ordinary least squares estimator (OLSE), Weighted least squares estimator (WLSE), Cramer-von-Mises estimator (CME), Maximum product of spacings estimator (MPSE) and Bayesian estimation has been discussed. In Section 5, a Monte Carlo simulation study has been carried out to assess the performances of the above cited classical and the Bayes estimators of the survival and hazard rate functions in terms of corresponding Mean squared error (MSE)s. Also, we assessed the performances of ACIs and the Bayes credible intervals of the model parameters in terms of Coverage probability (CP)s and Average width (AW)s. For illustrative purposes, two real data sets are analyzed in Section 6. Finally, concluding remarks are given in Section 7.

2. Generalized Inverse Xgamma Distribution

The IXGD is actually the inverted version of XGD [see, Sen et al. (2016)], is recently proposed by Yadav et al. (2018). They have studied the different statistical properties and estimation of the unknown parameter using different methods of estimation. They have mentioned that IXGD possesses non-monotone hazard rates (upside-down bathtub) and also shows the superiority of IXGD among the inverted family of distributions. As we know that, the shape parameter play an important role in flexibility of any lifetime model and hence it becomes more realistic for use in any real life situation. In this present article, we have proposed a more flexible model by adding one more parameter α, the shape parameter, as the power of IXGD variable.

If X be a random variable having PDF and CDF of XGD mentioned in Equations (1) and (2) respectively, then GIXGD is obtained by using power transformation $Y = X^{1/\alpha}$, where α is the shape parameter. Hence, the PDF and CDF of GIXGD are obtained as

$$f(y \mid \alpha,\theta) = \frac{\alpha\theta^2}{1+\theta}\frac{1}{y^{(\alpha+1)}}\left(1+\frac{\theta}{2y^{2\alpha}}\right)e^{-\theta/y^{\alpha}}; \quad y>0,\ \alpha>0,\ \theta>0, \tag{3}$$

and

$$F(y \mid \alpha,\theta) = \left(1+\frac{\theta^2}{2(\theta+1)}\frac{1}{y^{(2\alpha)}}+\frac{\theta}{\theta+1}\frac{1}{y^{\alpha}}\right)e^{-\theta/y^{\alpha}}; \quad y>0,\ \alpha>0,\ \theta>0, \tag{4}$$

respectively. In particular, if $\alpha = 1$, then GIXGD is coincide with IXGD. Survival characteristics of any life time model is often measured in terms of it's Survival function (SF) and HRF. The SF $S(t \mid \alpha,\theta)$ and HRF $H(t \mid \alpha,\theta)$ for specified value at $x = t$ of GIXGD are, respectively, obtained as:

$$S(t \mid \alpha, \theta) \quad = \quad 1 - F(t \mid \alpha, \theta),$$

and

$$H(t \mid \alpha, \theta) \quad = \quad \frac{f(t \mid \alpha, \theta)}{S(t \mid \alpha, \theta)}.$$

A typical graphs of PDF and HRF are displayed in Figures 1 and 2 for different choices of shape and scale parameters respectively. From the shape of density function, it is clearly observable that GIXGD is positively skewed and uni-modal distribution. Also, from the shape of HRFs, it is noted that initially HRF is increasing and reaches to a peak after that declined slowly, which indicates that the model possesses the hump or upside-down bathtub property of hazard rate. Such behaviour of HRFs are quite common in reliability studies and clinical trial studies etc.

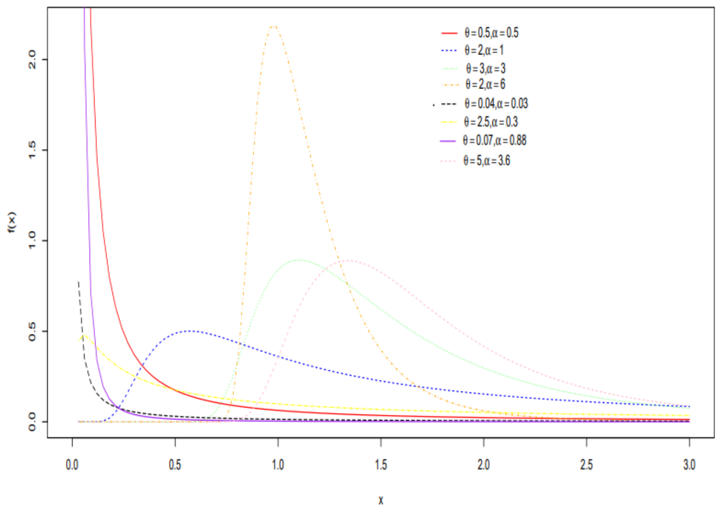

Figure 1 PDF of GIXGD for different values of shape and scale parameters.

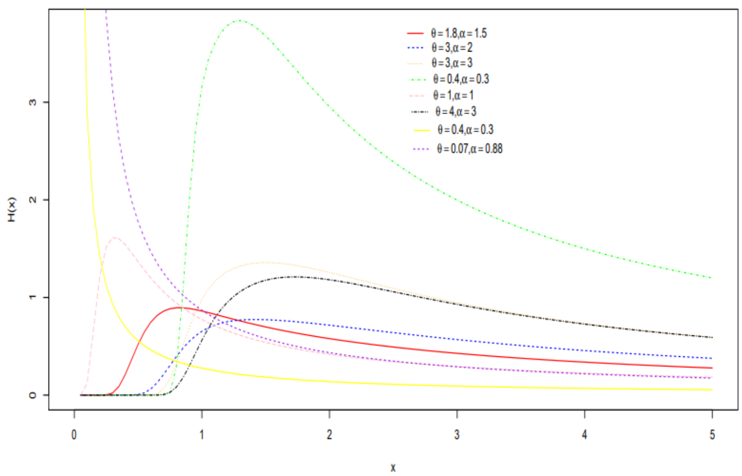

Figure 2 HRF of GIXGD for different values of shape and scale parameters.

3. Some Statistical Properties of GIXGD

In the following subsections, the associated distributional properties of the proposed distribution have been derived.

3.1. Moments

Moments are very useful to determine the various properties of a model. Here, We are in interested in investigating the raw moments of GIXGD. Expression of c-th order raw moment is given below

$$\begin{aligned} E(Y^c) &= \int_0^\infty y^c f(y \mid \alpha, \theta) dy, \\ &= \int_0^\infty y^c \frac{\alpha\theta^2}{1+\theta} \frac{1}{y^{(\alpha+1)}} \left(1 + \frac{\theta}{2y^{2\alpha}}\right) e^{-\theta/y^\alpha} dy, \\ &= \frac{\theta^{(c/\alpha)+1}}{(1+\theta)} \Gamma[1-(c/\alpha)] + \frac{1}{2} \frac{\theta^{(c/\alpha)}}{(\theta+1)} \Gamma[3-(c/\alpha)]. \end{aligned}$$

c-th order raw moment exists iff $\frac{c}{\alpha} < 1$ and first four central moments can be easily obtained by using the relationship between raw moments and central moments. Hence, Pearson measures of skewness (SK) and kurtosis (K) based on moments can be obtained by using following formulae

$$\text{SK} = \frac{\mu_3^2}{\mu_2^3} \text{ and } \text{K} = \frac{\mu_4}{\mu_2^2}$$

where, μ_2, μ_3 and μ_4 are the second, third and fourth central moments respectively.

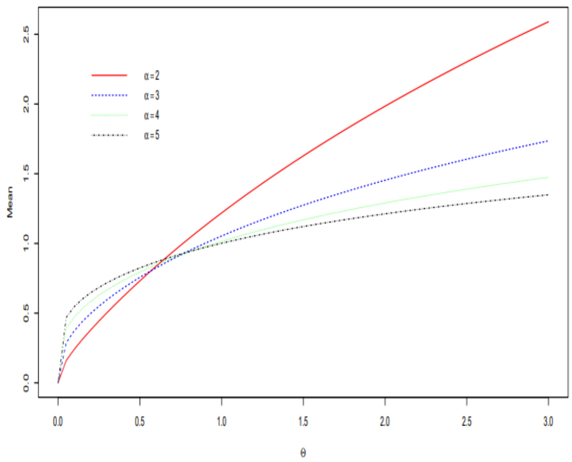

Figure 3 Mean of GIXGD.

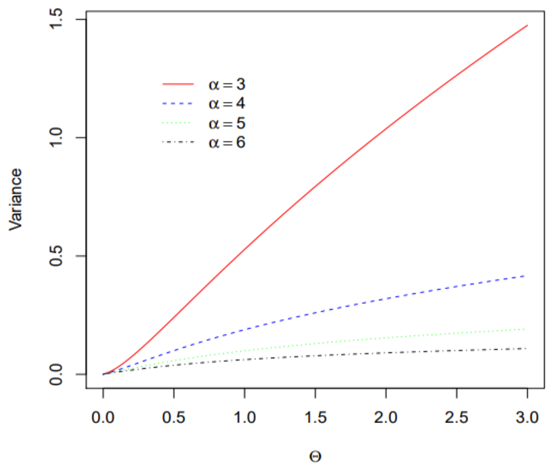

Figure 4 Variance of GIXGD.

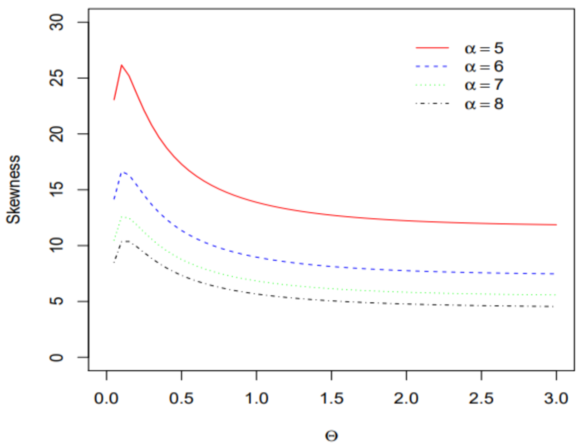

Figure 5 Skewness of GIXGD.

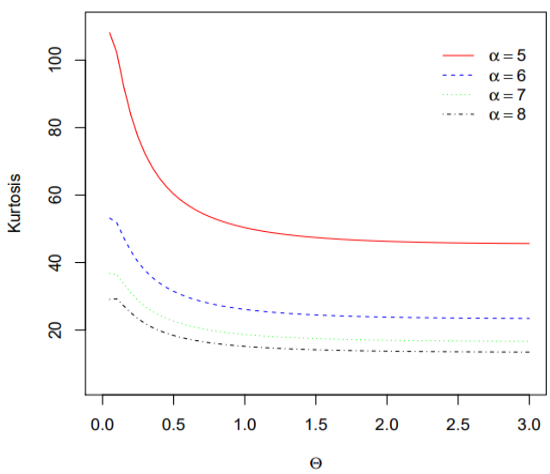

Figure 6 Kurtosis of GIXGD.

From Figure 3 and Figure 4, we have observed that the mean and variance of proposed distribution have increasing nature for each value of shape parameter. From Figure 5, we can observed that distribution is positively skewed and trend of skewness is decreasing for the considered value of the shape parameter for certain value of θ and from Figure 6, we have noticed that kurtosis of GIXGD is decreasing in nature for the all considered value of shape parameter α.

3.2. Inverse moments

The c-th order inverse moments about origin of GIXGD is given as

$$\begin{aligned} E\left(\frac{1}{Y^c}\right) &= \int_0^\infty \frac{1}{y^c} f(y \mid \alpha, \theta) dy, \\ &= \int_0^\infty \frac{1}{y^c} \frac{\alpha\theta^2}{1+\theta} \frac{1}{y^{(\alpha+1)}} \left(1 + \frac{\theta}{2y^{2\alpha}}\right) e^{-\theta/y^\alpha} dy, \\ &= \frac{\theta^2}{2(1+\theta)} \left(\frac{2\Gamma[1+(c/\alpha)]}{\theta^{[1+(c/\alpha)]}} + \frac{\Gamma[3+(c/\alpha)]}{\theta^{[2+(c/\alpha)]}}\right) \quad c = 1, 2, 3, \ldots \ . \end{aligned} \tag{5}$$

The harmonic mean for the random variable can be computed from the Equation (5) by putting $c = 1$. Hence, after simplification, we get

$$E\left(\frac{1}{Y}\right) = \frac{\theta^2}{2(1+\theta)} \left(\frac{2\Gamma[1+(1/\alpha)]}{\theta^{[1+(1/\alpha)]}} + \frac{\Gamma[3+(1/\alpha)]}{\theta^{[2+(1/\alpha)]}}\right).$$

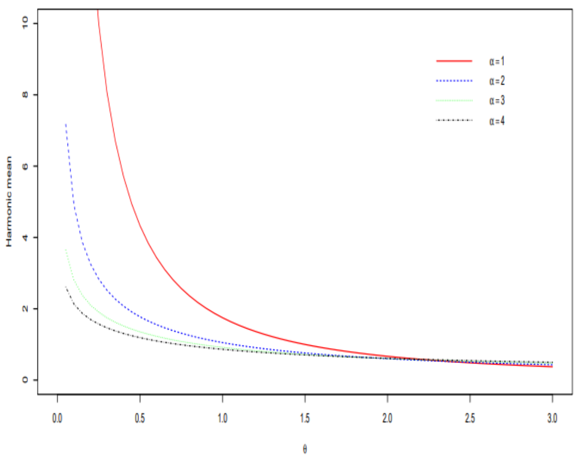

Figure 7 Harmonic mean of GIXGD.

Figure 7 is the graphical representation of the harmonic mean or first inverse raw moment of GIXGD for different combination of α. From this plot, it has been noticed that the value of harmonic mean is decreases as the parametric value increases.

3.3. Conditional moment

Conditional moments about origin of GIXGD is obtained as

$$\begin{aligned} E(Y^n | Y > y) &= \int_y^\infty y^n \frac{f(y \mid \alpha, \theta)}{1 - F(y \mid \alpha, \theta)} dy, \\ &= \frac{1}{1 - F(y \mid \alpha, \theta)} \left[\frac{\theta^{(n/\alpha)+1}}{\theta+1} \gamma\left(1 - \frac{n}{\alpha}, \frac{\theta}{y^\alpha}\right) + \frac{\theta^{(n/\alpha)}}{\theta+1} \gamma\left(3 - \frac{n}{\alpha}, \frac{\theta}{y^\alpha}\right)\right]. \end{aligned} \tag{6}$$

We choose such value of shape parameter α which makes the gamma function $\gamma(.,.)$ positive in Equation (6). In other words we can say that the conditional moments exists with the one restriction over the choices of shape parameter α, i.e., $\frac{n}{\alpha} < 1$.

3.4. Mean deviation

The mean deviation about mean of random variable Y, having density function (3) is obtained as

$$M.D.(Y) = \int_0^\infty |(y-\mu)| f(y \mid \alpha, \theta) dy.$$

where $\mu = E(Y)$. On simplification

$$M.D = 2\mu F(\mu) - 2\mu + 2\int_\mu^\infty y f(y \mid \alpha, \theta) dy,$$

where $F(\mu)$ stands for CDF of Y up to point μ and $\int_\mu^\infty y f(y;\alpha,\theta)dy$ is obtained through use of conditional distribution

$$\int_\mu^\infty y f(y \mid \alpha, \theta) dy = [1 - F(\mu)] E(Y/Y > \mu).$$

3.5. Quantile function

If $Q(p)$ be the quantile of order p $(0 < p < 1)$ of the random variable Y, then it will be the solution of

$$F(Q(p)) \quad = \quad \left(1 + \frac{\theta^2}{2(\theta+1)} \frac{1}{Q(p)^{(2\alpha)}} + \frac{\theta}{\theta+1} \frac{1}{Q(p)^\alpha}\right) e^{-\theta/Q(p)^\alpha} = p.$$

The degree of long-tail is measured by skewness and while the degree of tail heaviness is measured by kurtosis of the random variable. The Bowley measure of skewness [see, Bowley (1920)] and Moors measure of kurtosis [see, Moors (1988)] based on quantile can be used and are given as

$$\text{SK} = \frac{Q(\frac{3}{4}) - 2Q(\frac{1}{2}) + Q(\frac{1}{4})}{Q(\frac{3}{4}) - Q(\frac{1}{4})}$$

$$\text{K} = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{2}{8})}.$$

3.6. Bonferroni and Lorenz curves

Bonferroni and Lorenz curves introduced by Kleiber and Kotz (2003). These curves are very useful in field of income, poverty, reliability, demography and insurance. Let Y be random variable with PDF given Equation (3), then Bonferroni and Lorenz curves are respectively defined as

$$B(p) = \frac{1}{p\mu}\int_0^q y f(y \mid \alpha, \theta) dy = \frac{1}{p\mu}\left\{\mu - \left[\frac{\theta^{(1/\alpha)+1}}{\theta+1}\gamma\left(1 - \frac{1}{\alpha}, \frac{\theta}{q^\alpha}\right) + \frac{\theta^{(1/\alpha)}}{\theta+1}\gamma\left(3 - \frac{1}{\alpha}, \frac{\theta}{q^\alpha}\right)\right]\right\}$$

and

$$L(p) = \frac{1}{\mu}\int_0^q y f(y \mid \alpha, \theta) dy = \frac{1}{\mu}\left\{\mu - \left[\frac{\theta^{(1/\alpha)+1}}{\theta+1}\gamma\left(1 - \frac{1}{\alpha}, \frac{\theta}{q^\alpha}\right) + \frac{\theta^{(1/\alpha)}}{\theta+1}\gamma\left(3 - \frac{1}{\alpha}, \frac{\theta}{q^\alpha}\right)\right]\right\}.$$

and the indices based on these two curves are obtained as

$$B = 1 - \int_0^1 B(p)dp \quad \text{and} \quad G = 1 - 2\int_0^1 L(p)dp$$

respectively, where B and G represents the Bonferroni and Gini indices.

3.7. Entropies

Entropies are the one of the most essential aspect in the model study. Entropy measures the information regarding the uncertainty of random experiment. Application of entropies are very spacious and applicable in the fields like finance, physics, molecular imaging of tumors, statistics, economics and sparse kernel density estimation. Here, we discussed the two important entropies viz., Renyi entropy [see, Renyi (1961)] and Shannon's entropy [see, Shannon (1951)].

Renyi entropy:

Renyi entropy (RE) is defined as:

$$\text{RE} = \frac{1}{1-\gamma}\log\left(\int_0^\infty f^\gamma(y \mid \alpha,\theta)dx\right)$$

where $\gamma > 0$ and $\gamma \neq 1$.

Theorem 1 *If Y follow the GIXGD then expression of Renyi entropy is:*

$$\text{RE} = \frac{1}{1-\gamma}\log\left\{\left(\frac{\theta^2}{1+\theta}\right)^\gamma \alpha^{\gamma-1}\sum_{i=0}^{\infty}\binom{\gamma}{i}\left(\frac{\theta}{2}\right)^i \frac{\Gamma[\gamma(1+\frac{1}{\alpha})+2i-(1+\frac{1}{\alpha})]}{(\theta\gamma)^{\gamma(1+\frac{1}{\alpha})+2i-(1+\frac{1}{\alpha})}}\right\}.$$

Proof:

$$\begin{aligned}
\text{RE} &= \frac{1}{1-\gamma}\log\left\{\int_0^\infty \left[\frac{\alpha\theta^2}{1+\theta}\frac{1}{y^{(\alpha+1)}}\left(1+\frac{\theta}{2y^{2\alpha}}\right)e^{-\theta/y^\alpha}\right]^\gamma dy\right\} \\
\text{RE} &= \frac{1}{1-\gamma}\log\left\{\left(\frac{\alpha\theta^2}{1+\theta}\right)^\gamma \int_0^\infty \left[\frac{1}{y^{\gamma(\alpha+1)}}e^{\frac{-\theta\gamma}{y^\alpha}}\left(1+\frac{\theta}{2y^{2\alpha}}\right)^\gamma dy\right]\right\}
\end{aligned}$$

Final expression of RE is obtained by the solving the above expression of RE and the RE is given in Equation in (7):

$$\text{RE} = \frac{1}{1-\gamma}\log\left\{\left(\frac{\theta^2}{1+\theta}\right)^\gamma \alpha^{\gamma-1}\sum_{i=0}^{\infty}\binom{\gamma}{i}\left(\frac{\theta}{2}\right)^i \frac{\Gamma[\gamma(1+\frac{1}{\alpha})+2i-(1+\frac{1}{\alpha})]}{(\theta\gamma)^{\gamma(1+\frac{1}{\alpha})+2i-(1+\frac{1}{\alpha})}}\right\}. \tag{7}$$

Shannon Entropy:

Shannon entropy defined by the E$[-\log f(y \mid \alpha,\theta)]$ and this is the particular case of Renyi entropy when $\gamma \uparrow 1$. Limiting $\gamma \uparrow 1$ in Equation (7) and using L'Hospitals rule, one obtains after considerable algebraic manipulation of E$[-\log f(y \mid \alpha,\theta)]$. The expression of the $-\log f(y \mid \alpha,\theta)$ is:

$$-\log f(y \mid \alpha,\theta) = -\log\frac{\theta^2}{1+\theta} - \log\frac{1}{y^{\alpha+1}} + \frac{\theta}{y^\alpha} - \log\left(1+\frac{\theta}{2y^{2\alpha}}\right).$$

Now, the expectation of above written expression of $-\log f(x)$ is:

$$E[-\log f(y \mid \alpha,\theta)] = -\log\frac{\theta^2}{1+\theta} - E\left(\log\frac{1}{y^{\alpha+1}}\right) + \theta E\left(\frac{1}{y^\alpha}\right) - E\left[\log\left(1+\frac{\theta}{2y^{2\alpha}}\right)\right]. \tag{8}$$

To get the final solution of the Equation (8), we have to solve the second, the third and the fourth term of the Equation (8). Now second term is:

$$E\left(\log\frac{1}{Y^{\alpha+1}}\right) = E\left[\log\left(1+\frac{1}{y^{\alpha+1}}-1\right)\right] = E\left[\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}\left(\frac{1}{y^{\alpha+1}}-1\right)^{i}\right].$$

To simplify the above equation, we use the following expansion:

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.$$

$$\begin{aligned}E\left(\log\frac{1}{Y^{\alpha+1}}\right) &= E\left[\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}\sum_{k=0}^{i}\binom{i}{k}\left(\frac{1}{y^{\alpha}}\right)^{i-k}(-1)^k\right] \\ &= \sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}\sum_{k=0}^{i}\binom{i}{k}(-1)^k\frac{\alpha\theta^2}{1+\theta}\times \\ &\left[\int_0^{\infty}\left(\frac{1}{y^{\alpha+1}}\right)^{i-k}\frac{1}{y^{\alpha+1}}e^{\frac{-\theta}{y^{\alpha}}}dy+\int_0^{\infty}\left(\frac{1}{y^{\alpha+1}}\right)^{i-k}\frac{1}{y^{\alpha+1}}e^{\frac{-\theta}{y^{\alpha}}}\frac{\theta}{2y^{2\alpha}}dy\right]. \qquad (9)\end{aligned}$$

After solving the two integrals which are involve in above written Equation (9) then final expression of second term $E\left(\log\frac{1}{y^{\alpha+1}}\right)$ is given below:

$$E\left(\log\frac{1}{Y^{\alpha+1}}\right) = \sum_{i=1}^{\infty}\sum_{k=0}^{i}\binom{i}{k}\frac{(-1)^{i+k+1}}{i}\frac{\theta^2}{1+\theta}\left[\frac{\Gamma(1+\frac{1}{\alpha})(i-k+1)-(1+\frac{1}{\alpha})}{\theta^{\Gamma(1+\frac{1}{\alpha})(i-k+1)-(1+\frac{1}{\alpha})}}+\frac{\theta}{2}\frac{\Gamma(1+\frac{1}{\alpha})(i-k+1)+2-(1+\frac{1}{\alpha})}{\theta^{\Gamma(1+\frac{1}{\alpha})(i-k+1)+2-(1+\frac{1}{\alpha})}}\right].$$

Now, the third term $E\left(\frac{1}{Y}^{\alpha}\right)$ can be solved by the inverse moment of the GIXGD and the final expression of third term [see, Equation (10)] after replacing c by α in the expression of inverse moment, then,

$$E\left(\frac{1}{Y^{\alpha}}\right) = \frac{\theta+3}{\theta(\theta+1)}. \qquad (10)$$

The last, i.e., fourth term of the Equation (8) can be solved by follow the steps of the second term and the expression is:

$$E\left[\log\left(1+\frac{\theta}{2Y^{2\alpha}}\right)\right] = \sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}\frac{\theta^2}{1+\theta}\left(\frac{\theta}{2}\right)^i\left[\frac{\Gamma 2i}{\theta^{2i}}+\frac{\theta}{2}\frac{\Gamma 2i+2}{\theta^{2i+2}}\right].$$

Now, substitute the values of expectations obtained above in the Equation (8) to get the final solution of Shannon entropy is:

$$\begin{aligned}E[-\log f(y\mid\alpha,\theta)] &= -\log\frac{\theta^2}{1+\theta}+\frac{\theta+3}{\theta(\theta+1)}-\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}\frac{\theta^2}{1+\theta}\left(\frac{\theta}{2}\right)^i\left[\frac{\Gamma 2i}{\theta^{2i}}+\frac{\theta}{2}\frac{\Gamma 2i+2}{\theta^{2i+2}}\right] \\ &-\sum_{i=1}^{\infty}\sum_{k=0}^{i}\binom{i}{k}\frac{(-1)^{i+k+1}}{i}\frac{\theta^2}{1+\theta} \\ &\left[\frac{\Gamma(1+\frac{1}{\alpha})(i-k+1)-(1+\frac{1}{\alpha})}{\theta^{\Gamma(1+\frac{1}{\alpha})(i-k+1)-(1+\frac{1}{\alpha})}}+\frac{\theta}{2}\frac{\Gamma(1+\frac{1}{\alpha})(i-k+1)+2-(1+\frac{1}{\alpha})}{\theta^{\Gamma(1+\frac{1}{\alpha})(i-k+1)+2-(1+\frac{1}{\alpha})}}\right].\end{aligned}$$

3.8. Order statistic

Let $Y_1,\ Y_2,\ Y_3, ...,\ Y_n$ is a random sample of size n, obtained from GIXGD. Then, the ordered observations as $Y_{(1)} < Y_{(2)} < Y_{(3)} < ..... < Y_{(n)}$ constitute the order statistic(s). Let $Y_{(r;n)}$ denotes the r-th order statistic, then the PDF and CDF of r-th order statistic are computed as

$$f_r(Y_{r:n}) = \frac{n!}{(n-r)!(r-1)!}\sum_{l=0}^{n-r}\binom{n-r}{l}(-1)^l F(y \mid \alpha,\theta)^l F(y \mid \alpha,\theta)^{(r-1)} f(y \mid \alpha,\theta)$$

and

$$F(Y_{r:n}) = \sum_{j=r}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l F(y \mid \alpha,\theta)^{j+l},$$

respectively, obtained with help proposed model given in Equations (3) and (4).

3.9. Random number generation

To generate random number from GIXGD (α,θ). The following steps may be used.

1. Generate U_i from Uniform$(0,1)$ distribution $(i = 1,2,3...,n)$.
2. Generate V_i from Gamma$(1,\theta)$ distribution $(i = 1,2,3...,n)$.
3. Generate W_i from Gamma$(3,\theta)$ distribution $(i = 1,2,3...,n)$.
4. If $U_i \le \frac{\theta}{\theta+1}$, set $Z_i = V_i$, otherwise set $Z_i = W_i$.
5. $Y = \left(\frac{1}{Z}\right)^{1/\alpha}$ be random numbers GIXGD.

If we take $\alpha = 1$, then the algorithm of generating random number from GIXGD is same as that of IXGD.

4. Methods of Estimation

Here, we briefly described different classical methods of estimation, namely, MLE, OLSE and WLSE CME and MPSE of the parameters as well as the estimators of SF $S(t \mid \alpha,\theta)$ and HRF $H(t \mid \alpha,\theta)$ respectively.

4.1. Maximum likelihood estimator

Let $Y_1,\ Y_2,\cdots,\ Y_n$ be a random sample of size n, obtained from Equation (3). Then, the likelihood function for the observed random sample $y_1,\ y_2,\cdots,\ y_n$ is given as

$$L(\alpha,\theta|y) = \prod_{i=1}^{n}\left[\frac{\alpha\theta^2}{1+\theta}\frac{1}{y_i^{(\alpha+1)}}\left(1+\frac{\theta}{2y_i^{2\alpha}}\right)e^{-\theta/y_i^{\alpha}}\right]. \quad (11)$$

Taking logarithm on both sides of Equation (11), we have

$$\log L(\alpha,\theta|y) = n\log(\alpha) + 2n\log\theta - n\log(\theta+1) + \sum_{i=1}^{n}\log(\frac{1}{y_i^{(\alpha+1)}}) + \sum_{i=1}^{n}\log(1+\frac{\theta}{2y_i^{2\alpha}}) - \sum_{i=1}^{n}\frac{\theta}{y_i^{\alpha}}.$$

Partial derivatives of the log-likelihood function with respect to α and θ and equating to zero yield the estimate of α and θ respectively, i.e.,

$$\frac{\partial\log L(\alpha,\theta|y)}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\log y_i - \sum_{i=1}^{n}\frac{\theta y^{-2\alpha}\log y}{\left(1+\frac{\theta}{2y_i^{2\alpha}}\right)} + \theta\sum_{i=1}^{n}y_i^{-\alpha}\log y_i = 0, \quad (12)$$

$$\frac{\partial\log L(\alpha,\theta|y)}{\partial\theta} = \frac{2n}{\theta} - \sum_{i=1}^{n}\frac{1}{y_i^{\alpha}} + \sum_{i=1}^{n}\frac{(1/2y_i^{2\alpha})}{\left(1+\frac{\theta}{2y_i^{2\alpha}}\right)} - \frac{n}{\theta+1} = 0. \quad (13)$$

Now, equating these partial derivatives to zero which do not yield closed form solutions for the MLEs of α, θ and thus a numerical method is used to obtain MLEs $(\hat{\alpha}_{mle}, \hat{\theta}_{mle})$ for (α, θ) from Equations (12) and (13) simultaneously. Using the invariance property of MLE, we can get the estimators of $S(t \mid \alpha, \theta)$ and $H(t \mid \alpha, \theta)$, given as

$$\hat{S}(t \mid \hat{\alpha}, \hat{\theta})_{mle} = 1 - \left[1 + \frac{\hat{\theta}^2_{mle}}{2(\hat{\theta}_{mle}+1)} \frac{1}{t^{(2\hat{\alpha}_{mle})}} + \frac{\hat{\theta}_{mle}}{\hat{\theta}_{mle}+1} \frac{1}{t^{\hat{\alpha}_{mle}}}\right] e^{-\hat{\theta}_{mle}/t^{\hat{\alpha}_{mle}}},$$

and

$$\hat{H}(t \mid \hat{\alpha}, \hat{\theta})_{mle} = \left[\frac{\frac{\hat{\alpha}_{mle}\hat{\theta}^2_{mle}}{1+\hat{\theta}_{mle}} \frac{1}{t^{(\hat{\alpha}_{mle}+1)}} \left(1 + \frac{\hat{\theta}_{mle}}{2t^{2\hat{\alpha}_{mle}}}\right) e^{-\hat{\theta}_{mle}/t^{\hat{\alpha}_{mle}}}}{1 - \left[1 + \frac{\hat{\theta}^2_{mle}}{2(\hat{\theta}_{mle}+1)} \frac{1}{t^{(2\hat{\alpha}_{mle})}} + \frac{\hat{\theta}_{mle}}{\hat{\theta}_{mle}+1} \frac{1}{t^{\hat{\alpha}_{mle}}}\right] e^{-\hat{\theta}_{mle}/t^{\hat{\alpha}_{mle}}}}\right],$$

respectively.

4.1.1 Asymptotic Confidence Interval (ACI) of parameters

From subsection (4.1), we observed that obtaining the exact confidence intervals for the parameters $\Theta = (\alpha,\, \theta)$ is too difficult. To avoid this difficulty, we used the asymptotic normality assumption of the MLE to compute the confidence intervals of the parameters α and θ respectively. As we know that for large sample, $\sqrt{n}(\hat{\Theta} - \Theta) \sim \text{AN}\left(0, \frac{1}{\eta}\right)$, where, $\hat{\Theta}$ is the MLE of Θ, η is the observed Fisher information matrix, and and η^{-1} is the inverse of the observed Fisher information matrix and η is given by;

$$\eta = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix}$$

The elements of η are given as follows:

$$\begin{aligned}
\eta_{11} &= \left| -\frac{\partial^2}{\partial \alpha^2} \log L(\alpha, \theta | y) \right|_{\alpha=\hat{\alpha}, \theta=\hat{\theta}}, \\
&= \left| \frac{n}{\alpha^2} + \theta \sum_{i=1}^{n} y_i^{-\alpha} (\log y_i)^2 - \sum_{i=1}^{n} \left[-\frac{\theta^2 y^{-4\alpha} (\log y)^2}{\left(1 + \frac{\theta}{2y_i^{2\alpha}}\right)^2} + \frac{2\theta y^{-2\alpha} (\log y)^2}{\left(1 + \frac{\theta}{2y_i^{2\alpha}}\right)} \right] \right|_{\alpha=\hat{\alpha}, \theta=\hat{\theta}}, \\
\eta_{22} &= \left| -\frac{\partial^2}{\partial \theta^2} \log L(\alpha, \theta | y) \right|_{\alpha=\hat{\alpha}, \theta=\hat{\theta}}, \\
&= \left| \frac{2n}{\theta^2} + \sum_{i=1}^{n} \frac{(1/4y_i^{4\alpha})}{\left(1 + \frac{\theta}{2y_i^{2\alpha}}\right)^2} - \frac{n}{(\theta+1)^2} \right|_{\alpha=\hat{\alpha}, \theta=\hat{\theta}}, \\
\eta_{12} = \eta_{21} &= \left| -\frac{\partial^2}{\partial \alpha \partial \theta} \log L(\alpha, \theta | y) \right|_{\alpha=\hat{\alpha}, \theta=\hat{\theta}}, \\
&= \left| -\sum_{i=1}^{n} y_i^{-\alpha} \log y_i - \sum_{i=1}^{n} \left[\frac{\theta y^{-4\alpha} \log y}{2\left(1 + \frac{\theta}{2y_i^{2\alpha}}\right)^2} - \frac{y^{-2\alpha} \log y}{\left(1 + \frac{\theta}{2y_i^{2\alpha}}\right)} \right] \right|_{\alpha=\hat{\alpha}, \theta=\hat{\theta}}.
\end{aligned}$$

The estimated elements of variance covariance matrix of the parameters α and θ can be calculated by inverting η as follows:

$$\psi^{-1} = \begin{pmatrix} \widehat{var(\alpha)} & \widehat{cov(\alpha)} \\ \widehat{cov(\alpha)} & \widehat{var(\theta)} \end{pmatrix} = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix}^{-1}$$

The diagonal elements $\widehat{var(\alpha)}$ and $\widehat{var(\theta)}$ of the matrix are the asymptotic variance of the variance of α and θ respectively. Thus, the asymptotic 100(1-a)% confidence interval for α and θ are given by

$$\left(\hat{\alpha}_{(L,U)} = \hat{\alpha} \mp Z_{\frac{a}{2}} \sqrt{\widehat{var(\alpha)}}, \qquad \hat{\theta}_{(L,U)} = \hat{\theta} \mp Z_{\frac{a}{2}} \sqrt{\widehat{var(\theta)}}\right).$$

4.2. Ordinary and weighted least square estimators

The OLSE and the WLSE were proposed by Swain et al. (1988) to estimate the parameters of Beta distributions. Suppose $F(y_{(i:n)};\alpha,\theta)$ denotes the distribution function of the ordered random variables $Y_{(1:n)} < Y_{(2:n)} < \cdots < Y_{(n:n)}$ of size n from a distribution function $F(\cdot)$ from Equation (4). Then, the OLSEs of the parameters α and θ, say, $\hat{\alpha}_{olse}$ and $\hat{\theta}_{olse}$ are obtained by minimizing

$$\mathcal{L}(\alpha,\theta)=\sum_{i=1}^{n}\left(F(y_{(i:n)}\mid\alpha,\theta)-\frac{i}{n+1}\right)^{2}.$$

i.e., by solving the following non-linear equations:

$$\sum_{i=1}^{n}\left[F(y_{(i:n)}\mid\alpha,\theta)-\frac{i}{n+1}\right]\psi_1(y_{(i:n)}\mid\alpha,\theta) \quad = \quad 0, \tag{14}$$

and

$$\sum_{i=1}^{n}\left[F(y_{(i:n)}\mid\alpha,\theta)-\frac{i}{n+1}\right]\psi_2(y_{(i:n)}\mid\alpha,\theta) \quad = \quad 0, \tag{15}$$

where, $\psi_1(y_{(i:n)}\mid\alpha,\theta)$ and $\psi_2(y_{(i:n)}\mid\alpha,\theta)$ are the first derivatives of $F(y_{(i:n)}\mid\alpha,\theta)$ with respect to α and θ respectively, given as follows:

$$\begin{aligned}
\psi_1(y_{(i:n)}\mid\alpha,\theta) &= \frac{-\theta e^{-\theta/y^{\alpha}}}{y^{\alpha}}\log\frac{1}{y}\left(1+\frac{\theta^2}{2(\theta+1)}\frac{1}{y^{2\alpha}}+\frac{\theta}{(\theta+1)}\frac{1}{y^{\alpha}}\right)+\frac{e^{-\theta/y^{\alpha}}}{y^{\alpha}}\\
&\quad\times\log\frac{1}{y}\left(\frac{\theta^2}{(1+\theta)}\frac{1}{y^{\alpha}}+\frac{\theta}{\theta+1}\right),\\
\psi_2(y_{(i:n)}\mid\alpha,\theta) &= \frac{-e^{-\theta/y^{\alpha}}}{y^{\alpha}}+\frac{1}{y^{2\alpha}}\left[\frac{-\theta^2 e^{-\theta/y^{\alpha}}}{(\theta+1)^2}+\frac{1}{\theta+1}\left(2\theta e^{-\theta/y^{\alpha}}-\frac{-\theta^2 e^{-\theta/y^{\alpha}}}{y^{\alpha}}\right)\right]\\
&\quad+\frac{1}{y^{\alpha}}\left[\frac{-\theta e^{-\theta/y^{\alpha}}}{(1+\theta)^2}+\frac{1}{\theta+1}\left(e^{-\theta/y^{\alpha}}-\frac{-\theta^2 e^{-\theta/y^{\alpha}}}{y^{\alpha}}\right)\right].
\end{aligned}$$

The above normal Equations (14) and (15) cannot be solved analytically, therefore, we used NLM (Non-Linear Minimization) function [see, Dannis and Schnabel (1983)] to obtained the solutions. Substituting the OLSEs, we can get the estimators of $S(t\mid\alpha,\theta)$ and $H(t\mid\alpha,\theta)$ as

$$\hat{S}(t\mid\hat{\alpha},\hat{\theta})_{olse}=1-\left(1+\frac{\hat{\theta}_{olse}^2}{2(\hat{\theta}_{olse}+1)}\frac{1}{t^{(2\hat{\alpha}_{olse})}}+\frac{\hat{\theta}_{olse}}{\hat{\theta}_{olse}+1}\frac{1}{t^{\hat{\alpha}_{olse}}}\right)e^{-\hat{\theta}_{olse}/t^{\hat{\alpha}_{olse}}},$$

and

$$\hat{H}(t\mid\hat{\alpha},\hat{\theta})_{olse}=\left(\frac{\frac{\hat{\alpha}_{olse}\hat{\theta}_{olse}^2}{1+\hat{\theta}_{olse}}\frac{1}{t^{(\hat{\alpha}_{olse}+1)}}\left(1+\frac{\hat{\theta}_{olse}}{2t^{2\hat{\alpha}_{olse}}}\right)e^{-\hat{\theta}_{olse}/t^{\hat{\alpha}_{olse}}}}{1-\left(1+\frac{\hat{\theta}_{olse}^2}{2(\hat{\theta}_{olse}+1)}\frac{1}{t^{(2\hat{\alpha}_{olse})}}+\frac{\hat{\theta}_{olse}}{\hat{\theta}_{olse}+1}\frac{1}{t^{\hat{\alpha}_{olse}}}\right)e^{-\hat{\theta}_{olse}/t^{\hat{\alpha}_{olse}}}}\right),$$

respectively. The WLSEs of the parameters α and θ can be obtained by minimising

$$\mathcal{W}(\alpha,\theta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(F(y_{(i:n)}\mid\alpha,\theta)-\frac{i}{n+1}\right)^{2}.$$

These estimators can also be obtained by solving the following normal equations:

$$\begin{aligned}
\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(F(y_{(i:n)}\mid\alpha,\theta)-\frac{i}{n+1}\right)\psi_1(y_{(i:n)}\mid\alpha,\theta) &= 0,\\
\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(F(y_{(i:n)}\mid\alpha,\theta)-\frac{i}{n+1}\right)\psi_2(y_{(i:n)}\mid\alpha,\theta) &= 0.
\end{aligned}$$

Substituting the WLSEs, we can get the estimators of $S(t \mid \alpha, \theta)$ and $H(t \mid \alpha, \theta)$, respectively, as

$$\hat{S}(t \mid \hat{\alpha}, \hat{\theta})_{wlse} = 1 - \left(1 + \frac{\hat{\theta}^2_{wlse}}{2(\hat{\theta}_{wlse}+1)} \frac{1}{t^{(2\hat{\alpha}_{wlse})}} + \frac{\hat{\theta}_{wlse}}{\hat{\theta}_{wlse}+1} \frac{1}{t^{\hat{\alpha}_{wlse}}}\right) e^{-\hat{\theta}_{wlse}/t^{\hat{\alpha}_{wlse}}},$$

and

$$\hat{H}(t \mid \hat{\alpha}, \hat{\theta})_{wlse} = \left(\frac{\frac{\hat{\alpha}_{wlse}\hat{\theta}^2_{wlse}}{1+\hat{\theta}_{wlse}} \frac{1}{t^{(\hat{\alpha}_{wlse}+1)}} \left(1 + \frac{\hat{\theta}_{wlse}}{2t^{2\hat{\alpha}_{wlse}}}\right) e^{-\hat{\theta}_{wlse}/t^{\hat{\alpha}_{wlse}}}}{1 - \left(1 + \frac{\hat{\theta}^2_{wlse}}{2(\hat{\theta}_{wlse}+1)} \frac{1}{t^{(2\hat{\alpha}_{wlse})}} + \frac{\hat{\theta}_{wlse}}{\hat{\theta}_{wlse}+1} \frac{1}{t^{\hat{\alpha}_{wlse}}}\right) e^{-\hat{\theta}_{wlse}/t^{\hat{\alpha}_{wlse}}}} \right).$$

4.3. Cramèr-von-Mises estimator

To motivate our choice of Cramèr-von Mises type minimum distance estimators, MacDonald (1971) provided empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators. Thus, the Cramèr-von Mises estimators $\hat{\alpha}_{cme}$ and $\hat{\theta}_{cme}$ of the parameter α, θ are obtained by minimizing the following equation:

$$\mathcal{C}(\alpha, \theta) = \frac{1}{12n} + \sum_{i=1}^{n} \left(F(y_{(i:n)} \mid \alpha, \theta) - \frac{2i-1}{2n} \right)^2. \tag{16}$$

The minimization of the Equation (16) yields

$$\sum_{i=1}^{n} \left(F(y_{(i:n)} \mid \alpha, \theta) - \frac{2i-1}{2n} \right) \psi_1(y_{(i:n)} \mid \alpha, \theta) = 0,$$

and

$$\sum_{i=1}^{n} \left(F(y_{(i:n)} \mid \alpha, \theta) - \frac{2i-1}{2n} \right) \psi_2(y_{(i:n)} \mid \alpha, \theta) = 0.$$

Hence, substituting the CMEs, we can get the estimators of $S(t \mid \alpha, \theta)$ and $H(t \mid \alpha, \theta)$, respectively, given as

$$\hat{S}(t \mid \hat{\alpha}, \hat{\theta})_{cme} = 1 - \left(1 + \frac{\hat{\theta}^2_{cme}}{2(\hat{\theta}_{cme}+1)} \frac{1}{t^{(2\hat{\alpha}_{cme})}} + \frac{\hat{\theta}_{cme}}{\hat{\theta}_{cme}+1} \frac{1}{t^{\hat{\alpha}_{cme}}}\right) e^{-\hat{\theta}_{cme}/t^{\hat{\alpha}_{cme}}},$$

and

$$\hat{H}(t \mid \hat{\alpha}, \hat{\theta})_{cme} = \left(\frac{\frac{\hat{\alpha}_{cme}\hat{\theta}^2_{cme}}{1+\hat{\theta}_{cme}} \frac{1}{t^{(\hat{\alpha}_{cme}+1)}} \left(1 + \frac{\hat{\theta}_{cme}}{2t^{2\hat{\alpha}_{cme}}}\right) e^{-\hat{\theta}_{cme}/t^{\hat{\alpha}_{cme}}}}{1 - \left(1 + \frac{\hat{\theta}^2_{cme}}{2(\hat{\theta}_{cme}+1)} \frac{1}{t^{(2\hat{\alpha}_{cme})}} + \frac{\hat{\theta}_{cme}}{\hat{\theta}_{cme}+1} \frac{1}{t^{\hat{\alpha}_{cme}}}\right) e^{-\hat{\theta}_{cme}/t^{\hat{\alpha}_{cme}}}} \right).$$

4.4. Maximum product of spacings estimator

This Method was introduced by Cheng and Amin (1979) as an alternative to the method of MLE. The method is briefly described as follows. The CDF of the propose distribution is given in the Equation (4), using the same notations in subsection (4.2), define the uniform spacings of a random sample from GIXGD distribution as:

$$\mathcal{D}_i(\alpha, \theta) \quad = \quad F(y_{(i:n)} \mid \alpha, \theta) - F(y_{(i-1:n)} \mid \alpha, \theta); i = 1,\ 2,\ ...,\ n+1,$$

where, $F(y_{(0:n)} \mid \alpha, \theta) = 0$ and $F(y_{(n+1:n)} \mid \alpha, \theta) = 1 - F(y_{(n:n)} \mid \alpha, \theta)$. Clearly $\sum_{i=1}^{n+1} \mathcal{D}_i(\alpha, \theta) = 1$. The MPSEs $\hat{\alpha}_{mpse}$ and $\hat{\theta}_{mpse}$, of the parameters α and θ are obtained by maximizing with respect to α and θ, the geometric mean of the spacings:

$$\mathcal{G} = \sqrt[n+1]{\left(\prod_{i=1}^{n+1} \mathcal{D}_i(\alpha, \theta) \right)}. \tag{17}$$

Taking logarithm on both sides of Equation (17), we get,

$$\ln \mathcal{G} \quad = \quad \frac{1}{(n+1)} \sum_{i=1}^{n+1} \ln \mathcal{D}_i(\alpha, \theta).$$

The MPSEs are obtained by solving the following non-linear equations:

$$\sum_{i=1}^{n+1} \frac{1}{\mathcal{D}_i(\alpha,\theta)} \left[\psi_1(y_{(i:n)} \mid \alpha,\theta) - \psi_1(y_{(i-1:n)} \mid \alpha,\theta)\right] = 0,$$

and

$$\sum_{i=1}^{n+1} \frac{1}{D_i(\alpha,\theta)} \left[\psi_2(y_{(i:n)} \mid \alpha,\theta) - \psi_2(y_{(i-1:n)} \mid \alpha,\theta)\right] = 0,$$

where, $\psi_1(y_{(i:n)} \mid \alpha, \theta)$ and $\psi_2(y_{(i:n)} \mid \alpha, \theta)$ are the first derivatives of $\mathcal{D}_i(\alpha, \theta)$ with respect to α and θ respectively. Substituting the MPSEs, we can get the estimators of $S(t \mid \alpha, \theta)$ and $H(t \mid \alpha, \theta)$ as

$$\hat{S}(t \mid \hat{\alpha}, \hat{\theta})_{mpse} = 1 - \left(1 + \frac{\hat{\theta}^2_{mpse}}{2(\hat{\theta}_{mpse}+1)} \frac{1}{t^{(2\hat{\alpha}_{mpse})}} + \frac{\hat{\theta}_{mpse}}{\hat{\theta}_{mpse}+1} \frac{1}{t^{\hat{\alpha}_{mpse}}}\right) e^{-\hat{\theta}_{mpse}/t^{\hat{\alpha}_{mpse}}},$$

and

$$\hat{H}(t \mid \hat{\alpha}, \hat{\theta})_{mpse} = \left(\frac{\frac{\hat{\alpha}_{mpse}\hat{\theta}^2_{mpse}}{1+\hat{\theta}_{mpse}} \frac{1}{t^{(\hat{\alpha}_{mpse}+1)}} \left(1 + \frac{\hat{\theta}_{mpse}}{2t^{2\hat{\alpha}_{mpse}}}\right) e^{-\hat{\theta}_{mpse}/t^{\hat{\alpha}_{mpse}}}}{1 - \left(1 + \frac{\hat{\theta}^2_{mpse}}{2(\hat{\theta}_{mpse}+1)} \frac{1}{t^{(2\hat{\alpha}_{mpse})}} + \frac{\hat{\theta}_{mpse}}{\hat{\theta}_{mpse}+1} \frac{1}{t^{\hat{\alpha}_{mpse}}}\right) e^{-\hat{\theta}_{mpse}/t^{\hat{\alpha}_{mpse}}}} \right),$$

respectively.

4.5. Bayesian method of estimation

Here, we have developed the Bayesian estimation procedure to estimate α, θ, SF and HRF. As we know that, in the Bayesian analysis the model parameters (α and θ) are treated as random variable and follows some standard distribution, called as prior distribution. Here, we assume that the parameters α and θ follow independent gamma priors, i.e., $\alpha \sim gamma(a_1, b_1)$ and $\theta \sim gamma(a_2, b_2)$ respectively. Then, the joint prior distribution $\pi(\alpha, \theta)$ turned out to be

$$\pi(\alpha, \theta) \quad = \quad \frac{b_1^{a_1} b_2^{a_2}}{\Gamma(a_1)\Gamma(a_2)} \alpha^{a_1-1} \theta^{a_2-1} e^{-b_1\alpha - b_2\theta}; \quad \theta > 0, \alpha > 0, \tag{18}$$

where, a_1, a_2, b_1 and b_2 are the hyper-parameters and are assumed to be known. Since, the Bayesian procedure utilize the information, supplied by the sample and prior distribution which is combined by using the concept of the Bayes theorem. Hence, by using Equations (11) and (18), the joint posterior distribution is obtained as

$$p(\alpha, \theta | y_i) \quad = \quad K^{-1} \alpha^{n+a_1-1} \theta^{2n+a_2-1} (1+\theta)^{-n} e^{-b_1\alpha - b_2\theta} e^{-\theta\left(\sum\limits_{i=1}^{n} \frac{1}{y_i^{\alpha}}\right)} \prod_{i=1}^{n} \left[\left(\frac{1}{y_i^{\alpha+1}}\right) \left(1 + \frac{\theta}{2y_i^{2\alpha}}\right) \right],$$

where,

$$K^{-1} = \int_0^{\infty}\int_0^{\infty} \left(\frac{\alpha\theta^2}{1+\theta}\right)^n \alpha^{a_1-1} \theta^{a_2-1} e^{-b_1\alpha - \theta b_2} e^{-\theta\left(\sum\limits_{i=1}^{n} \frac{1}{y_i^{\alpha}}\right)} \prod_{i=1}^{n} \left[\left(\frac{1}{y_i^{\alpha+1}}\right) \left(1 + \frac{\theta}{2y_i^{2\alpha}}\right) \right] \partial\alpha\partial\theta.$$

is a normalizing constant. In the Bayes point estimation theory, the selection of appropriate loss function is also a important task . Since, different loss functions are available in literature and used according to the need of the study. Here, we took most popular and widely used symmetric loss function, named as, the Squared error loss function (SELF). Bayes estimates of the parameters under the SELF are the means of their respective marginals posteriors. Therefore, the Bayes estimators of α, θ, $S(t; \alpha, \theta)$ and $H(t; \alpha, \theta)$ are given as follows:

$$\hat{\alpha}_{Bayes} = K^{-1}\int_0^{\infty}\int_0^{\infty}\frac{\alpha^{n+a_1}\theta^{2n+a_2-1}}{(1+\theta)^{-n}}e^{-b_1\alpha-b_2\theta}e^{-\theta\left(\sum_{i=1}^{n}\frac{1}{y_i^{\alpha}}\right)}\prod_{i=1}^{n}\left[\left(\frac{1}{y_i^{\alpha+1}}\right)\left(1+\frac{\theta}{2y_i^{2\alpha}}\right)\right]\partial\alpha\partial\theta, \quad (19)$$

$$\hat{\theta}_{Bayes} = K^{-1}\int_0^{\infty}\int_0^{\infty}\frac{\alpha^{n+a_1-1}\theta^{2n+a_2}}{(1+\theta)^{-n}}e^{-b_1\alpha-b_2\theta}e^{-\theta\left(\sum_{i=1}^{n}\frac{1}{y_i^{\alpha}}\right)}\prod_{i=1}^{n}\left[\left(\frac{1}{y_i^{\alpha+1}}\right)\left(1+\frac{\theta}{2y_i^{2\alpha}}\right)\right]\partial\alpha\partial\theta, \quad (20)$$

$$\hat{S}(t \mid \hat{\alpha},\hat{\theta})_{Bayes} = K^{-1}\int_0^{\infty}\int_0^{\infty}\frac{\alpha^{n+a_1-1}\theta^{2n+a_2-1}}{(1+\theta)^{-n}}e^{-b_1\alpha-b_2\theta}e^{-\theta\left(\sum_{i=1}^{n}\frac{1}{y_i^{\alpha}}\right)}\prod_{i=1}^{n}\left[\left(\frac{1}{t^{\alpha+1}}\right)\left(1+\frac{\theta}{2t^{2\alpha}}\right)\right]$$
$$\times\left[1-\left[1+\frac{\theta^2}{2(\theta+1)}\frac{1}{t^{(2\alpha)}}+\frac{\theta}{\theta+1}\frac{1}{t^{\alpha}}\right]e^{-\theta/t^{\alpha}}\right]\partial\alpha\partial\theta, \quad (21)$$

$$\hat{H}(t \mid \hat{\alpha},\hat{\theta})_{Bayes} = K^{-1}\int_0^{\infty}\int_0^{\infty}\frac{\alpha^{n+a_1-1}\theta^{2n+a_2-1}}{(1+\theta)^{-n}}e^{-b_1\alpha-b_2\theta}e^{-\theta\left(\sum_{i=1}^{n}\frac{1}{y_i^{\alpha}}\right)}\prod_{i=1}^{n}\left[\left(\frac{1}{t^{\alpha+1}}\right)\left(1+\frac{\theta}{2t^{2\alpha}}\right)\right]$$
$$\times\left[\frac{\frac{\alpha\theta^2}{1+\theta}\frac{1}{t^{(\alpha+1)}}\left(1+\frac{\theta}{2t^{2\alpha}}\right)e^{-\theta/t^{\alpha}}}{1-\left[1+\frac{\theta^2}{2(\theta+1)}\frac{1}{t^{(2\alpha)}}+\frac{\theta}{\theta+1}\frac{1}{t^{\alpha}}\right]e^{-\theta/t^{\alpha}}}\right]\partial\alpha\partial\theta. \quad (22)$$

From the above expressions, given in the Equations (19)-(22), it is clearly observed that the analytical solutions are not possible due to involvement of ratio of two integrals. Hence, any Bayes computational technique may be used to obtain the required estimates. Here, we used Markov Chain Monte Carlo (MCMC) method to get the Bayes estimates.

MCMC method

Here, we have considered MCMC method to compute the Bayes estimate of α, θ, $S(t \mid \alpha,\theta)$ and $H(t \mid \alpha,\theta)$ as well as credible interval of the parameters α and θ based on generated posterior samples. For more details about MCMC method, the readers may follow the articles, given by Robert and Smith (1993), Hastings (1970), Upadhyay et al. (2001) and many more. To implement the MCMC algorithm, the full conditional density of α and θ can be written as

$$\phi_1(\alpha|\theta,y) \propto \alpha^{n+a_1-1}e^{-b_1\alpha}e^{-\theta\left(\sum_{i=1}^{n}\frac{1}{y_i^{\alpha}}\right)}\prod_{i=1}^{n}\left[\left(\frac{1}{y^{\alpha+1}}\right)\left(1+\frac{\theta}{2y^{2\alpha}}\right)\right], \quad (23)$$

$$\phi_2(\theta|\alpha,y) \propto \theta^{2n+a_2-1}(1+\theta)^{-n}e^{-b_2\theta}e^{-\theta\left(\sum_{i=1}^{n}\frac{1}{y_i^{\alpha}}\right)}\prod_{i=1}^{n}\left(1+\frac{\theta}{2y^{2\alpha}}\right). \quad (24)$$

Now to generate the samples from above conditional densities, given in Equations (23) and (24), the following steps are used:

1. Set the initial guess value $\alpha^{(0)}$, $\theta^{(0)}$ of α and θ respectively.

2. Begin with $j = 1$,

3. Generate a new sample for α and θ as follows:
   $\alpha^{(j)} \sim \phi_1(x \mid \theta^{(j-1)},\alpha)$
   $\theta^{(j)} \sim \phi_2(x \mid \alpha^{(j)},\theta)$

4. Now again repeat step 2-3 for all $j = 1,2,3,......,R(=10,000)$ times and obtain posterior samples of size R for parameters α and θ.

5. Using the above sequence of samples in step 4, we can obtain the sequence of $S(t \mid \alpha,\theta)^{(j)}$ and $H(t \mid \alpha,\theta)^{(j)}$.
   After obtaining the posterior samples, the Bayes estimates of α, θ, $S(t \mid \alpha,\theta)$ and $H(t \mid \alpha,\theta)$ under SELF are obtained as

$$\hat{\alpha}^{MCMC}=E(\alpha\mid y)\approx\frac{1}{R-R_0}\sum_{j=1}^{R-R_0}\alpha^{(j)},\quad \hat{\theta}^{MCMC}=E(\theta\mid y)\approx\frac{1}{R-R_0}\sum_{j=1}^{R-R_0}\theta^{(j)}$$

$$\hat{S}(t\mid\hat{\alpha},\hat{\theta})^{MCMC}\approx\frac{1}{R-R_0}\sum_{j=1}^{R-R_0}S(t\mid\alpha,\theta)^{(j)},\quad \hat{H}(t\mid\hat{\alpha},\hat{\theta})^{MCMC}\approx\frac{1}{R-R_0}\sum_{j=1}^{R-R_0}H(t\mid\alpha,\theta)^{(j)}$$

respectively, where, R_0 is the burn-in-period of Markov Chain and here R_0 is taken to be 500.

6. Now we can get the $100(1-a)\%$ HPD credible intervals for α and θ by using the algorithm of Chen and Shao (1999).

5. Simulation Study

In this section, we have executed a Monte Carlo simulation study to ascertain the performances of the four classical methods of estimation (MLE, OLSE, WLSE, CME and MPSE) and the Bayesian method of estimation for both informative (Prior-I) and non-informative (Prior-0) prior distributions of the model parameters, SF and HRF of the proposed model. The performance of the estimators is examined in terms of their average MSEs. Besides, we have also constructed ACIs and HPD credible intervals for the parameters, and compared in terms of AWs and corresponding CPs. For this purpose, we have taken different choices of parameters, such as, (α,θ)=[$(0.75,1.5)$, $(1.0,2.0)$, $(1.5,2.0)$, $(2.0,2.0)$, $(2.0,3.0)$] along with the sample sizes $n=10,20,30,50$ and 100 respectively. To evaluate the estimates of SF and HRF, different choices of t are as $t=2,\ 3,\ 4,\ 2,\ 1$. For each design, sample with each of size n are drawn from the original sample and replicated $3,000$ times. For the Bayesian computation, the values of hyper-parameters are chosen such that either the prior variances are very large. Following cases are considered regarding selection of hyper-parameters values: (i) for non-informative prior (Prior-0), we took $a_1=b_1=a_2=b_2=0.000001$, (ii) for informative prior (Prior-I), we took as $(a_1,b_1)=(0.56,0.75),(1,1),(2.25,1.5),(4,2),(4,2)$ and $(a_2,b_2)=(2.25,1.5),(4,2),(4,2),(4,2),(9,3)$ respectively. We generate a chain of 10,000 estimated values of parameters in the model. and we repeat this procedure 1000 times. Each time we took 500 burn-in for each parameter. For an MCMC chain in a model with only two dimensions, it is quite enough to reach a stationary state. That's why there is no need for any other method to check the convergence of the chain. All computations are performed by using programs, written in the open source statistical package R [see, Ihaka and Gentleman (1996)]. For the analysis purpose, we have used several statistical package like VGAM, boa etc..For each set up, we calculated the average estimates and corresponding MSEs of the considered characteristics using MLE, OLSE, WLSE, CME, MPSE and results are reported in Table 1 and 2 respectively. From Table 1, we can see that, in saome cases, for both the parameters (α,θ) MLE provides the leaset MSEs for samll samples but as we increases the sample size MPSE provides the least MSEs for almost all the cases as compared to other classical methods. Also we can see the simmiler trend From Table 2, for the MSEs of survival and hazard rate functions. So, we can say that for all most all the considered cases, MPSE provides the least MSEs as compared to other classical methods (MLE, OLSE, WLSE and CME) of estimation, and efficiency of these methods can be considered as $MPSE<MLE<WLSE<OLSE<CME.$ It is also observed that as the sample sizes increases, the MSEs of all the estimators are decreases, which ensured the consistency of the proposed estimators. Table 3 depicts the Bayes estimates under SELF with Prior-I and Prior-0 respectively. After analyzing the simulation results, we observed that the Bayes estimator with Prior-I have least MSEs for all the parameters set-up as compared to different classical estimators and the Bayes estimator with Prior-0. Table 4, shows the ACIs and HPD credible intervals of parameters for the same variations of $n,\alpha,\theta.$ After analyzing Table 4, we found that the AWs of Bayes credible intervals are smaller than the AWs of ACIs and Bayes procedure with Prior-0, and decreases as the sample size increases.

6. Applications

- **Data Set I:** Data Set I, initially considered by Bjerkedal (1960) which represents the survival times (in days) guinea pigs with different doses of tubercle bacilli. The regimen is common logarithmic of number of bacillary units per $0.5\ ml$. Corresponding to 6.6 regimen, there were 72 observations given below.
  12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58,
  58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83,
  84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175,
  211, 233, 258, 258, 263, 297, 341, 341, 376.

Table 1 True value of (α, θ) and their estimates by different methods of estimation along with the corresponding MSEs.

n	(α, θ)	Estimates & MSEs of α					Estimates & MSEs of θ				
		MLE (MSE)	OLSE (MSE)	WLSE (MSE)	CME (MSE)	MPSE (MSE)	MLE (MSE)	OLSE (MSE)	WLSE (MSE)	CME (MSE)	MPSE (MSE)
10		0.865649	0.751833	0.756890	0.893809	0.873935	1.550106	1.557190	1.564608	1.629190	1.305405
		(0.074445)	(0.078749)	(0.074703)	(0.161331)	(0.081592)	(0.241499)	(0.263179)	(0.389927)	(1.156089)	(0.202691)
20		0.804894	0.750812	0.757294	0.811735	0.798181	1.514127	1.521966	1.521480	1.534066	1.385473
		(0.023882)	(0.028872)	(0.024317)	(0.040647)	(0.023648)	(0.092543)	(0.097470)	(0.092113)	(0.114841)	(0.086681)
30	(0.75, 1.5)	0.786356	0.749776	0.755981	0.788208	0.778980	1.507299	1.509805	1.510503	1.514795	1.418376
		(0.014118)	(0.018365)	(0.015703)	(0.022388)	(0.013720)	(0.057477)	(0.058686)	(0.056130)	(0.064865)	(0.055435)
50		0.771919	0.751096	0.756425	0.773674	0.764367	1.505348	1.507963	1.507473	1.510035	1.450069
		(0.007723)	(0.010377)	(0.008775)	(0.011726)	(0.007393)	(0.032929)	(0.033471)	(0.032626)	(0.035883)	(0.031910)
100		0.762211	0.752028	0.755599	0.763101	0.756337	1.501141	1.502702	1.502260	1.503493	1.472397
		(0.003417)	(0.005073)	(0.004078)	(0.005427)	(0.003282)	(0.015693)	(0.016364)	(0.015839)	(0.016921)	(0.015514)
10		1.160140	1.020408	1.022213	1.217839	1.180978	2.169420	2.112148	2.090227	2.386885	1.815111
		(0.138343)	(0.190704)	(0.154780)	(0.381698)	(0.164887)	(0.624675)	(2.310865)	(0.673626)	(8.536457)	(0.443966)
20		1.070673	0.993109	1.003300	1.073325	1.065521	2.075391	2.037211	2.041877	2.102838	1.884168
		(0.042664)	(0.052398)	(0.045969)	(0.071095)	(0.043382)	0.174976)	(0.182629)	(0.169295)	(0.258918)	(0.147730)
30	(1, 2)	1.045164	0.996517	1.005537	1.048529	1.036518	2.036871	2.013072	2.017122	2.051000	1.904106
		(0.025679)	(0.034648)	(0.030109)	(0.041905)	(0.025312)	(0.091029)	(0.094673)	(0.089297)	(0.114527)	(0.084055)
50		1.027639	1.000884	1.008425	1.031634	1.018181	2.021100	2.008002	2.011867	2.029420	1.936277
		(0.013496)	(0.019573)	(0.016454)	(0.022168)	(0.013019)	(0.053269)	(0.057873)	(0.054348)	(0.064568)	(0.050855)
100		1.013354	0.999760	1.005129	1.014784	1.005815	2.013282	2.007096	2.009786	2.017363	1.967445
		(0.006055)	(0.009153)	(0.007317)	(0.009709)	(0.005867)	(0.024822)	(0.027588)	(0.025756)	(0.029137)	(0.024029)
10		1.744390	1.522428	1.527635	1.805770	1.769699	2.175038	2.095452	2.098770	2.332936	1.822319
		(0.321700)	(0.368604)	(0.325676)	(0.717189)	(0.361475)	(0.647127)	(0.657429)	(0.680819)	(4.657543)	(0.459950)
20		1.617787	1.498580	1.514619	1.620192	1.608679	2.069134	2.025707	2.032029	2.089939	1.879417
		(0.104465)	(0.120072)	(0.106846)	(0.163958)	(0.104581)	(0.182475)	(0.168727)	(0.164785)	(0.236312)	(0.155097)
30	(1.5, 2)	1.568336	1.496788	1.508657	1.575529	1.554558	2.042620	2.021554	2.024277	2.060784	1.909422
		(0.056298)	(0.073771)	(0.062874)	(0.091131)	(0.054915)	(0.099023)	(0.101992)	(0.097142)	(0.123996)	(0.089538)
50		1.540250	1.497417	1.508338	1.543206	1.526443	2.021151	2.008446	2.011549	2.029889	1.936374
		(0.030610)	(0.041579)	(0.035646)	(0.046752)	(0.029433)	(0.054674)	(0.058924)	(0.056131)	(0.065699)	(0.052189)
100		1.520494	1.499633	1.507513	1.522174	1.509516	2.010021	2.002713	2.005850	2.012877	1.964331
		(0.013595)	(0.020202)	(0.016373)	(0.021479)	(0.013158)	(0.024389)	(0.027276)	(0.025372)	(0.028724)	(0.023895)
10		2.323778	2.011712	2.021689	2.368737	2.366541	2.159547	2.077017	2.078881	2.264344	1.806076
		(0.536275)	(0.588663)	(0.541377)	(1.052318)	(0.624744)	(0.589608)	(0.775758)	(0.761893)	(1.860478)	(0.433238)
20		2.149430	2.009054	2.024375	2.172652	2.139909	2.072104	2.031710	2.035665	2.097489	1.881041
		(0.176657)	(0.227336)	(0.197961)	(0.311441)	(0.179418)	(0.176022)	(0.175924)	(0.169060)	(0.254053)	(0.149501)
30	(2, 2)	2.096556	1.992429	2.011558	2.096054	2.078568	2.030884	2.008128	2.012102	2.045820	1.898836
		(0.103057)	(0.133070)	(0.114533)	(0.160765)	(0.101278)	(0.095685)	(0.098818)	(0.093248)	(0.119273)	(0.089637)
50		2.060431	2.001292	2.017367	2.062440	2.042504	2.025553	2.011145	2.015253	2.032681	1.940518
		(0.054170)	(0.075533)	(0.063200)	(0.085546)	(0.052180)	(0.054708)	(0.059880)	(0.055632)	(0.067047)	(0.051406)
100		2.028848	1.996069	2.008060	2.026022	2.013810	2.015037	2.006864	2.010110	2.017112	1.969280
		(0.026259)	(0.038248)	(0.031064)	(0.040379)	(0.025433)	0.026605)	(0.029188)	(0.027311)	(0.030809)	(0.025530)
10		2.317817	1.996565	1.999273	2.350468	2.367326	3.587563	3.247982	3.215915	3.963875	2.992683
		(0.525366)	(0.545379)	(0.457818)	(0.986813)	(0.611145)	(3.407707)	(4.650749	(3.879001)	(20.756270)	(2.051921)
20		2.143164	1.985263	2.005668	2.144900	2.139122	3.206811	3.044063	3.065071	3.258530	2.887652
		(0.179173)	(0.208782)	(0.186348)	(0.276913)	(0.184672)	(0.568596)	(0.647916)	(0.600020)	(1.049474)	(0.432981)
30	(2, 3)	2.095184	1.996925	2.015139	2.100727	2.081210	3.124837	3.035954	3.048917	3.168730	2.896647
		(0.105218)	(0.131920)	(0.115477)	(0.160041)	(0.103690)	(0.287470)	(0.365067)	(0.323123)	(0.493119)	(0.237879)
50		2.054472	1.995856	2.010696	2.056524	2.038802	3.092558	3.039571	3.051871	3.115066	2.941821
		(0.055127)	(0.076417)	0(.064075)	(0.085528)	(0.053846)	(0.157087)	(0.202483)	(0.177130)	(0.243232)	(0.133104)
100		2.030763	2.004043	2.013748	2.033579	2.017009	3.043351	3.017951	3.026830	3.053255	2.959237
		(0.025694)	(0.036882)	(0.030533)	(0.040030)	(0.024754)	(0.066437)	(0.084720)	(0.072575)	(0.093427)	(0.061475)

Table 2 True value of SF, HRF and their estimates by different methods of estimation along with their corresponding MSEs.

n	(α, θ) $S(t \mid \alpha, \theta)$, $H(t \mid \alpha, \theta)$	t	SF MLE (MSE)	SF OLSE (MSE)	SF WLSE (MSE)	SF CME (MSE)	SF MPSE (MSE)	HRF MLE (MSE)	HRF OLSE (MSE)	HRF WLSE (MSE)	HRF CME (MSE)	HRF MPSE (MSE)
10			0.241250	0.273599	0.271860	0.246549	0.196158	0.180063	0.153292	0.154275	0.187089	0.187625
			(0.012528)	(0.013366)	(0.012914)	(0.015095)	(0.012342)	(0.005112)	(0.005070)	(0.004823)	(0.010602)	(0.005864)
20			0.245389	0.262634	0.259977	0.248366	0.222740	0.165038	0.152264	0.153673	0.166680	0.166351
			(0.006137)	(0.006712)	(0.006379)	(0.007095)	(0.006092)	(0.001641)	(0.001857)	(0.001593)	(0.002668)	(0.001686)
30	(0.75, 1.5)	4	0.247951	0.259226	0.257228	0.249655	0.232705	0.160440	0.151901	0.153256	0.160985	0.160696
	$S(t \mid \alpha, \theta)$ =0.253670		(0.004225)	(0.004643)	(0.004409)	(0.004812)	(0.004152)	(0.000991)	(0.001197)	(0.001051)	(0.001481)	(0.000988)
50	$H(t \mid \alpha, \theta)$)=0.201053		0.250315	0.256960	0.255190	0.251121	0.241516	0.156799	0.151917	0.153128	0.157249	0.156255
			(0.002534)	(0.002791)	(0.002631)	(0.002852)	(0.002482)	(0.000540)	(0.000676)	(0.000589)	(0.000770)	(0.000524)
100			0.003417	0.005073	0.004078	0.005427	0.003282	0.015693	0.016364	0.015839	0.016921	0.015514
			(0.001206)	(0.001397)	(0.001285)	(0.001416)	(0.001195)	(0.000237)	(0.000332)	(0.000275)	(0.000357)	(0.000230)
10			0.253858	0.283762	0.281977	0.257026	0.205027	0.241195	0.208514	0.208641	0.255824	0.254234
			(0.013603)	(0.013971)	(0.013499)	(0.016076)	(0.013711)	(0.009755)	(0.012687)	(0.010433)	(0.025570)	(0.012152)
20			0.261615	0.278624	0.276196	0.264713	0.236325	0.218520	0.200206	0.202458	0.219370	0.221494
			(0.006347)	(0.006845)	(0.006538)	(0.007229)	(0.006284)	(0.003029)	(0.003448)	(0.003088)	(0.004782)	(0.003185)
30	(1, 2)	4	0.262486	0.273642	0.271422	0.264174	0.245811	0.212472	0.201034	0.203071	0.213461	0.213263
	$S(t \mid \alpha, \theta)$ =0.267108		(0.004301)	(0.004805)	(0.004545)	(0.005011)	(0.004251)	(0.001837)	(0.002319)	(0.002059)	(0.002856)	(0.001854)
50	$H(t \mid \alpha, \theta)$ =0.201053		0.263728	0.269971	0.268191	0.264151	0.254031	0.208030	0.201773	0.203486	0.209119	0.207550
			(0.002495)	(0.002867)	(0.002678)	(0.002955)	(0.002476)	(0.000956)	(0.001301)	(0.001122)	(0.001490)	(0.000937)
100			0.265992	0.269233	0.267959	0.266320	0.261405	0.204347	0.201156	0.202390	0.204736	0.203479
			(0.001273)	(0.001457)	(0.001344)	(0.001475)	(0.001251)	(0.000435)	(0.000611)	(0.000506)	(0.000653)	(0.000424)
10			0.337726	0.361881	0.361043	0.342292	0.277654	0.673272	0.578056	0.579239	0.701687	0.716102
			(0.017300)	(0.015849)	(0.015799)	(0.019774)	(0.018257)	(0.083577)	(0.089436)	(0.079540)	(0.179072)	(0.100251)
20			0.341920	0.355372	0.353652	0.344726	0.310655	0.615072	0.564197	0.570571	0.616699	0.627252
			(0.008269)	(0.008147)	(0.007986)	(0.009105)	(0.008561)	(0.026711)	(0.028244)	(0.025723)	(0.03945)2	(0.028276)
30	(1.5, 2)	2	0.345847	0.354733	0.353136	0.347553	0.324620	0.592126	0.561474	0.566295	0.595277	0.597220
	$S(t \mid \alpha, \theta)$ =0.349624		(0.005264)	(0.005419)	(0.005226)	(0.005825)	(0.005342)	(0.014461)	(0.017207)	(0.015159)	(0.021581)	(0.014711)
50	$H(t \mid \alpha, \theta)$ =0.560935		0.346691	0.352014	0.350644	0.347635	0.333832	0.579564	0.561252	0.565799	0.580855	0.580467
			(0.003076)	(0.003239)	(0.003119)	(0.003385)	(0.003094)	(0.007831)	(0.009674)	(0.008605)	(0.010992)	(0.007710)
100			0.348009	0.350403	0.349582	0.348183	0.341657	0.570403	0.561631	0.564856	0.571267	0.569260
			(0.001481)	(0.001593)	(0.001524)	(0.001631)	(0.001482)	(0.003499)	(0.004688)	(0.003994)	(0.005012)	(0.003424)
10			0.252893	0.284169	0.282640	0.258094	0.203807	0.967273	0.818846	0.822679	0.990240	1.019945
			(0.013475)	(0.013661)	(0.013272)	(0.015382)	(0.013722)	0.152244)	(0.156071)	(0.144607)	(0.285096)	(0.186062)
20			0.260124	0.275695	0.273621	0.261713	0.234861	0.878619	0.813091	0.819583	0.891511	0.890861
			(0.006429)	(0.007180)	(0.006806)	(0.007660)	(0.006432)	(0.050136)	(0.061121)	(0.054067)	(0.085093)	(0.052803)
30	(2, 2)	2	0.260685	0.272705	0.270387	0.263235	0.244233	0.853551	0.804124	0.812799	0.853649	0.856307
	$S(t \mid \alpha, \theta)$ =0.267108		(0.004325)	(0.004688)	(0.004468)	(0.004894)	(0.004328)	(0.029632)	(0.035479)	(0.031401)	(0.043629)	(0.029858)
50	$H(t \mid \alpha, \theta)$ =0.804215		0.263684	0.270455	0.268623	0.264676	0.253843	0.834316	0.806481	0.813814	0.835671	0.832913
			(0.002557)	(0.002874)	(0.002710)	(0.002953)	(0.002537)	(0.015490)	(0.020177)	(0.017406)	(0.023061)	(0.015191)
100			0.265919	0.269751	0.268366	0.266847	0.261345	0.818378	0.802962	0.808482	0.817226	0.814916
			(0.001303)	(0.001524)	(0.001402)	(0.001540)	(0.001280)	(0.007438)	(0.010230)	(0.008565)	(0.010861)	(0.007245)
10			0.396998	0.415082	0.414676	0.400973	0.327935	0.854846	0.724007	0.723728	0.870353	0.927449
			(0.018148)	(0.015659)	(0.015349)	(0.020205)	(0.020777)	(0.130584)	(0.124541)	(0.105284)	(0.236603)	(0.169208)
20			0.400117	0.409417	0.408681	0.401718	0.363371	0.780081	0.716000	0.723466	0.781718	0.804435
			(0.008644)	(0.008211)	(0.008069)	(0.009384)	(0.009400)	(0.045368)	(0.047280)	(0.043300)	(0.064792)	(0.050288)
30	(2, 3)	2	0.401184	0.408080	0.406751	0.402959	0.376160	0.759046	0.718575	0.725802	0.761061	0.770952
	$S(t \mid \alpha, \theta)$ =0.405851		(0.005625)	(0.005671)	(0.005503)	(0.006203)	(0.005965)	(0.026589)	(0.029766)	(0.027020)	(0.036815)	(0.027709)
50	$H(t \mid \alpha, \theta)$ =0.716065		0.405819	0.410045	0.409057	0.407007	0.390285	0.738728	0.714801	0.720608	0.739450	0.743424
			(0.003373)	(0.003643)	(0.003474)	(0.003832)	(0.003413)	(0.013868)	(0.017309)	(0.015156)	(0.019537)	(0.014033)
100			0.404848	0.406630	0.406044	0.405066	0.397011	0.729326	0.718575	0.722390	0.730625	0.729456
			(0.001582)	(0.001682)	(0.001617)	(0.001729)	(0.001596)	(0.006377)	(0.008261)	(0.007200)	(0.008942)	(0.006267)

Table 3 Bayes estimates of α, θ, SF and HRF along with their corresponding MSEs.

n	(α, θ) $S(t \mid \alpha, \theta)$, $H(t \mid \alpha, \theta)$	t	Prior-I $\hat{\alpha}$ (MSE)	Prior-I $\hat{\theta}$ (MSE)	Prior-0 $\hat{\alpha}$ (MSE)	Prior-0 $\hat{\theta}$ (MSE)	Prior-I $\hat{S}(t \mid \hat{\alpha}, \hat{\theta})$ (MSE)	Prior-I $\hat{H}(t \mid \hat{\alpha}, \hat{\theta})$ (MSE)	Prior-0 $\hat{S}(t \mid \hat{\alpha}, \hat{\theta})$ (MSE)	Prior-0 $\hat{H}(t \mid \hat{\alpha}, \hat{\theta})$ (MSE)
10			0.836438	1.548480	0.849862	1.563076	0.252254	0.173964	0.253642	0.177246
			(0.007472)	(0.002350)	(0.009972)	(0.003979)	(0.007927)	(0.003472)	(0.010216)	(0.004338)
20			0.790134	1.520728	0.793328	1.523435	0.253596	0.161960	0.253904	0.162760
			(0.001611)	(0.000430)	(0.001877)	(0.000549)	(0.004900)	(0.001300)	(0.005634)	(0.001447)
30	(0.75, 1.5)	4	0.775320	1.518059	0.776757	1.519017	0.255001	0.158016	0.255053	0.158377
	$S(t \mid \alpha, \theta)$=0.253670		(0.000641)	(0.000326)	(0.000716)	(0.000362)	(0.003596)	(0.000856)	(0.003941)	(0.000910)
50	$H(t \mid \alpha, \theta)$=0.151365		0.762885	1.500631	0.763708	1.500299	0.253090	0.155024	0.252887	0.155240
			(0.000166)	(3.98E-07)	(0.000188)	(8.92E-08)	(0.002175)	(0.000437)	(0.002302)	(0.000454)
100			0.758543	1.500055	0.759195	1.499434	0.252819	0.153711	0.252510	0.153878
			(7.30E-05)	(2.98E-09)	(8.45E-05)	(3.21E-07)	(0.001122)	(0.000224)	(0.001155)	(0.000228)
10			1.103567	2.090701	1.135338	2.176307	0.265211	0.229232	0.266804	0.236478
			(0.010726)	(0.008227)	(0.018316)	(0.031084)	(0.008225)	(0.005782)	(0.011200)	(0.007992)
20			1.051606	2.055206	1.057356	2.070039	0.268054	0.214983	0.268961	0.216262
			(0.002663)	(0.003048)	(0.003290)	(0.004905)	(0.005138)	(0.002637)	(0.006090)	(0.002966)
30	(1, 2)	4	1.037681	2.025645	1.040414	2.031050	0.265234	0.211390	0.265472	0.212024
	$S(t \mid \alpha, \theta)$=0.267108		(0.001420)	(0.000658)	(0.001633)	(0.000964)	(0.003427)	(0.001585)	(0.003817)	(0.001700)
50	$H(t \mid \alpha, \theta)$=0.201053		1.021132	2.025541	1.022195	2.027846	0.267857	0.206662	0.267979	0.206900
			(0.000447)	(0.000652)	(0.000493)	(0.000775)	(0.002229)	(0.000892)	(0.002393)	(0.000930)
100			1.009489	2.016018	1.009900	2.016403	0.268296	0.203511	0.268251	0.203610
			(9.00E-05)	(0.000257)	(9.80E-05)	(0.000269)	(0.001193)	(0.000416)	(0.001241)	(0.000424)
10			1.678203	2.094757	1.749974	2.178859	0.338574	0.650839	0.337870	0.683491
			(0.031756)	(0.008979)	(0.062487)	(0.031990)	(0.009156)	(0.044978)	(0.014149)	(0.076028)
20			1.590603	2.037558	1.606196	2.049330	0.342634	0.607751	0.342129	0.615263
			(0.008209)	0.001411)	(0.011278)	(0.002433)	(0.005901)	(0.021320)	(0.007402)	(0.026739)
30	(1.5, 2)	2	1.551812	2.027531	1.557937	2.031915	0.346772	0.587962	0.346554	0.590939
	$S(t \mid \alpha, \theta)$=0.349624		(0.002684)	(0.000758)	(0.003357)	(0.001019)	(0.004270)	(0.012984)	(0.004963)	(0.014861)
50	$H(t \mid \alpha, \theta)$=0.560935		1.538737	2.008901	1.541145	2.010043	0.345216	0.581313	0.345041	0.582506
			(0.001501)	(7.92E-05)	(0.001693)	(0.000101)	(0.002785)	(0.007220)	(0.003069)	(0.007798)
100			1.510926	2.009991	1.511683	2.010111	0.349879	0.566683	0.349776	0.567064
			(0.000119)	(9.98E-05)	(0.000136)	(0.000102)	(0.001267)	(0.003206)	(0.001333)	(0.003323)
10			2.148241	2.10603	2.251504	2.178121	0.271203	0.882769	0.271097	0.935397
			(0.021975)	(0.011242)	(0.063254)	(0.031727)	(0.007058)	(0.059652)	(0.011370)	(0.128977)
20			2.099377	2.06299	2.125035	2.077623	0.267993	0.856212	0.267962	0.869400
			(0.009876)	(0.003968)	(0.015634)	(0.006025)	(0.004346)	(0.032569)	(0.005636)	(0.045125)
30	(2, 2)	2	2.070756	2.036979	2.082374	2.041919	0.267151	0.841917	0.267028	0.848016
	$S(t \mid \alpha, \theta)$=0.267108		(0.005006)	(0.001367)	(0.006785)	(0.001757)	(0.003514)	(0.023567)	(0.004193)	(0.028814)
50	$H(t \mid \alpha, \theta)$=0.804215		2.033008	2.016431	2.036939	2.017395	0.267549	0.822600	0.267398	0.824736
			(0.00109)	(0.00027)	(0.001364)	(0.000303)	(0.002033)	(0.012568)	(0.002267)	(0.014073)
100			2.018860	2.013384	2.020018	2.013409	0.267857	0.814201	0.267771	0.814847
			(0.000356)	(0.000179)	(0.000401)	(0.00018)	(0.001158)	(0.006455)	(0.001227)	(0.006830)
10			2.121600	3.127936	2.284998	3.577317	0.399974	0.785537	0.399270	0.853558
			(0.014787)	(0.016368)	(0.081224)	(0.333295)	(0.008465)	(0.055940)	(0.015428)	(0.138651)
20			2.088915	3.110418	2.131873	3.217139	0.401503	0.763041	0.401627	0.780170
			(0.007906)	(0.012192)	(0.017391)	(0.047149)	(0.005517)	(0.029570)	(0.007544)	(0.042295)
30	(2, 3)	2	2.079403	3.076428	2.100260	3.124297	0.398864	0.757933	0.398726	0.766207
	$S(t \mid \alpha, \theta)$=0.405851		(0.006305)	(0.005841)	(0.010052)	(0.015450)	(0.004248)	(0.021818)	(0.005282)	(0.026910)
50	$H(t \mid \alpha, \theta)$=0.716065		2.057193	3.056766	2.065180	3.075160	0.400886	0.745432	0.400918	0.748525
			(0.003271)	(0.003222)	(0.004248)	(0.005649)	(0.002838)	(0.013638)	(0.003245)	(0.015307)
100			2.023651	3.024940	2.025332	3.029397	0.403775	0.728462	0.403851	0.729057
			(0.000559)	(0.000622)	(0.000642)	(0.000864)	(0.001437)	(0.005810)	(0.001533)	(0.006086)

Table 4 AWs and CPs of ACI and HPD credible interval of parameters α and θ using MCMC method.

n	(α, θ)	ACI				Prior-I				Prior-0			
		α		θ		α		θ		α		θ	
		AW	CP	AW	CP	AW	CP	AW	CP	AW	CP	AW	CP
10	(0.75, 1.5)	0.804394	0.943	1.614525	0.923	0.751094	0.962	1.435246	0.957	0.779379	0.945	1.551978	0.932
20		0.522340	0.947	1.094618	0.927	0.506780	0.954	1.039685	0.949	0.514308	0.945	1.080489	0.941
30		0.412411	0.960	0.895072	0.947	0.406216	0.948	0.860792	0.951	0.409049	0.945	0.881122	0.944
50		0.316678	0.952	0.691065	0.950	0.309961	0.959	0.668726	0.953	0.311080	0.952	0.678699	0.945
100		0.219606	0.946	0.486787	0.942	0.219251	0.950	0.478725	0.949	0.218434	0.952	0.480100	0.952
10	(1, 2)	1.124764	0.940	2.225816	0.945	1.009759	0.963	1.804272	0.965	1.070111	0.947	2.132914	0.927
20		0.716450	0.962	1.417274	0.945	0.686603	0.950	1.314480	0.963	0.700970	0.946	1.403010	0.947
30		0.566039	0.948	1.142941	0.945	0.554699	0.954	1.075075	0.954	0.562825	0.953	1.120718	0.944
50		0.429064	0.948	0.877008	0.947	0.423616	0.944	0.845378	0.948	0.426201	0.941	0.867288	0.938
100		0.300547	0.955	0.614094	0.940	0.297137	0.942	0.602577	0.957	0.296750	0.943	0.609093	0.954
10	(1.5, 2)	1.669196	0.943	2.272327	0.955	1.484300	0.973	1.802143	0.977	1.644833	0.948	2.134324	0.938
20		1.074175	0.949	1.437838	0.948	1.022275	0.962	1.302062	0.957	1.066392	0.950	1.391390	0.948
30		0.854912	0.958	1.137295	0.952	0.821761	0.943	1.077847	0.954	0.841250	0.939	1.123494	0.942
50		0.651178	0.943	0.874738	0.934	0.633667	0.952	0.839405	0.950	0.641966	0.951	0.860520	0.941
100		0.450518	0.950	0.613328	0.946	0.442153	0.951	0.599786	0.959	0.444942	0.943	0.607233	0.958
10	(2, 2)	2.247930	0.954	2.247504	0.946	1.828998	0.975	1.804247	0.971	2.125924	0.945	2.110935	0.945
20		1.438726	0.940	1.434001	0.956	1.318418	0.972	1.315905	0.949	1.409659	0.955	1.406959	0.929
30		1.136120	0.950	1.144725	0.940	1.076768	0.961	1.080403	0.956	1.121931	0.954	1.127468	0.944
50		0.865551	0.944	0.880451	0.939	0.828442	0.961	0.841363	0.957	0.848441	0.957	0.863411	0.951
100		0.601326	0.948	0.613938	0.957	0.589317	0.959	0.601265	0.952	0.595938	0.954	0.607930	0.951
10	(2, 3)	2.229278	0.953	4.428954	0.969	1.795622	0.976	2.502107	0.985	2.179635	0.938	3.994696	0.959
20		1.443669	0.944	2.418672	0.952	1.320840	0.970	1.956297	0.973	1.432360	0.956	2.356517	0.947
30		1.153372	0.948	1.878874	0.957	1.091003	0.960	1.639249	0.954	1.146947	0.947	1.842227	0.944
50		0.876436	0.945	1.405472	0.965	0.846812	0.949	1.300576	0.953	0.871061	0.945	1.389403	0.936
100		0.607776	0.959	0.969953	0.945	0.596412	0.952	0.931906	0.947	0.603663	0.950	0.959899	0.939

- **Data Set II:** Second data shows survival time of 44 patients suffering from head and neck cancer disease and were treated using combined radiotherapy and chemotherapy [See, Efron (1988)]. Values of data set 2 are given below-
$12.20,\ 23.56,\ 23.74,\ 25.87,\ 31.98,\ 37,\ 41.35,\ 47.38,\ 55.46,\ 58.36,\ 63.47,\ 68.46,\ 78.26,$
$74.47,\ 81.43,\ 84,\ 92,\ 94,\ 110,\ 112,\ 119,\ 127,\ 130,\ 133,\ 140,\ 146,\ 155,\ 159,\ 173,\ 179,$
$194,\ 195,\ 209,\ 249,\ 281,\ 319,\ 339,\ 432,\ 469,\ 519,\ 633,\ 725,\ 817,\ 1776.$

From the summary of the both data sets, given in Table 5, it is observed that the Coefficient of skewness (CS) and Coefficient of kurtosis (CK) for both data set are positive. Hence we may conclude that the considered data sets are compatible for the proposed model even though data set contains outliers, see in Figure 10. At first we have checked whether the considered data set is actually comes from GIXGD or not by goodness-of-fit test and compared the fit with the following lifetime distributions: GIXGD, ILD,IXGD, IWD, IED, Generalized exponential distribution (GED) [see Gupta and Kundu (2001)], Gamma distribution (GD) [see Thom et al. (1958)]. This procedure is based on the Kolmogorov-Smirnov (K-S) statistic and it compares an empirical and a theoretical model by computing the maximum absolute difference between the empirical and theoretical CDFs. Note that, K-S statistic to be used only to verify the goodness-of-fit and not as a discrimination criteria. Therefore, we consider four discrimination criteria based on the log-likelihood function evaluated at the maximum likelihood estimates of the parameters. The criteria are: Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC), Bayesian information criterion (BIC). The model with least AIC, CAIC, HQIC, BIC and K-S is treated as best model. The obtained measures are reported in Table 6 which indicates that the GIXGD is best choices among one parameter as well as two parameters family of distributions. Further, density with histogram, empirical CDF and normal probability plot (P-P) plot are also displayed in Figures 8-9 for the considered data sets, these plots reveals the same results as we seen in Table 6. Hence, GIXGD might be chosen as an alternative model. The classical and Bayes estimates of the parameters, SF and HRF for specified value of $t(=\ 54,\ 70,\ 99,\ 112)$ are obtained and reported in Tables 7-8 respectively. We have provided the trace plots in Figure 11 for the parameters only for data set I due to constrant of length of the paper. Figure 11 shows that the parameters are convergence enough through MCMC method.

Table 5 Descriptive statistics of the considered data sets.

Data set	Minimum	1st Quartile	Median	Mean	3rd Quartile	Maximum	CS	CK
I	12	54.75	70	99.82	112.80	376	1.796245	5.614438
II	12.20	67.21	128.50	223.50	219	1776	3.38382	16.5596

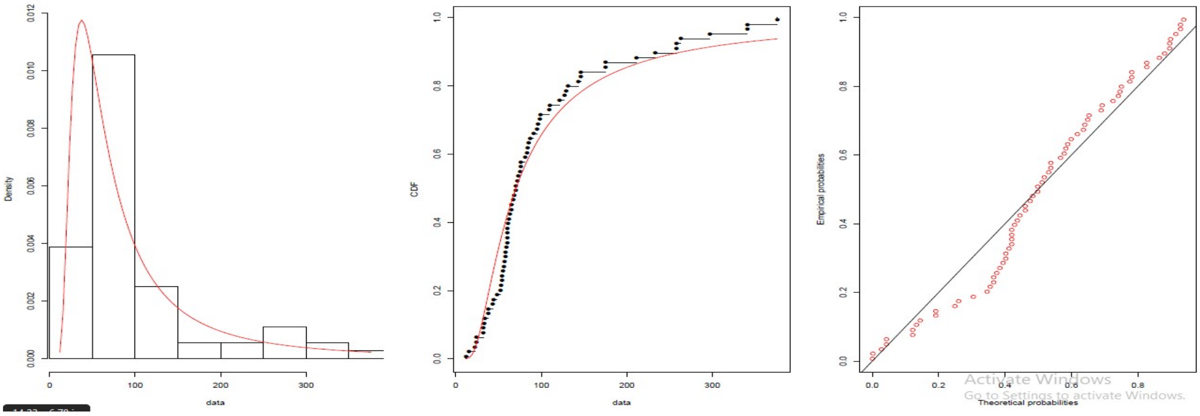

Figure 8 Histogram-density, CDFs and P-P plot of data set I.

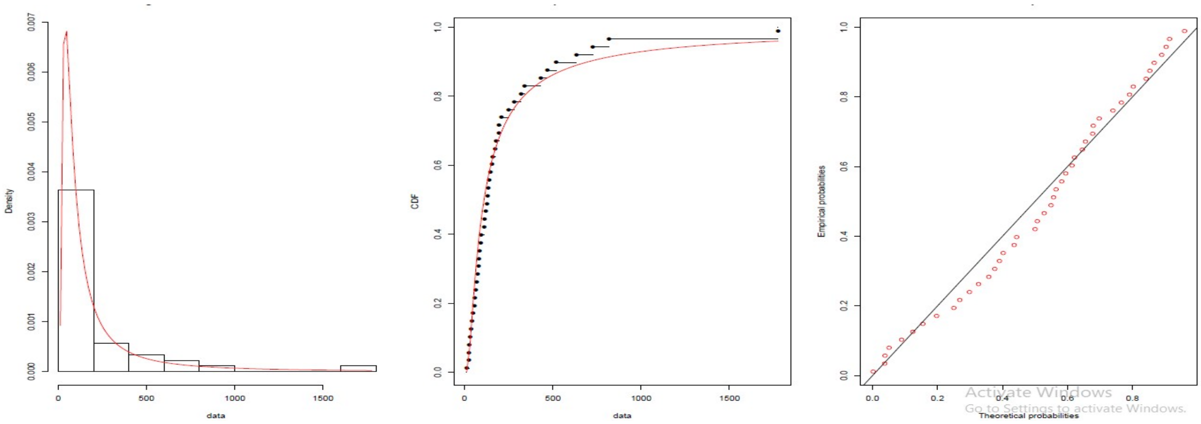

Figure 9 Histogram-density, CDFs and P-P plot of data set II.

Table 6 The model fitting summary for the considered data sets I and II.

Data Set	Model	MLEs	-LogL	AIC	BIC	HQIC	CAIC	K-S
I	GIXGD	[1.416598, 287.9991]	395.5712	795.1423	799.6957	796.955	800.6957	0.137367
	ILD	61.06575	402.6685	807.3371	809.6137	808.2434	810.6137	0.184594
	IXGD	61.844	402.8761	807.7522	810.0289	808.6585	811.0289	0.187181
	IWD	[1.414755, 283.831]	420.1391	844.2782	848.8316	846.0909	849.8316	0.138098
	IED	0.01663913	402.6718	807.3437	809.6203	808.2500	810.6203	0.184658
Data Set	Model	MLEs	-LogL	AIC	BIC	HQIC	CAIC	K-S
II	GIXGD	[1.019225, 84.66623]	279.4906	562.9863	566.5547	564.3096	567.5547	0.08167396
	IW	[1.013332, 80.76181]	280.142	564.2841	567.8525	565.6074	568.8525	0.08318193
	GED	[1.071444, 213.3867]	281.9558	567.9116	571.48	569.235	572.48	0.1497917
	GD	[1.023544, 0.004579491]	282.0028	568.0055	571.5739	569.3288	572.5739	0.9999908
	Weibull	[0.9409097, 216.1249]	281.8427	567.6854	571.2538	569.0088	572.2538	0.130701

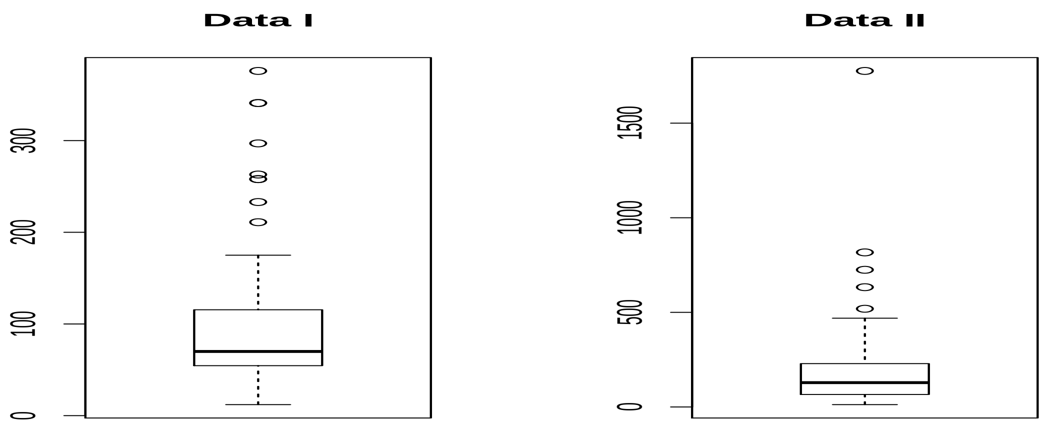

Figure 10 Box plot of considered data sets I and II.

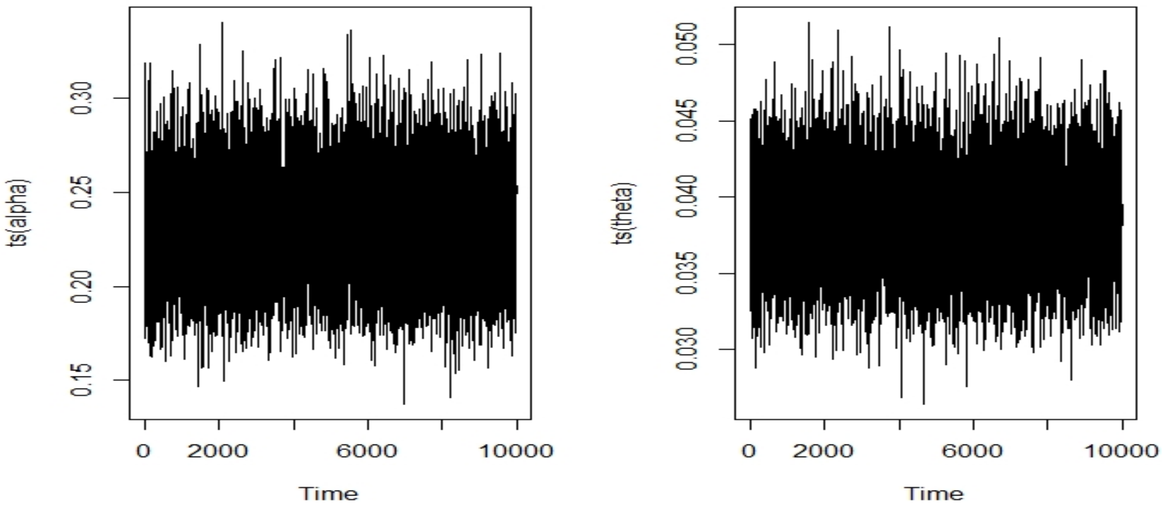

Figure 11 Trace plot of considered data sets I.

Table 7 Classical estimates of (α, θ), $S(t;\alpha,\theta)$, $H(t;\alpha,\theta)$ using different methods of estimation.

Data Set		Estimates of the parameters						$\hat{S}(t;\hat{\alpha},\hat{\theta})$ (Ist row) and $\hat{H}(t;\hat{\alpha},\hat{\theta})$ (IInd row)				
		MLE	OLSE	WLSE	CME	MPSE	t	MLE	OLSE	WLSE	CME	MPSE
	$\hat{\alpha}$	1.623886	0.918475	0.859473	1.947916	1.372743	54	0.627236	0.078093	0.069858	0.707156	0.620275
	$\hat{\theta}$	643.3561	4.004110	3.009880	2910.965	232.7604		0.017622	0.016165	0.015168	0.018342	0.015038
	$\hat{\alpha}$	1.623886	0.918475	0.859473	1.947916	1.372743	70	0.476717	0.062184	0.056422	0.523296	0.492638
	$\hat{\theta}$	643.3561	4.004110	3.009880	2910.965	232.7604		0.016483	0.012605	0.011813	0.018778	0.013684
I	$\hat{\alpha}$	1.623886	0.918475	0.859473	1.947916	1.372743	99	0.308540	0.045719	0.042298	0.314196	0.344168
	$\hat{\theta}$	643.3561	4.004110	3.009880	2910.965	232.7604		0.013559	0.009010	0.008435	0.016197	0.011136
	$\hat{\alpha}$	1.623886	0.918475	0.859473	1.947916	1.372743	112	0.260647	0.040947	0.038151	0.256654	0.299651
	$\hat{\theta}$	643.3561	4.004110	3.009880	2910.965	232.7604		0.012417	0.007989	0.007478	0.014939	0.010195
	$\hat{\alpha}$	1.061182	0.957568	0.616333	1.151660	0.989437	67	0.677399	0.022223	0.152573	0.736633	0.657048
	$\hat{\theta}$	99.58223	1.925138	3.021227	170.7906	70.09361		0.008486	0.014051	0.008245	0.008168	0.008188
II	$\hat{\alpha}$	1.061182	0.957568	0.616333	1.151660	0.989437	128	0.434914	0.022224	0.105983	0.469724	0.432215
	$\hat{\theta}$	99.58223	1.925138	3.021227	170.7906	70.09361		0.006131	0.006405	0.004470	0.006431	0.005725
	$\hat{\alpha}$	1.061182	0.957568	0.616333	1.151660	0.989437	223	0.271730	0.022222	0.076865	0.284667	0.279169
	$\hat{\theta}$	99.58223	1.925138	3.021227	170.7906	70.09361		0.004038	0.003308	0.002621	0.004343	0.003742
	$\hat{\alpha}$	1.061182	0.957568	0.616333	1.151660	0.989437	219	0.276187	0.022222	0.077683	0.289692	0.283410
	$\hat{\theta}$	99.58223	1.925138	3.021227	170.7906	70.09361		0.004098	0.003379	0.002667	0.004406	0.003798

Table 8 Bayes estimates of parameters (α, θ), SF and HRF for real data sets.

Data Set	Bayes Estimates		t	Bayes Estimates	
	$\hat{\alpha}$	$\hat{\theta}$		$\hat{S}(t;\hat{\alpha},\hat{\theta})$	$\hat{H}(t;\hat{\alpha},\hat{\theta})$
I	1.34436	221.0235	t=54	0.642714	0.014210
			t=70	0.516444	0.013044
			t=99	0.366328	0.010705
			t=112	0.320606	0.009823
II	1.012478	88.609970	t=67	0.675770	0.008071
			t=128	0.444028	0.005847
			t=223	0.289459	0.003807
			t=219	0.293801	0.003861

Table 9 Widths of ACI and HPD credible interval of the parameters α and θ using MCMC method for the considered data sets I and II.

Data Set	Width of ACI		Width of HPD credible interval	
	α	θ	α	θ
I	0.456567	495.7005	0.374586	431.1524
II	0.431425	146.6619	0.382655	129.4561

7. Concluding Remarks

In this article, we have proposed a new positively skewed probability distribution, namely, GIXGD by considering the power transformation of IXGD, introduced by Yadav et al. (2018). Several distributional properties viz., moments, conditional moments, quantile function, Bonferroni and Lorentz curve, entropy etc., have been derived. Also shape of HRF and SF have been studied through graphical representation. Next, the different classical and the Bayesian estimation procedures for the parameters and SF, HRF are considered. Further, interval estimates (ACIs and HPD) are also constructed based on MLE and posterior samples respectively. The Monte Carlo simulation study has been performed to compare the performance of the classical and the Bayes estimators for the different variations of n, α, θ in terms of average mean squared error. Finally, two real data sets have been analyzed for illustration purposes of the proposed study. Estimation of the parameters and the reliability characteristics may be further studied under different types of censoring scheme in future.

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