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The Marshall-Olkin-Gompertz-Exponentiated Half Logistic-G Family of Distributions: Model, Properties and Applications

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Abstract

In this work, a new family of distributions is introduced by combining Marshall-Olkin and Gompertz-exponentiated half logistic-G distributions. The new distribution is an infinite linear combination of the exponentiated-G family of distributions. Some of key properties including order statistics, ordinary moments, quantiles and moment generating function are derived. To estimate the model parameters, the maximum likelihood method is also applied by means of Monte Carlo simulation study. Applications of the proposed family is demonstrated in many fields such as survival analysis and univariate data fitting. Empirical results show that the proposed models provide better fits than other well-known classes of distributions in different application fields.

Keywords: Marshall-Olkin-G, Gompertz-G distribution, exponentiated half logistic-G, Monte Carlo simulation, order statistics, Rnyi entropy, stochastic ordering.

1. Introduction

The critical limitations and problems of classic statistical distributions to model real data sets lead statistical researchers to introduce new flexible distributions. New generalized families of continuous distributions are often developed through classic distributions and provide the necessary flexibility in data modeling.

The Marshall-Olkin-G (MO-G) family of distributions has played crucial role in statistical modelling of data in several areas including reliability, engineering and biological studies. Marshall and Olkin (1997) introduced an important method of adding an extra shape parameter to a given baseline model thus defining an extended distribution. The Marshall and Olkin transformation provides variety of behaviors with respect to the baseline distribution. Recently the model has been used to model data that has monotonic and non-monotonic hazard rate functions. The Marshal Olkin extended family of distributions has the cumulative density function (cdf) and probability density function (pdf) given by

$$F_{MO-G}(x;\delta,\Psi)=\frac{G(x;\Psi)}{1-\overline{\delta G}(x;\Psi)}, \tag{1}$$

and

$$f_{MO-G}(x;\delta,\Psi)=\frac{\delta g(x;\Psi)}{[1-\overline{\delta F}(x;\Psi)]^2}, \tag{2}$$

respectively, where δ is the tilt parameter and $G(x;\Psi)$ is the baseline distribution.

The readers are referred to Marshall-Olkin extended Weibull by Santos Neto et al. (2014). Marshall-Olkin-Gompertz-Weibull by Chipepa and Oluyede (2021), Marshall-Olkin odd exponentiated half logistic-log-logistic by Oluyede and Chipepa (2021), Marshall-Olkin extended Pareto by Ghitany and Marshall-Olki (2005), MarshallOlkin extended Lomax by Ghitany et al. (2007), MarshallOlkin gamma by Ristic et al. (2007), and Marshall-Olkin generalized exponential by Ristic and Kundu (2015) for more details.

Alizadeh et al. (2017) proposed the Gompertz-G generated family of distributions with cdf and pdf given by

$$F(x;\lambda,\gamma,\xi)=1-\exp\left[\frac{\lambda}{\gamma}\left(1-\left(1-G(x;\xi)\right)^{-\gamma}\right)\right] \tag{3}$$

and

$$f(x;\lambda,\gamma,\xi)=\lambda g(x;\xi)\Big(1-G(x;\xi)\Big)^{-\gamma-1}\exp\left[\frac{\lambda}{\gamma}\left(1-\left(1-G(x;\xi)\right)^{-\gamma}\right)\right], \tag{4}$$

respectively, for $\lambda,\gamma>0$ and ξ a parameter vector from the baseline distribution. In this note, we will take $\lambda=1$.

Other well known Gompertz distribution type generalizations in the literature are the generalized Gompertz distribution by El-Gohary et al. (2013), beta-Gompertz by Jafari et al. (2014), Gompertz power series distribution by Jafari and Tahmasebi (2016) and a power Gompertz distribution by Ieren et al. (2019).

Introducing new distributions requires us to have good motivations. The newly proposed family of distributions reduces the modeling error of the several interesting data sets such as left and right-tailed data sets. The MO-Gom-EHL-G family of distributions gives the desirable properties and flexibility to model these kind of data sets. Moreover, the distribution exhibit monotonic and non-monotonic hazard rate function.

In this paper, we develop the new family of distributions, called the Marshall-Olkin-Gompertz-Exponentiated Half Logistic-G (MO-Gom-EHL-G) family of distributions. In Section 2, we present the new generalized family of distributions and its linear representation. Statistical properties of the MO-Gom-EHL-G family of distributions are derived in Section 3. In Section 4, we present some special cases. Maximum likelihood estimates of the MO-Gom-EHL-G family of distributions are presented in Section 5. Monte Carlo simulation study is conducted to examine the consistency of the maximum likelihood estimators for each parameter in Section 6. Applications of the special case of Marshall-Olkin-Gompertz-Exponentiated Half Logistic-log-logistic (MO-Gom-EHL-LLoG) to real data sets are given in Section 7, followed by concluding remarks.

2. The Model and Linear Representation

We employ Marshall-Olkin transformation by Marshall and Olkin (1997) given in equation (1) to propose a new family of distributions, namely, Marshall-Olkin-Gompertz-Exponentiated Half Logistic-G (MO-Gom-EHL-G) family of distributions. Therefore, the cdf, pdf and hazard rate function (hrf) of the MO-Gom-EHL-G family of distributions are given by

$$F_{MO-Gom-EHL-G}(x)=\frac{1-\left[H(x;\alpha,\gamma,\Psi)\right]}{1-\bar{\delta}\left[H(x;\alpha,\gamma,\Psi)\right]}, \tag{5}$$

$$
\begin{aligned}
f_{MO-Gom-EHL-G}(x) &= \frac{2\alpha\delta g(x;\Psi)G^{\alpha-1}(x;\Psi)}{(1+\bar{G}(x;\Psi))^{\alpha+1}}\big[H(x;\alpha,\gamma,\Psi)\big] \\
&\times \left\{1-\bar{\delta}\big[H(x;\alpha,\gamma,\Psi)\big]\right\}^{-2}\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma-1} \qquad (6)
\end{aligned}
$$

and

$$
\begin{aligned}
h_{MO-Gom-EHL-G}(x) &= \frac{2\alpha\delta g(x;\Psi)G^{\alpha-1}(x;\Psi)}{(1+\bar{G}(x;\Psi))^{\alpha+1}}\big[H(x;\alpha,\gamma,\Psi)\big] \\
&\times \frac{1-\big[H(x;\alpha,\gamma,\Psi)\big]}{1-\bar{\delta}\big[H(x;\alpha,\gamma,\Psi)\big]}\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma-1},
\end{aligned}
$$

respectively, where $H(x;\alpha,\gamma,\Psi)=\exp\left\{\frac{1}{\gamma}\left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)\right\}$ for $\alpha,\,\delta,\,\gamma>0,\,\bar{\delta}=1-\delta$ and Ψ is a vector of parameters from the baseline distribution function $G(.)$.

2.1. Linear representation

In this section, a useful linear representation for the MO-Gom-EHL-G pdf is derived. The pdf in equation (6) can be expressed as

$$
\begin{aligned}
f(x) &= \frac{2\alpha\delta g(x;\Psi)G^{\alpha-1}(x;\Psi)}{(1+\bar{G}(x;\Psi))^{\alpha+1}}\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma-1} \\
&\times \exp\left\{\frac{1}{\gamma}\left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)\right\} \\
&\times \underbrace{\left\{1-\bar{\delta}\left[\exp\left\{\frac{1}{\gamma}\left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)\right\}\right]\right\}^{-2}}_{B}.
\end{aligned}
$$

Using the series expansion $(1-z)^{-k}=\sum_{j=0}^{\infty}\frac{\Gamma(k+j)}{\Gamma(k)j!}z^j$ for $|z|<1$,

$$
B=\sum_{j=0}^{\infty}\frac{\Gamma(2+j)}{\Gamma(2)}\bar{\delta}^j\left[\exp\left\{\frac{1}{\gamma}\left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)\right\}\right]^j,
$$

so we can write the pdf of the MO-Gom-EHL-G family of distributions as:

$$
\begin{aligned}
f(x) &= \frac{2\alpha\delta g(x;\Psi)G^{\alpha-1}(x;\Psi)}{(1+\bar{G}(x;\Psi))^{\alpha+1}}\sum_{j=0}^{\infty}\frac{\Gamma(2+j)}{\Gamma(2)}\bar{\delta}^j\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma-1} \\
&\times \underbrace{\exp\left\{\frac{j+1}{\gamma}\left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)\right\}}_{C}.
\end{aligned}
$$

Applying the generalized binomial series expansion on C, we obtain

$$
C = \sum_{q=0}^{\infty}\frac{(j+1)^q}{\gamma^q q!}\left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)^q.
$$

Thus, we get

$$
\begin{aligned}
f(x) &= \frac{2\alpha\delta g(x;\Psi)G^{\alpha-1}(x;\Psi)}{(1+\bar{G}(x;\Psi))^{\alpha+1}}\sum_{j,q=0}^{\infty}\frac{\Gamma(2+j)\bar{\delta}^j(j+1)^q}{\Gamma(2)\gamma^q q!} \\
&\times \left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma-1}\left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)^q.
\end{aligned}
$$

Also, by applying the following generalised binomial series expansions:

$$\left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)^{q}=\sum_{w=0}^{\infty}(-1)^{w}\binom{q}{w}\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma w},$$

$$\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma(w+1)-1}=\sum_{m=0}^{\infty}(-1)^{m}\binom{-\gamma(w+1)-1}{m}\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha m},$$

$$\left(1+\bar{G}(x;\Psi)\right)^{-\alpha(m+1)-1}=\sum_{n=0}^{\infty}(-1)^{n}\binom{-\alpha(m+1)-1}{n}\bar{G}^{n}(x;\Psi),$$

$$(1-G(x;\Psi))^{n}=\sum_{s=0}^{\infty}(-1)^{s}\binom{n}{s}G^{s}(x;\Psi),$$

$$\left(1-\bar{G}(x;\Psi)\right)^{-\alpha(m+1)+s-2}=\sum_{t=0}^{\infty}(-1)^{t}\binom{-\alpha(m+1)+s-2}{t}\bar{G}^{t}(x;\Psi)$$

and

$$(\bar{G}(x;\Psi))^{t}=\sum_{v=0}^{\infty}(-1)^{v}\binom{t}{v}G^{v}(x;\Psi),$$

we can write

$$\begin{aligned}
f(x) &= 2\alpha\delta\bar{\delta}\sum_{j,q,w,m,n,s,t,v=0}^{\infty}\frac{(-1)^{w+m+n+s+t+v}\Gamma(2+j)(j+1)^{q}}{\Gamma(2)j!\gamma^{q}q!(v+1)}\binom{-\alpha(m+1)-1}{n}\\
&\times \binom{q}{w}\binom{n}{s}\binom{t}{v}\binom{-\gamma(w+1)-1}{m}\binom{-\alpha(m+1)+s-2}{t}g(x;\Psi)G^{v}(x;\Psi)\\
&= \sum_{v=0}^{\infty}c_{v+1}h_{v+1}(x;\Psi),
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
c_{v+1} &= 2\alpha\delta\bar{\delta}\sum_{j,q,w,m,n,s,t=0}^{\infty}\frac{(-1)^{w+m+n+s+t+v}\Gamma(2+j)(j+1)^{q}}{\Gamma(2)j!\gamma^{q}q!v+1}\binom{t}{v}\binom{q}{w}\binom{n}{s}\\
&\times \binom{-\gamma(w+1)-1}{m}\binom{-\alpha(m+1)-1}{n}\binom{-\alpha(m+1)+s-2}{t}
\end{aligned} \tag{8}$$

and $h_{v+1}(x;\Psi)=(v+1)g(x;\Psi)(G(x;\Psi))^{v}$ is the exp-G density with power parameter $(v+1)$. Thus, the MO-Gom-EHL-G family of distributions can be written as an infinite linear combination of exponentiated-G densities. The structural properties of the MO-Gom-EHL-G family of distributions follow directly from those of the exponentiated-G distribution.

3. Some Statistical Properties

We present some statistical properties of the MO-Gom-EHL-G family of distributions, which include distribution of the order statistics, entropy, moments, incomplete moments, generating function and stochastic ordering.

3.1. Order statistics

Let $X_1, X_2, ..., X_n$ be independent and identically distributed MO-Gom-EHL-G random variables. The pdf of the i^{th} order statistic, $X_{i:n}$ can be written as

$$f_{i:n}(x) = \frac{f(x)}{B(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} (F(x))^{j+i-1}, \tag{9}$$

where $B(.,.)$ is the beta function. Note that

$$\begin{aligned} f(x)F^{j+i-1}(x) &= \frac{2\alpha\delta g(x;\Psi)G^{\alpha-1}(x;\Psi)}{(1+\bar{G}(x;\Psi))^{\alpha+1}} \left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma-1} \\ &\times \left[H(x;\alpha,\gamma,\Psi)\right]\left\{1-\bar{\delta}\left[H(x;\alpha,\gamma,\Psi)\right]\right\}^{-2} \\ &\times \left(\frac{1-\left[H(x;\alpha,\gamma,\Psi)\right]}{1-\bar{\delta}\left[H(x;\alpha,\gamma,\Psi)\right]}\right)^{j+i-1}. \end{aligned}$$

Using the generalized binomial series expansions, we get

$$\begin{aligned} f(x)F^{j+i-1}(x) &= 2\alpha\delta g(x;\Psi)\sum_{k=0}^{\infty}\binom{-(j+i-1)}{k}\frac{(-1)^k\bar{\delta}^k G^{\alpha-1}(x,\Psi)}{(1+\bar{G}(x;\Psi))^{\alpha+1}} \\ &\times\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma-1}\left[H(x;\alpha,\gamma,\Psi)\right]^{k+1} \\ &\times(1-H(x;\alpha,\gamma,\Psi))^{j+i-1} \\ &= 2\alpha\delta g(x;\Psi)\sum_{k,q=0}^{\infty}\binom{-(j+i-1)}{k}\binom{j+i-1}{q}\left[H(x;\alpha,\gamma,\Psi)\right]^{k+q+1} \\ &\times\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma-1}\frac{(-1)^{k+q}\bar{\delta}^k G^{\alpha-1}(x,\Psi)}{(1+\bar{G}(x;\Psi))^{\alpha+1}} \\ &= 2\alpha\delta g(x;\Psi)\sum_{k,q,w=0}^{\infty}\frac{(k+q+1)^w(-1)^{k+q}\bar{\delta}^k G^{\alpha-1}(x,\Psi)}{\gamma^w w!(1+\bar{G}(x;\Psi))^{\alpha+1}} \\ &\times\binom{-(j+i-1)}{k}\binom{j+i-1}{q}\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma-1} \\ &\times\left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)^{w} \\ &= 2\alpha\delta g(x;\Psi)\sum_{k,q,w,m=0}^{\infty}\frac{(k+q+1)^w(-1)^{k+q+m}\bar{\delta}^k G^{\alpha-1}(x,\Psi)}{\gamma^w w!(1+\bar{G}(x;\Psi))^{\alpha+1}}\binom{w}{m} \\ &\times\binom{-(j+i-1)}{k}\binom{j+i-1}{q}\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma(m+1)-1} \\ &= 2\alpha\delta g(x;\Psi)\sum_{k,q,w,m,n=0}^{\infty}\frac{(k+q+1)^w(-1)^{k+q+m+n}\bar{\delta}^k G^{\alpha-1}(x,\Psi)}{\gamma^w w!} \\ &\times\binom{w}{m}\binom{-(j+i-1)}{k}\binom{j+i-1}{q}\binom{-\gamma(m+1)-1}{n} \\ &\times(1+\bar{G}(x;\Psi))^{-\alpha(n+1)-1}G^{-\alpha(n+1)-1}(x;\Psi) \end{aligned}$$

$$
\begin{aligned}
&= 2\alpha\delta g(x;\Psi) \sum_{k,q,w,m,n,s,p=0}^{\infty} \frac{(k+q+1)^w(-1)^{k+q+m+n+s+p}\bar{\delta}^k G^{\alpha-1}(x,\Psi)}{\gamma^w w!}\binom{w}{m} \\
&\quad \times \binom{-(j+i-1)}{k}\binom{j+i-1}{q}\binom{-\gamma(m+1)-1}{n}\binom{(-\alpha(n+1)-1)}{s}\binom{s}{p} \\
&\quad \times g(x;\Psi)G^{-\alpha(n+1)+p-1}(x;\Psi) \\
&= 2\alpha\delta g(x;\Psi) \sum_{k,q,w,m,n,s,p,t=0}^{\infty} \frac{(k+q+1)^w(-1)^{k+q+m+n+s+p+t}\bar{\delta}^k}{\gamma^w w!}\binom{w}{m}\binom{s}{p} \\
&\quad \times \binom{-(j+i-1)}{k}\binom{j+i-1}{q}\binom{-\gamma(m+1)-1}{n}\binom{(-\alpha(n+1)-1)}{s} \\
&\quad \times \binom{-\alpha(n+1)+p-1}{t} g(x;\Psi)\bar{G}^t(x;\Psi)
\end{aligned}
$$

$$
\begin{aligned}
&= 2\alpha\delta g(x;\Psi) \sum_{k,q,w,m,n,s,p,t,l=0}^{\infty} \frac{(k+q+1)^w(-1)^{k+q+m+n+s+p+t+l}\bar{\delta}^k}{\gamma^w w!}\binom{w}{m}\binom{s}{p} \\
&\quad \times \binom{-(j+i-1)}{k}\binom{j+i-1}{q}\binom{-\gamma(m+1)-1}{n}\binom{(-\alpha(n+1)-1)}{s} \\
&\quad \times \binom{-\alpha(n+1)+p-1}{t}\binom{t}{l} g(x;\Psi)G^l(x;\Psi). \qquad (10)
\end{aligned}
$$

Substituting Equation (10) in Equation (9), the pdf of $X_{i:n}$ can be expressed as

$$
f_{i:n}(x) = \sum_{p=0}^{\infty} c_{l+1} h_{l+1}(x;\Psi), \qquad (11)
$$

where $h_{l+1}(x;\Psi)$ is the Exp-G density function with power parameter $(l+1)$ and

$$
\begin{aligned}
c_{l+1} &= 2\alpha\delta g(x;\Psi) \sum_{k,q,w,m,n,s,p,t,l=0}^{\infty} \frac{(k+q+1)^w(-1)^{k+q+m+n+s+p+t+l}\bar{\delta}^k}{\gamma^w w!}\binom{w}{m} \\
&\times \binom{-(j+i-1)}{k}\binom{j+i-1}{q}\binom{-\gamma(m+1)-1}{n}\binom{(-\alpha(n+1)-1)}{s} \\
&\times \binom{s}{p}\binom{-\alpha(n+1)+p-1}{t}\binom{t}{l}.
\end{aligned}
$$

It follows that the pdf of the i^{th} order statistics can be expressed as an infinite linear combination of exponentiated-G densities.

3.2. Renyi entropy

Rényi entropy (Rényi (1961)) is an extension of Shannon entropy. Rényi entropy is defined to be

$$
I_R(v) = \frac{1}{1-v}\log\left(\int_0^{\infty}[f(x;\alpha,\delta,\gamma,\Psi)]^v dx\right), v \neq 1, v > 0. \qquad (12)
$$

Rényi entropy tends to Shannon entropy as $v \to 1$. Set $[f(x;\alpha,\delta,\gamma,\Psi)]^v = [f(x)]^v$. From Equation (6),

$$\begin{aligned} f^{\nu}(x) &= \frac{(2\alpha\delta)^{\nu} g^{\nu}(x;\Psi) G^{\nu(\alpha-1)}(x;\Psi)}{(1+\bar{G}(x;\Psi))^{\nu(\alpha+1)}} \left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{\nu(-\gamma-1)} \\ &\times \ \exp\left\{\frac{\nu}{\gamma}\left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)\right\} \\ &\times \ \underbrace{\left\{1-\bar{\delta}\left[\exp\left\{\frac{1}{\gamma}\left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)\right\}\right]\right\}^{-2\nu}}_{D}. \end{aligned}$$

Using the series expansion $(1-z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{\Gamma(k) j!} z^j$, for $|z| < 1$,

$$D = \sum_{j=0}^{\infty} \frac{\Gamma(2\nu+j)}{\Gamma(2\nu)} \bar{\delta}^j \left[\exp\left\{\frac{1}{\gamma}\left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)\right\}\right]^j,$$

so we can write

$$\begin{aligned} f^{\nu}(x) &= \frac{(2\alpha\delta)^{\nu} g^{\nu}(x;\Psi) G^{\nu(\alpha-1)}(x;\Psi)}{(1+\bar{G}(x;\Psi))^{\nu(\alpha+1)}} \sum_{j=0}^{\infty} \frac{\Gamma(2\nu+j)}{\Gamma(2\nu)} \bar{\delta}^j \left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{\nu(-\gamma-1)} \\ &\times \ \underbrace{\exp\left\{\frac{j+\nu}{\gamma}\left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)\right\}}_{E}. \end{aligned}$$

Applying the generalized binomial series expansion on E, we obtain

$$E = \sum_{q=0}^{\infty} \frac{(j+\nu)^q}{\gamma^q q!} \left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)^q.$$

Thus, we get

$$\begin{aligned} f^{\nu}(x) &= \frac{(2\alpha\delta)^{\nu} g^{\nu}(x;\Psi) G^{\nu(\alpha-1)}(x;\Psi)}{(1+\bar{G}(x;\Psi))^{\nu(\alpha+1)}} \sum_{j,q=0}^{\infty} \frac{\Gamma(2\nu+j)\bar{\delta}^j (j+\nu)^q}{\Gamma(2\nu)\gamma^q q!} \left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{\nu(-\gamma-1)} \\ &\times \ \left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)^q. \end{aligned}$$

Also, by applying the following generalised binomial series expansions:

$$\begin{aligned} \left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)^q &= \sum_{w=0}^{\infty} (-1)^w \binom{q}{w} \left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma w}, \\ \left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma(w+\nu)-\nu} &= \sum_{n=0}^{\infty} (-1)^n \binom{-\gamma(w+\nu)-\nu}{n} \left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha n}, \\ \left(1+\bar{G}(x;\Psi)\right)^{-\alpha(n+\nu)-\nu} &= \sum_{m=0}^{\infty} (-1)^m \binom{-\alpha(n+\nu)-\nu}{m} \bar{G}^m(x;\Psi), \\ (1-G(x;\Psi))^m &= \sum_{s=0}^{\infty} (-1)^s \binom{m}{s} G^s(x;\Psi), \end{aligned}$$

$$\left(1-\bar{G}(x;\Psi)\right)^{-\alpha(m+\nu)+s-2}=\sum_{t=0}^{\infty}(-1)^t\binom{-\alpha(m+\nu)+s-2}{t}\bar{G}^t(x;\Psi)$$

and

$$(\bar{G}(x;\Psi))^t=\sum_{i=0}^{\infty}(-1)^i\binom{t}{i}G^i(x;\Psi),$$

we can write

$$\begin{aligned}
f^{\nu}(x) &= (2\alpha\delta)^{\nu}\bar{\delta}\sum_{j,q,w,n,m,s,t,i=0}^{\infty}\frac{(-1)^{w+m+n+s+t+i}\Gamma(2\nu+j)(j+\nu)^q}{\Gamma(2\nu)j!\gamma^q q!}\binom{q}{w}\binom{n}{s}\\
&\times \binom{-\alpha(m+\nu)-\nu}{n}\binom{-\gamma(w+\nu)-\nu}{m}\binom{-\alpha(m+\nu)+s-2}{t}\binom{t}{i}\\
&\times g^{\nu}(x;\Psi)G^i(x;\Psi). \qquad (13)
\end{aligned}$$

Thus, the Rényi entropy of the MO-Gom-EHL-G family of distributions can be expressed as

$$I_R(v)=\frac{1}{1-v}\log\Big[\sum_{i=0}^{\infty}b_i\left(\int_0^{\infty}[g(x;\Psi)]^vG^i(x,\Psi)dx\right)\Big], \qquad (14)$$

where

$$\begin{aligned}
b_i &= (2\alpha\delta)^{\nu}\bar{\delta}\sum_{j,q,w,n,m,s,t=0}^{\infty}\frac{(-1)^{w+m+n+s+t+i}\Gamma(2\nu+j)(j+\nu)^q}{\Gamma(2\nu)j!\gamma^q q!}\binom{q}{w}\binom{n}{s}\\
&\times\binom{-\alpha(m+\nu)-\nu}{n}\binom{-\gamma(w+\nu)-\nu}{m}\binom{-\alpha(m+\nu)+s-2}{t}\binom{t}{i}.
\end{aligned}$$

Note that, $\int_0^{\infty}[g(x;\Psi)]^vG^i(x,\Psi)dx$ can be obtained numerically. Rényi entropy of the MO-Gom-EHL-G family of distributions can be obtained directly from that of the exponentiated-G distribution as follows

$$I_R(v)=\frac{1}{1-v}\log\left[\sum_{i=0}^{\infty}\omega_i e^{(1-v)I_{REG}}\right], \qquad (15)$$

where

$$\begin{aligned}
\omega_i &= (2\alpha\delta)^{\nu}\bar{\delta}\sum_{j,q,w,n,m,s,t=0}^{\infty}\frac{(-1)^{w+m+n+s+t+i}\Gamma(2\nu+j)(j+\nu)^q}{\Gamma(2\nu)j!\gamma^q q!}\binom{q}{w}\binom{n}{s}\\
&\times\binom{-\alpha(m+\nu)-\nu}{n}\binom{-\gamma(w+\nu)-\nu}{m}\binom{-\alpha(m+\nu)+s-2}{t}\binom{t}{i},
\end{aligned}$$

and $I_{REG}=\int_0^{\infty}[(\frac{i}{\nu}+1)g(x;\Psi)G^{\frac{i}{\nu}}(x;\Psi)]^{\nu}dx$ is the Rényi entropy of the exponentiated-G distribution with power parameter $\frac{i}{\nu}$.

3.3. Moments and generating function

Let $X\sim MO-Gom-EHL-G(\alpha,\delta,\gamma,\Psi)$, then the r^{th} moment can be obtained from Equation (7). For

$$E(X^r)=\sum_{v=0}^{\infty}c_{v+1}E(W_{v+1}^r),$$

where c_{v+1} is as defined in Equation (8) and $E(W_{v+1}^r)$ denotes the r^{th} moment of W_{v+1} which follows an Exp-G distribution with power parameter $v+1$.

The incomplete moments can be obtained as follows. The s^{th} incomplete moment is

$$I_X(t) = \int_0^t x^s f_{MO-Gom-EHL-G}(x;\psi)dx = \sum_{v=0}^{\infty} c_{v+1} I_{v+1}(t),$$

where $I_{v+1}(t) = \int_0^t x^s g_{v+1}(x;\Psi)dx$ and c_{v+1} is as defined in Equation (8). The moment generating function (mgf) of X is given by

$$M_X(t) = \sum_{v=0}^{\infty} c_{v+1} E(e^{tW_{v+1}}),$$

where $E(e^{tW_{v+1}})$ is the mgf of the Exp-G distribution with power parameter $(v+1)$ and c_{v+1} is as defined in Equation (8).

Furthermore, we can obtain the characteristic function given by $\phi(t) = E(e^{itX})$, where $i = \sqrt{-1}$. For

$$\phi(t) = \sum_{v=0}^{\infty} c_{v+1}\phi_{v+1}(t),$$

where $\phi_{v+1}(t)$ is the characteristic function of Exp-G distribution with power parameter $(v+1)$ and c_{v+1} is as defined in Equation (8).

The coefficients of variation (CV), skewness (CS) and kurtosis (CK) can be readily obtained. The variance (σ^2), Standard deviation (SD=σ), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by

$$\sigma^2 = \mu_2' - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu_2' - \mu^2}}{\mu} = \sqrt{\frac{\mu_2'}{\mu^2} - 1},$$

$$CS = \frac{E\left[(X-\mu)^3\right]}{\left[E(X-\mu)^2\right]^{3/2}} = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}},$$

and

$$CK = \frac{E\left[(X-\mu)^4\right]}{\left[E(X-\mu)^2\right]^2} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2' - \mu^2)^2},$$

respectively.

Note that the r^{th} cumulant of the random variable X can be readily obtained from the recursive relationship: $\kappa_r = \mu_r' - \sum_{s=1}^{r-1}\binom{r-1}{s-1}\mu_{r-s}'\kappa_s$, where $\mu_r' = E(X-\mu_1')^r$, so that the CS and CK are given by $\tau_1 = \frac{\kappa_3}{\kappa_2^{3/2}}$ and $\tau_2 = \frac{\kappa_4}{\kappa_2^2}$. A table of moments, SD, CV, CS, and CK for selected parameter values of the special case of the Marshall-Olkin-Gompertz-Exponentiated Half Logistic-log-logistic (MO-Gom-EHL-LLoG) distribution are given in Table 1.

The MO-Gom-EHL-LLoG distribution applies to various levels of skewness and kurtosis as shown by the values of CS and CK in Table 1.

Table 1 Moments of the MO-Gom-EHL-LLoG distribution for some parameter values

	(1.5,0.5,0.5,1)	(0.5,1.5,1,1.5)	(0.5,1.5,1,1.5)	(1,1.5,1,2.5)	(1.3,1.5,1.5,2.5)
E(X)	0.1707	0.3003	0.3003	0.2118	0.1583
E(X^2)	0.1113	0.1921	0.1921	0.1625	0.1260
E(X^3)	0.0818	0.1403	0.1403	0.1316	0.1045
E(X^4)	0.0644	0.1101	0.1101	0.1105	0.0892
E(X^5)	0.0530	0.0904	0.0904	0.0951	0.0778
SD	0.2866	0.3193	0.3193	0.3430	0.3177
CV	1.6787	1.0632	1.0632	1.6199	2.0068
CS	1.4752	0.6558	0.6558	1.1733	1.6402
CK	3.7729	2.0318	2.0318	2.6488	3.9351

3.4. Quantile function

The quantile function for the MO-Gom-EHL-G family of distributions is obtained as follows:

$$\frac{1-\big[H(x;\alpha,\gamma,\Psi)\big]}{1-\bar{\delta}\big[H(x;\alpha,\gamma,\Psi)\big]}=u,$$

for $0 \leq u \leq 1$, so that

$$\frac{1-u}{(1-u\bar{\delta})}=\exp\left\{\frac{1}{\gamma}\left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)\right\}.$$

Therefore, the quantiles of the MO-Gom-EHL-G family of distributions may be determined by solving the non-linear equation

$$x(u)=G^{-1}\left(\frac{2\left(1-\left[1-\gamma\log\{\frac{1-u}{(1-u\bar{\delta})}\}\right]^{\frac{1}{\gamma}}\right)^{\frac{1}{\alpha}}}{1+\left(1-\left[1-\gamma\log\{\frac{1-u}{(1-u\bar{\delta})}\}\right]^{\frac{1}{\gamma}}\right)^{\frac{1}{\alpha}}}\right), \tag{16}$$

via iterative methods in R or Matlab software.

Quantiles for selected parameter values for the Marshall-Olkin-Gompertz-Exponentiated Half Logistic-log-logistic (MO-Gom-EHL-LLoG) distribution are shown in Table 2.

Table 2 Table of quantiles for selected parameters of the MO-Gom-EHL-LLoG distribution

u	(1.5,1.5,1.5,1.5)	(1.5,1,1.5,1.5)	(1.5,0.5,1.5,2)	(1.5,1.5,2,1.5)	(1,1.5,1,2.5)
0.1	0.7793	0.6445	0.5633	0.7658	0.6246
0.2	1.1277	0.9398	0.7498	1.0900	0.8349
0.3	1.4189	1.1956	0.9052	1.3508	0.9971
0.4	1.6880	1.4402	1.0522	1.5838	1.1396
0.5	1.9517	1.6880	1.2016	1.8056	1.2742
0.6	2.2242	1.9517	1.3622	2.0284	1.4092
0.7	2.5228	2.2488	1.5454	2.2660	1.5535
0.8	2.8794	2.6122	1.7722	2.5418	1.7221
0.9	3.3848	3.1375	2.1008	2.9197	1.9556

3.5. Stochastic ordering

The concept of stochastic ordering is an important tool in the comparison of probability models in areas including reliability, survival analysis, finance, risks and economics. For more details on stochastic ordering, see Shaked and Shanthikumar Shaked and Shanthikumar (1994). A random variable X_1 is said to be stochastically smaller than X_2, $(X_1 <_{st} X_2)$ in the

- Stochastic order $(X_1 <_{st} X_2)$ if $F_{X_1}(x) \geq F_{X_2}(x)$ for all x.
- Hazard rate order $(X_1 <_{hr} X_2)$ if $h_{X_1}(x) \geq h_{X_2}(x)$ for all x.
- Likelihood ratio order $(X_1 <_{lr} X_2)$ if $\frac{f_{X_1}(x)}{f_{X_2}(x)}$ decreasing in x.

The stochastic orders given above are associated and it holds that $X_1 <_{lr} X_2 \Rightarrow X_1 <_{hr} X_2 \Rightarrow X_1 <_{st} X_2$.

Theorem 1 *Let $X_1 \sim MO-Gom-EHL-G(\delta_1, \alpha, \gamma, \Psi)$ and $X_2 \sim MO-Gom-EHL-G(\delta_2, \alpha, \gamma, \Psi)$. If $\delta_1 < \delta_2$, then the ratio $\frac{f_1(x;\delta_1,\alpha,\gamma,\Psi)}{f_2(x;\delta_2,\alpha,\gamma,\Psi)}$ is decreasing in x.*

Proof: Consider the ratio:

$$\frac{f_1(x;\delta_1,\alpha,\gamma,\Psi)}{f_2(x;\delta_2,\alpha,\gamma,\Psi)} = \frac{\delta_1\left\{1-\bar{\delta}_1\left[\exp\left\{\frac{1}{\gamma}\left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)\right\}\right]\right\}^{-2}}{\delta_2\left\{1-\bar{\delta}_2\left[\exp\left\{\frac{1}{\gamma}\left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)\right\}\right]\right\}^{-2}}, \tag{17}$$

where $\overline{\delta}_1 = 1-\delta_1$, and $\overline{\delta}_2 = 1-\delta_2$. Taking derivative of Equation (17) with respect to x, we obtain

$$\frac{\partial}{\partial x}\left(\frac{f_1(x;\delta_1,\alpha,\gamma,\Psi)}{f_2(x;\delta_2,\alpha,\gamma,\Psi)}\right) = \frac{2\delta_1}{\delta_2}(\bar{\delta}_2-\bar{\delta}_1)\frac{(1-\overline{\delta}_2(W))}{(1-\overline{\delta}_1(W))^3}W',$$

where $W = \left[\exp\left\{\frac{1}{\gamma}\left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)\right\}\right]$ and $W' = \frac{dW}{dx}$.

If $\delta_1 < \delta_2$, then $\frac{\partial}{\partial x}\left(\frac{f_1(x;\delta_1,\alpha,\gamma,\Psi)}{f_2(x;\delta_2,\alpha,\gamma,\Psi)}\right) < 0$, which implies that $X_1 <_{lr} X_2$. Thus, X_1 and X_2 are stochastically ordered.

4. Some Special Cases

We present some special cases of the MO-Gom-EHL-G family of distributions in this section. We considered cases when the baseline distributions are log-logistic, Weibull, uniform and Kumaraswamy distributions.

4.1. Marshall-Olkin-Gompertz-Exponentiated half logistic-log-logistic (MO-Gom-EHL-LLoG) distribution

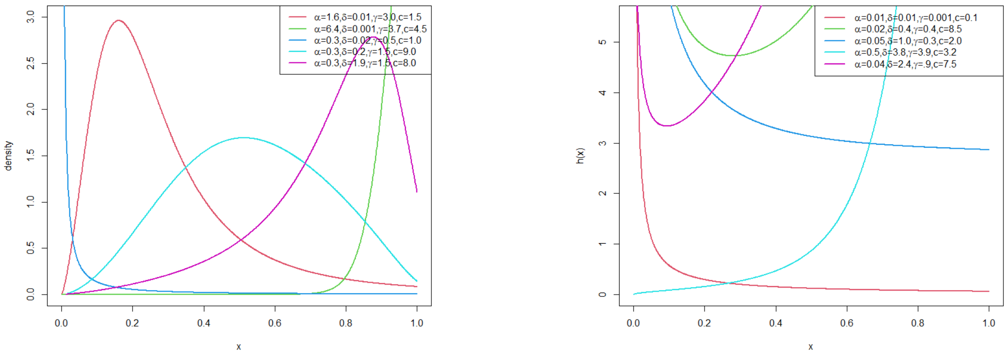
If we consider the log-logistic distribution as the baseline distribution with pdf and cdf given by $g(x) = cx^{c-1}(1+x^c)^{-2}$ and $G(x) = 1-(1+x^c)^{-1}$, for $c > 0$, respectively, we obtain the MO-Gom-EHL-LLoG distribution with cdf and pdf given by

$$F(x) = \frac{1-\exp\left\{\frac{1}{\gamma}\left(1-\left[1-\left(\frac{1-(1+x^c)^{-1}}{1+(1+x^c)^{-1}}\right)^{\alpha}\right]^{-\gamma}\right)\right\}}{1-(1-\delta)\exp\left\{\frac{1}{\gamma}\left(1-\left[1-\left(\frac{1-(1+x^c)^{-1}}{1+(1+x^c)^{-1}}\right)^{\alpha}\right]^{-\gamma}\right)\right\}}$$

and

$$\begin{aligned} f(x) &= \frac{2\alpha\delta c x^{c-1}(1+x^c)^{-2}(1-(1+x^c)^{-1})^{\alpha-1}}{(1+(1+x^c)^{-1})^{\alpha+1}}\big[H(x;\alpha,\gamma,c)\big] \\ &\times \left\{1-\bar{\delta}\big[H(x;\alpha,\gamma,c)\big]\right\}^{-2}\left[1-\left(\frac{1-(1+x^c)^{-1}}{1+(1+x^c)^{-1}}\right)^{\alpha}\right]^{-\gamma-1}, \end{aligned}$$

respectively, for $\alpha,\delta,\gamma,c>0$ and where $\big[H(x;\alpha,\gamma,c)\big]=\exp\left\{\frac{1}{\gamma}\left(1-\left[1-\left(\frac{1-(1+x^c)^{-1}}{1+(1+x^c)^{-1}}\right)^{\alpha}\right]^{-\gamma}\right)\right\}$.



(a) **(b)**

Figure 1 Plots of the pdf and hrf for the MO-Gom-EHL-LLoG distribution

Figures 1(a) and 1(b) show the plots of the pdfs and hrfs of the MO-Gom-EHL-LLoG distribution for selected parameters values. The pdf can take various shapes including reverse-J, J, almost symmetric, left-skewed and right-skewed. The hrf of the MO-Gom-EHL-LLoG distribution exhibits increasing, decreasing and bathtub hazard rate shapes.

4.2. Marshall-Olkin-Gompertz-Exponentiated half logistic-Weibull (MO-Gom-EHL-W) distribution

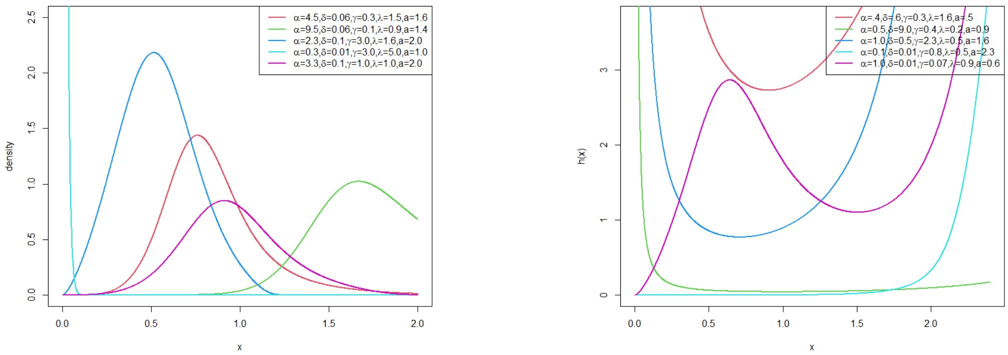
If we consider the Weibull distribution as the baseline distribution with pdf and cdf given by $g(x;\lambda,a)=\lambda a x^{a-1}e^{-\lambda x^a}$ and $G(x;\lambda,a)=1-e^{-\lambda x^a}$, respectively, for $\lambda,a>0$, we get the MO-Gom-EHL-W distribution with cdf and pdf given by

$$F(x)=\frac{1-\exp\left\{\frac{1}{\gamma}\left(1-\left[1-\left(\frac{1-e^{-\lambda x^a}}{1+e^{-\lambda x^a}}\right)^{\alpha}\right]^{-\gamma}\right)\right\}}{1-(1-\delta)\exp\left\{\frac{1}{\gamma}\left(1-\left[1-\left(\frac{1-e^{-\lambda x^a}}{1+e^{-\lambda x^a}}\right)^{\alpha}\right]^{-\gamma}\right)\right\}}$$

and

$$\begin{aligned} f(x) &= \frac{2\alpha\delta\lambda a x^{\beta-1}e^{-\lambda x^a}(1-e^{-\lambda x^a})^{\alpha-1}}{(1+e^{-\lambda x^a})^{\alpha+1}}\big[H(x;\alpha,\gamma,\lambda,a)\big] \\ &\times \left\{1-\bar{\delta}\big[H(x;\alpha,\gamma,\lambda,a)\big]\right\}^{-2}\left[1-\left(\frac{1-e^{-\lambda x^a}}{1+e^{-\lambda x^a}}\right)^{\alpha}\right]^{-\gamma-1}, \end{aligned}$$

respectively, for $\alpha,\delta,\gamma,\lambda,a>0$, and where
$\big[H(x;\alpha,\gamma,\lambda,a)\big]=\exp\left\{\frac{1}{\gamma}\left(1-\left[1-\left(\frac{1-e^{-\lambda x^a}}{1+e^{-\lambda x^a}}\right)^{\alpha}\right]^{-\gamma}\right)\right\}$.



(a) **(b)**

Figure 2 Plots of the pdf and hrf for the MO-Gom-EHL-W distribution

Figures 2(a) and 2(b) show the plots of the pdfs and hrfs of the MO-Gom-EHL-W distribution for selected parameters values. The pdf can take various shapes including reverse-J, almost symmetric, left-skewed and right-skewed. The hrf of the MO-Gom-EHL-W distribution exhibits decreasing, bathtub, upside bathtub followed by bathtub and increasing shapes.

4.3. Marshall-Olkin-Gompertz-Exponentiated half logistic-uniform (MO-Gom-EHL-U) distribution

By considering the uniform distribution as the baseline distribution with pdf and cdf given by $g(x) = (1/\beta)$ and $G(x) = (x/\beta)$, for $\beta > 0$, respectively, we obtain the MO-Gom-EHL-U distribution with cdf and pdf given by

$$F(x) = 1 - \frac{1 - \exp\left\{\frac{1}{\gamma}\left(1 - \left[1 - \left(\frac{(x/\beta)}{1+(1-(x/\beta))}\right)^{\alpha}\right]^{-\gamma}\right)\right\}}{1 - (1-\delta)\exp\left\{\frac{1}{\gamma}\left(1 - \left[1 - \left(\frac{(x/\beta)}{1+(1-(x/\beta))}\right)^{\alpha}\right]^{-\gamma}\right)\right\}} \tag{18}$$

and

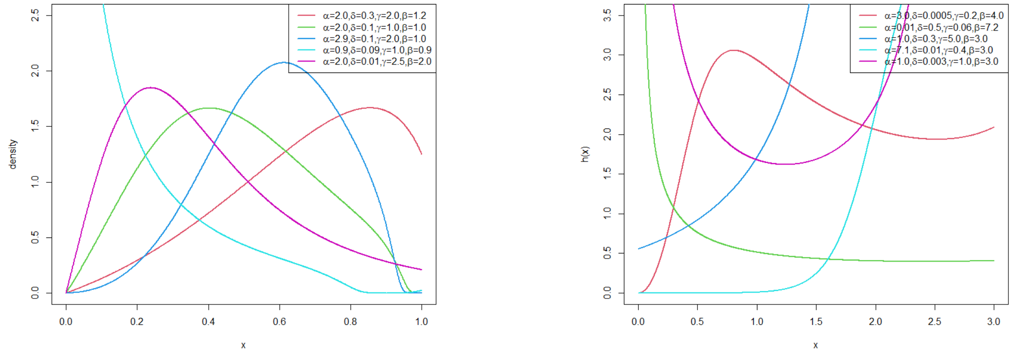
$$\begin{aligned} f(x) &= \frac{2\alpha\delta(1/\beta)((x/\beta))^{\alpha-1}}{(1+(1-(x/\beta)))^{\alpha+1}}\left[H(x;\alpha,\gamma,\beta)\right] \\ &\times \left\{1 - \bar{\delta}\left[H(x;\alpha,\gamma,\beta)\right]\right\}^{-2}\left[1 - \left(\frac{(x/\beta)}{1+(1-(x/\beta))}\right)^{\alpha}\right]^{-\gamma-1} \end{aligned} \tag{19}$$

respectively for $\alpha, \delta, \gamma, \beta > 0$ and where $\left[H(x;\alpha,\gamma,\beta)\right] = \exp\left\{\frac{1}{\gamma}\left(1 - \left[1 - \left(\frac{(x/\beta)}{1+(1-(x/\beta))}\right)^{\alpha}\right]^{-\gamma}\right)\right\}$.

Figures 3(a) and 3(b) show the plots of the pdfs and hrfs of the MO-Gom-EHL-U distribution for selected parameters values. The pdf can take various shapes including reverse-J, almost symmetric, left-skewed and right-skewed. The hrf of the MO-Gom-EHL-U distribution exhibits both monotonic and non-monotonic shapes.

4.4. Marshall-Olkin-Gompertz-Exponentiated half logistic-Kumaraswamy (MO-Gom-EHL-K) distribution

By considering the Kumaraswamy distribution as the baseline distribution with pdf and cdf given by $g(x) = \lambda c x^{c-1}(1-x^c)^{\lambda-1}$ and $G(x) = 1 - (1-x^c)^{\lambda}$, for $\lambda, c > 0$, respectively, we obtain the MO-Gom-EHL-K distribution with cdf and pdf given by



(a) **(b)**

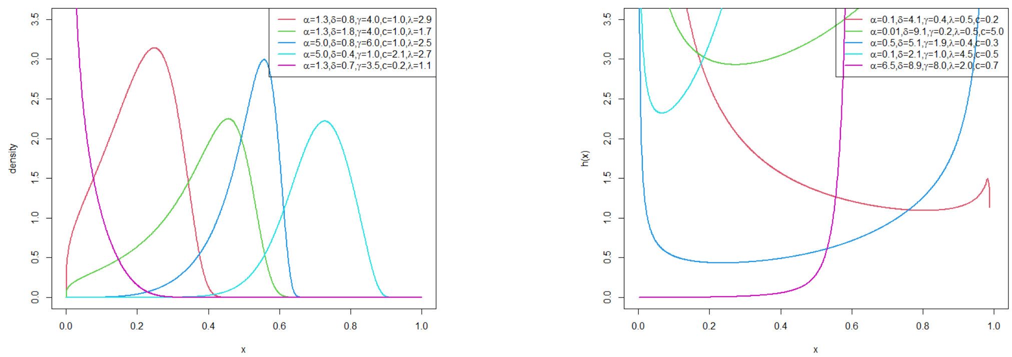
Figure 3 Plots of the pdf and hrf for the MO-Gom-EHL-U distribution

$$F(x) = \frac{1 - \exp\left\{\frac{1}{\gamma}\left(1 - \left[1 - \left(\frac{1-(1-x^c)^\lambda}{1+(1-x^c)^\lambda}\right)^\alpha\right]^{-\gamma}\right)\right\}}{1 - (1-\delta)\exp\left\{\frac{1}{\gamma}\left(1 - \left[1 - \left(\frac{1-(1-x^c)^\lambda}{1+(1-x^c)^\lambda}\right)^\alpha\right]^{-\gamma}\right)\right\}}$$

and

$$\begin{aligned} f(x) &= \frac{2\alpha\delta\lambda c x^{c-1}(1-x^c)^{\lambda-1}(1-(1-x^c)^\lambda)^{\alpha-1}}{(1+(1-x^c)^\lambda)^{\alpha+1}}\left[H(x;\alpha,\gamma,\lambda,c)\right] \\ &\times \left\{1 - \bar{\delta}\left[H(x;\alpha,\gamma,\lambda,c)\right]\right\}^{-2}\left[1 - \left(\frac{1-(1-x^c)^\lambda}{1+(1-x^c)^\lambda}\right)^\alpha\right]^{-\gamma-1}, \end{aligned}$$

respectively for $\alpha, \delta, \gamma, \lambda, c > 0$ and where $H(x;\alpha,\gamma,\lambda,c) = \exp\left\{\frac{1}{\gamma}\left(1 - \left[1 - \left(\frac{1-(1-x^c)^\lambda}{1+(1-x^c)^\lambda}\right)^\alpha\right]^{-\gamma}\right)\right\}$.



(a) **(b)**

Figure 4 Plots of the pdf and hrf for the MO-Gom-EHL-K distribution

Figures 4(a) and 4(b) show the plots of the pdfs and hrfs of the MO-Gom-EHL-K distribution for selected parameters values. The pdf can take various shapes including reverse-J, left-skewed and right-skewed. The MO-Gom-EHL-K distribution exhibits increasing, U-shaped, bathtub and bathtub followed by upside bathtub hazard rate functions.

5. Maximum Likelihood Estimation

If $X_i \sim MO - Gom - EHL - G(\alpha, \delta, \gamma, \Psi)$ with the parameter vector $\Delta = (\alpha, \delta, \gamma, \Psi)^T$. The total log-likelihood $\ell = \ell(\Delta)$ from a random sample of size n is given by

$$\begin{aligned}
\ell &= n\log(2\alpha) + n\log\delta + \sum_{i=1}^{n}\log g(x_i;\Psi) + (\alpha-1)\sum_{i=1}^{n}\log G(x_i;\Psi) - (\alpha+1) \\
&\times \sum_{i=1}^{n}\log(1+\bar{G}(x_i;\Psi)) + (-\gamma-1)\sum_{i=1}^{n}\log\left[1-\left(\frac{G(x_i;\Psi)}{1+\bar{G}(x_i;\Psi)}\right)^{\alpha}\right] \\
&+ \sum_{i=1}^{n}\left\{\frac{1}{\gamma}\left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)\right\} \\
&- 2\sum_{i=1}^{n}\log\left\{1-\bar{\delta}\left\{\frac{1}{\gamma}\left(1-\left[1-\left(\frac{G(x;\Psi)}{1+\bar{G}(x;\Psi)}\right)^{\alpha}\right]^{-\gamma}\right)\right\}\right\}. \qquad (20)
\end{aligned}$$

The maximum likelihood estimates of the parameters of α, δ, γ and Ψ can be computed by maximizing the log likelihood function in (20). We used the routine **nlm** which is available in the R software.

6. Simulation Study

In this section, we conducted a simulation study for the MO-Gom-EHL-LLoG distribution to evaluate the consistency of the maximum likelihood estimators. We simulated for N=1000 times with sample size n= 25, 50, 100, 200, 400, 800 and 1000. Simulation results are shown in Table 3 and Table 4. From the Monte Carlo simulation results, we conclude that our model produces consistent results when estimating parameters of the distribution via the method of maximum likelihood estimation. Note that from the results in Table 3 and Table 4, the mean values approximate the true parameter values, the RMSE and bias decrease as sample size increase for all parameter values.

Table 3 Monte Carlo simulation results for MO-Gom-EHL-LLoG distribution: mean, RMSE and average bias

	I	$\alpha = 1.5, \delta = 1.5, \gamma = 0.2, c = 1.6$			II	$\alpha = 1.5, \delta = 1.0, \gamma = 0.5, c = 1.0$	
Parameter	n	Mean	RMSE	Bias	Mean	RMSE	Bias
	25	2.750093	1.776357	1.250093	2.665047	1.589801	1.165047
	50	2.528366	1.662557	1.028366	2.488597	1.529284	0.988597
	100	2.211464	1.442660	0.711464	2.233325	1.394821	0.733325
α	200	1.944399	1.110429	0.444399	2.036891	1.180217	0.536891
	400	1.768364	0.817308	0.268364	1.844095	0.912157	0.344094
	800	1.624468	0.501624	0.124468	1.688905	0.640701	0.188905
	1000	1.587228	0.381580	0.087228	1.617934	0.451746	0.117934
	25	1.818203	4.682717	0.318203	1.194642	3.146779	0.194642
	50	1.521544	2.418152	0.021544	1.089646	1.961471	0.089646
	100	1.577375	1.708379	0.077375	1.108147	1.384720	0.108147
δ	200	1.544625	1.309000	0.044625	1.030643	0.997808	0.030643
	400	1.494341	0.945075	-0.005659	0.984326	0.707344	-0.015674
	800	1.496849	0.708821	-0.003151	0.970283	0.485800	-0.029717
	1000	1.498819	0.606453	-0.001181	0.982009	0.415205	-0.017991
	25	3.904822	6.017167	3.704822	5.718296	9.430878	5.218296
	50	2.921365	5.231521	2.721365	4.525630	7.619818	4.025630
	100	2.038541	4.143690	1.838541	3.685887	6.493236	3.185887
γ	200	1.190683	3.088005	0.990683	2.935340	5.933786	2.435340
	400	0.690454	2.001924	0.490454	1.917682	4.524419	1.417682
	800	0.349893	0.916503	0.149893	1.170555	2.983945	0.670555
	1000	0.279247	0.542633	0.079247	0.795425	1.680079	0.295425
	25	1.148837	0.684526	-0.451163	0.739133	0.434386	-0.260867
	50	1.256779	0.601233	-0.343221	0.803306	0.394693	-0.196694
	100	1.378324	0.510475	-0.221676	0.871791	0.351009	-0.128209
c	200	1.465369	0.401509	-0.134631	0.901490	0.300692	-0.098511
	400	1.518076	0.308141	-0.081925	0.934822	0.230755	-0.065178
	800	1.556009	0.208920	-0.043991	0.959381	0.165399	-0.040619
	1000	1.569064	0.175073	-0.030936	0.974363	0.131258	-0.025637

Table 4 Monte Carlo simulation results for MO-Gom-EHL-LLoG distribution: mean, RMSE and average bias

	III	$\alpha = 1.5, \delta = 1.0, \gamma = 0.2, c = 1.6$			IV	$\alpha = 1.5, \delta = 0.5, \gamma = 0.5, c = 1.0$	
Parameter	n	Mean	RMSE	Bias	Mean	RMSE	Bias
	25	2.803065	1.678639	1.303065	2.663789	1.427577	1.163789
	50	2.578069	1.599283	1.078069	2.562032	1.448892	1.062032
	100	2.307557	1.456797	0.807557	2.349806	1.346380	0.849806
α	200	2.066219	1.216219	0.566219	2.091114	1.138647	0.591114
	400	1.829858	0.893509	0.329857	1.917349	0.928819	0.417349
	800	1.646360	0.544266	0.146360	1.714790	0.657598	0.214790
	1000	1.595395	0.394033	0.095395	1.665355	0.548751	0.165355
	25	0.799073	1.660021	-0.200927	0.381670	0.912027	-0.118330
	50	0.897142	1.535797	-0.102858	0.420078	0.816457	-0.079922
	100	0.971805	1.144372	-0.028195	0.439056	0.577958	-0.060944
δ	200	0.955247	0.849874	-0.044753	0.460836	0.438541	-0.039164
	400	0.970959	0.651570	-0.029041	0.453463	0.326114	-0.046537
	800	0.981326	0.454412	-0.018674	0.479817	0.248102	-0.020183
	1000	0.993229	0.405579	-0.006771	0.480120	0.216660	-0.019880
	25	4.274400	6.190377	4.074400	5.549881	9.795850	5.049881
	50	3.076837	5.806527	2.876837	4.946409	8.152921	4.446409
	100	2.161108	3.982793	1.961108	3.796418	5.739918	3.296418
γ	200	1.515962	3.324342	1.315962	2.819586	4.776999	2.319586
	400	0.813185	2.062377	0.613185	1.923031	3.481664	1.423031
	800	0.402664	0.998497	0.202664	1.194157	2.335138	0.694157
	1000	0.291628	0.458884	0.091628	0.981246	1.911774	0.481246
	25	1.063246	0.698844	-0.536754	0.695608	0.415582	-0.304392
	50	1.208411	0.621658	-0.391589	0.747879	0.408601	-0.252121
	100	1.337971	0.542015	-0.262029	0.815149	0.366273	-0.184851
c	200	1.414515	0.450419	-0.185485	0.874171	0.315887	-0.125829
	400	1.495934	0.345875	-0.104066	0.907184	0.254285	-0.092816
	800	1.545628	0.232473	-0.054372	0.951171	0.187646	-0.048829
	1000	1.564741	0.192468	-0.035259	0.962048	0.162711	-0.037952

7. Applications

In this section, we illustrate the performance of the MO-Gom-EHL-G family of distributions by fitting the special case of MO-Gom-EHL-LLoG distribution to two real data sets. Model performance was assessed using the well-recognized goodness-of-fit statistics, namely, -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramr-Von Mises (W^*) and Andersen-Darling (A^*) as described by Chen and Balakrishman (1985), Kolmogorov-Smirnov (KS) and its p-value. The model with the smallest values of these goodness-of-fit statistics and the largest p-value for the KS statistic is regarded as the best fitting model.

We used R software to estimate the model parameters via the **nlm** function. Model parameter estimates (standard errors in parenthesis) for the three data sets are shown in Tables 6, 9 and 13. The goodness-of-fit-statistics for the three data sets are shown in Tables 7, 10 and 14. We also present plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. (1983)) to show how well our model fits the observed data sets.

We compared the MO-Gom-EHL-LLoG distribution with some competitive distributions including the odd exponentiated half logistic-Burr XII (OEHLBXII) by Aldahlan et al. (2018), the Exponentiated Half Logistic odd Weibull-Topp-Leone-LLoG (EHLOW-TL-LLoG) by Chipepa et al. (2021), the Marshall-Olkin-Topp-Leone-half logistic-Weibull (MO-TL-HL-W) by Sengweni et al. (2021), Topp-Leone-Gompertz-Weibull (TL-Gom-W) by Oluyede et al. (2021) and the exponentiated half logistic log-logistic-Weibull (EHL-LLoGW) by Makubate et al. (2021). The pdfs of the distributions are as follows:

$$f_{OEHLBXII}(x;\alpha,\lambda,a,b)=\frac{2\alpha\lambda abx^{a-1}\exp(\lambda[1-(1+x^a)^b])(1-\exp(\lambda[1-(1+x^a)^b]))^{\alpha-1}}{(1+x^a)^{-b-1}(1+\exp(\lambda[1-(1+x^a)^b]))^{\alpha+1}}.$$

$x>0,\alpha,\lambda,a,b>0,$

$$\begin{aligned}f_{TL-Gom-W}(x;b,\gamma,\beta) &= 2b\beta x^{\beta-1}e^{-x^\beta}\left(e^{-x^\beta}\right)^{-\gamma-1}\exp\left[\frac{1}{\gamma}\left(1-\left(e^{-x^\beta}\right)^{-\gamma}\right)\right]\\ &\quad\times\left[1-\left(\exp\left[\frac{1}{\gamma}\left(1-\left(e^{-x^\beta}\right)^{-\gamma}\right)\right]\right)^2\right]^{b-1},\end{aligned}$$

respectively, for $b,\gamma,\beta>0,$

$$\begin{aligned}f_{MO-TL-HL-W}(x;b,\delta,\theta,\gamma) &= \frac{4b\delta\gamma\theta x^{\gamma-1}\exp(-2\theta x^\gamma)(1-\exp(-2\theta x^\gamma))^{b-1}}{(1+(1-[1-\exp(-2\theta x^\gamma)]^b))^2}\\ &\quad\times\left\{1-\bar{\delta}\left(1-\left[\frac{[1-\exp(-2\theta x^\gamma)]^b}{1+(1-[1-\exp(-2\theta x^\gamma)]^b)}\right]\right)\right\}^{-2},\end{aligned}$$

$x>0,b,\delta,\theta,\gamma>0,$

$$\begin{aligned}f_{EHLLLoGW}(x;\alpha,\beta,\delta,c) &= \frac{2\delta e^{-\alpha x^\beta}(1+x^c)^{-1}\left[\alpha x^{\beta-1}+\frac{cx^{c-1}}{(1+x^c)}\right]}{\left[1+(1+x^c)^{-1}e^{-\alpha x^\beta}\right]^2}\\ &\quad\times\left(\frac{1-(1+x^c)^{-1}e^{-\alpha x^\beta}}{1+(1+x^c)^{-1}e^{-\alpha x^\beta}}\right)^{\delta-1},\end{aligned}$$

$x>0,\alpha,\beta,\delta,c>0,$ and

$$
\begin{aligned}
f_{EHLOW-TL-BXII}(x;\alpha,\beta,\delta,\lambda,\gamma) &= \frac{4\alpha\beta\delta\lambda\gamma x^{\lambda-1}(1+x^{\lambda})^{-2\gamma-1}[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha\beta-1}}{(1-[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha})^{\beta+1}} \\
&\times \exp\left(-\left[\frac{[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha}}{1-[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha}}\right]^{\beta}\right) \\
&\times \left(1+\exp\left(-\left[\frac{[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha}}{1-[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha}}\right]^{\beta}\right)\right)^{-2} \\
&\times \left[\frac{1-\exp\left(-\left[\frac{[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha}}{1-[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha}}\right]^{\beta}\right)}{1+\exp\left(-\left[\frac{[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha}}{1-[1-(1+x^{\lambda})^{-2\gamma}]^{\alpha}}\right]^{\beta}\right)}\right]^{\delta-1},
\end{aligned}
$$

for $\alpha,\beta,\delta,\lambda,\gamma > 0$. By letting $\gamma = 1$, we obtain the exponentiated half logistic odd Weibull-Topp-Leone-log logistic (EHLOW-TL-LLoG) distribution from the EHLOW-TL-BXII distribution.

7.1. Head and neck cancer data

The data set was analyzed by Elgarhy et al. (2018) consisting of survival times (in days) for the patients in Arm A of the Head-and-Neck Cancer Trial and are given below: 7, 34, 42, 63, 64, 74*, 83, 84, 91, 108, 112, 129, 133, 133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 173, 176, 185*, 218, 225, 241, 248, 273, 277, 279*, 297, 319*, 405, 417, 420, 440, 523, 523*, 583, 594, 1101, 1116*, 1146, 1226*, 1349*, 1412*, 1417. * indicates observations lost to follow-up.

Table 5 Descriptive statistics head and neck cancer data set

Min	25th quartile	Median	Mean	75th quartile	Max	Skewness	Kurtosis
7	133.0	176.0	357.8	418.5	1417.0	1.6856	4.5766

The estimated variance-covariance matrix for MO-Gom-EHL-LLoG model on head and neck cancer data set is given by

$$
\begin{bmatrix}
15.974092 & -0.088698 & -0.313388 & 0.086198 \\
-0.088698 & 0.000751 & 0.001742 & -0.000175 \\
-0.313388 & 0.001742 & 0.006148 & -0.001689 \\
0.086198 & -0.000175 & -0.001689 & 0.000904
\end{bmatrix}
$$

and the 95% confidence intervals for the model parameters are given by
$\alpha \in [15.7794 \pm 7.8337]$, $\delta \in [0.0289 \pm 0.0537]$, $\gamma \in [18.3469 \pm 0.1537]$ and $c \in [0.3715 \pm 0.0589]$.

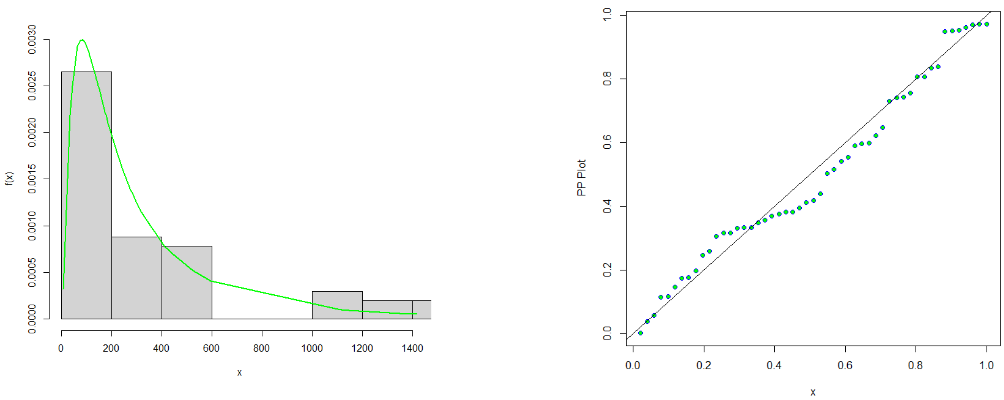
Table 6 shows that the proposed model gives the lowest values of the goodness-of-fit statistics, highest p-value for the Kolmogorov-Smirnov (KS) statistic and provides the overall best fit on head and neck cancer data set. Figures 5(a) and 5(b) displays clearly that the MO-Gom-EHL-LLoG distribution provide good fit and captures the head and neck cancer data set well. The histogram shows that the introduced model can well accommodate the extremely tailed data sets. From Figure 6, the Kaplan-Meier survival and ECDF curves shows that MO-Gom-EHL-LLoG model performs well. The TTT curve provide evidence that a bathtub hazard rate is adequate for head and neck cancer data set.

Table 6 Parameter estimates for various models fitted for head and neck cancer data set

Model	Estimates			
	α	δ	γ	c
MO-Gom-EHL-LLoG	15.7794	0.0289	18.3469	0.3715
	(3.9968)	(0.0274)	(0.0784)	(0.0301)
	b	β	δ	c
EHLOW-TL-LLoG	2.9965	3.8050	5.7797	0.0635
	(1.2095)	(5.9086)	(4.7884)	(0.0984)
	b	δ	θ	γ
MO-TL-HL-W	3.0205×10^{3}	2.6100×10^{4}	5.7774	0.0117
	(5.1712×10^{-5})	(5.9846×10^{-6})	(0.3439)	(0.0106)
	α	λ	a	b
OEHLBXII	0.4296	0.0001	1.6088	0.8925
	(0.0460)	(4.6805×10^{-5})	(0.0171)	(0.0312)
	γ	b	λ	
TL-Gom-W	4.2482×10^{-9}	3.8507	0.0919	-
	(0.0146)	(0.5467)	(0.0092)	
	α	β	δ	c
EHLLLoGW	0.3237	0.4039	12.3041	0.000
	(0.8021)	(0.2534)	(8.5889)	(1.0954)

Table 7 Goodness-of-fit statistics for various models fitted for head and neck cancer data set

Model	Statistics							
	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	$P-value$
MO-Gom-EHL-LLoG	695.4	703.4	704.3	711.2	0.1053	0.6831	0.0910	0.7920
EHLOW-TL-LLoG	696.2	704.2	705.1	711.9	0.1086	0.6975	0.1033	0.6484
MO-TL-HL-W	696.4	704.4	705.3	712.1	0.1105	0.6927	0.0996	0.6924
OEHLBXII	727.6	735.6	736.5	743.4	0.4469	2.6502	0.1942	0.0427
TL-Gom-W	881.1	887.1	887.6	892.9	0.1033	0.6809	0.7589	2.2×10^{-16}
EHLLLoGW	696.0	704.0	704.9	711.7	0.1087	0.6942	0.1048	0.6300



(a) **(b)**

Figure 5 Fitted densities and probability plots for head and neck cancer data

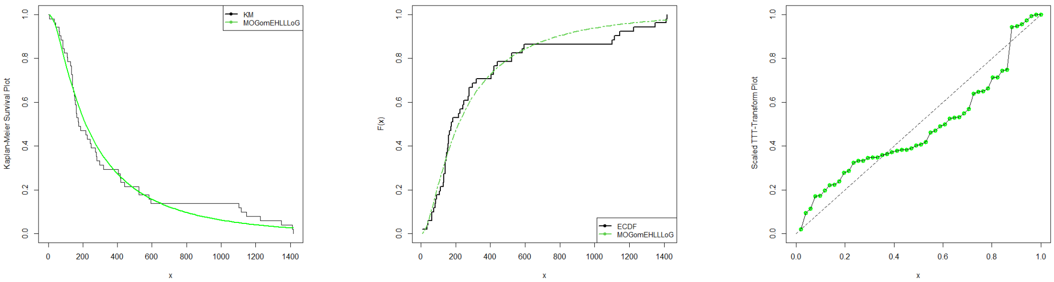


Figure 6 Kaplan-Meier survival, estimated cdf (ECDF) plots and the total time on test (TTT) plot of the MO-Gom-EHL-LLoG distribution for the head and neck cancer data set

7.2. Windshield data

We fit the MO-Gom-EHL-LLoG distribution to the data set reported by Silva et al. (2015). The observations are as follows 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

Table 8 Descriptive statistics windshield data set

Min	25th quartile	Median	Mean	75th quartile	Max	Skewness	Kurtosis
0.040	1.839	2.354	2.557	3.393	4.663	0.0995	2.3477

The estimated variance-covariance matrix for MO-Gom-EHL-LLoG model on windshield data set is given by

$$\begin{bmatrix} 0.123973 & -0.093889 & -0.000177 & 0.021578 \\ -0.093889 & 0.075269 & 0.000134 & -0.016060 \\ -0.000177 & 0.000134 & 0.000000 & -0.000031 \\ 0.021578 & -0.016060 & -0.000031 & 0.003819 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by
$\alpha \in [4.6664 \pm 0.6901]$, $\delta \in [0.2822 \pm 0.5377]$, $\gamma \in [407.6200 \pm 0.0010]$ and $c \in [0.2312 \pm 0.1211]$.

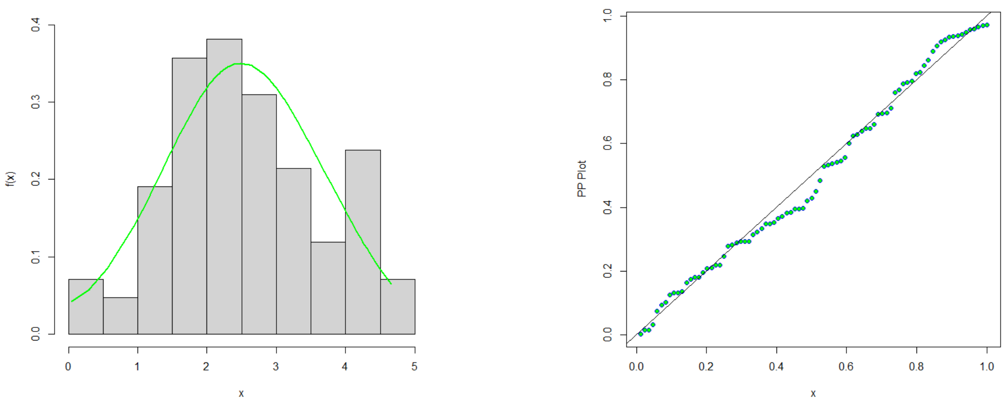
Based on the results shown in Table 9 we can conclude that the MO-Gom-EHL-LLoG model performs better on windshield data compared to the known competing non-nested models included in this paper. Furthermore, Figures 7(a) and 7(b) displays the flexibility enjoyed by fitting the windshield data set using the MO-Gom-EHL-LLoG distribution. From Figure 8, the Kaplan-Meier and ECDF plots give enough information about the closest fit of the MO-Gom-EHL-LLoG to the windshield data set. The TTT plot shows that the windshield has increasing hrf which means that the MO-Gom-EHL-LLoG distribution can be used to model this data.

Table 9 Parameter estimates for various models fitted for windshield data set

Model	Estimates			
	α	δ	γ	c
MO-Gom-EHL-LLoG	4.6664	0.2822	407.6200	0.2312
	(0.3521)	(0.2743)	(0.0005)	(0.0618)
	b	β	δ	c
EHLOW-TL-LLoG	5.8357	3.2226	0.3962	0.5712
	(2.4597)	(1.4050)	(0.2897)	(0.1769)
	b	δ	θ	γ
MO-TL-HL-W	0.3597	0.4785	0.0007	4.9003
	(0.0961)	(0.2679)	(0.0001)	(0.0261)
	α	λ	a	b
OEHLBXII	0.2383	0.0003	4.9436	1.2266
	(0.0590)	(0.0006)	(0.0237)	(0.2813)
	γ	b	λ	
TL-Gom-W	2.0697×10^{-9}	2.9477	4.5729	-
	(0.0080)	(0.3223)	(0.0372)	
	α	β	δ	c
EHLLLoGW	0.2347	1.8661	2.9978	0.2989
	(0.1441)	(0.3717)	(0.7399)	(0.2339)

Table 10 Goodness-of-fit statistics for various models fitted for windshield data set

Model	Statistics							
	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	$P-value$
MO-Gom-EHL-LLoG	252.3	260.3	260.8	270.0	0.0699	0.4996	0.080	0.6560
EHLOW-TL-LLoG	261.2	269.2	269.7	278.9	0.1143	0.8489	0.0858	0.5668
MO-TL-HL-W	255.3	263.3	263.8	273.1	0.1019	0.7053	0.0816	0.6306
OEHLBXII	270.1	278.1	278.6	287.8	0.1828	1.0936	0.0954	0.4295
TL-Gom-W	431.2	437.2	427.4	444.5	0.3477	2.5445	0.6275	2.2×10^{-16}
EHLLLoGW	254.1	262.1	262.6	271.8	0.0741	0.5251	0.0831	0.6080



(a) **(b)**

Figure 7 Fitted densities and probability plots for windshield data

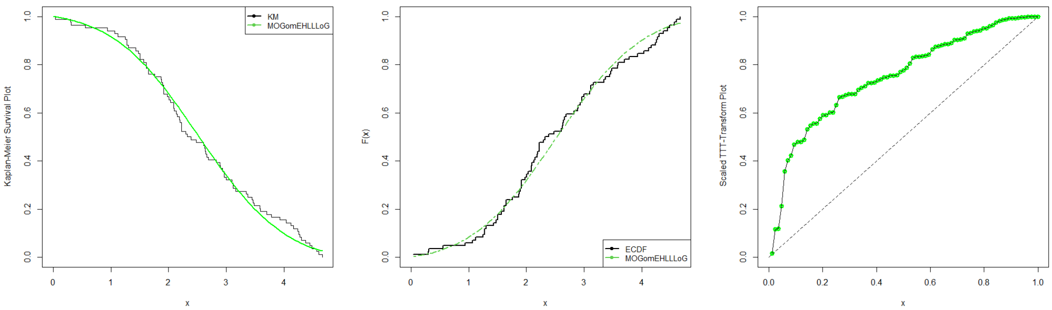


Figure 8 Kaplan-Meier survival, estimated cdf (ECDF) plots and the total time on test (TTT) plot of the MO-Gom-EHL-LLoG distribution for the windshield data set

7.3. Car price data

We fit the MO-Gom-EHL-LLoG distribution to the data set consists of prices (in dollars) of 428 new vehicles for the 2004 year (Kiplinger's Personal Finance, Dec 2003).

Table 11 Data set for prices (in 10 of thousands of dollars) of 428 new vehicles

1.028	1.0539	1.076	1.0995	1.1155	1.129	1.156	1.169	1.1839	1.1905	1.1939	1.2269	1.236
1.2585	1.274	1.28	1.2884	1.2965	1.327	1.327	1.358	1.367	1.373	1.3839	1.4085	1.4165
1.417	1.43	1.4385	1.45	1.461	1.4622	1.463	1.474	1.481	1.484	1.485	1.503	1.504
1.5295	1.5389	1.5389	1.546	1.5495	1.55	1.5568	1.558	1.5825	1.585	1.604	1.635	1.6385
1.6495	1.6497	1.653	1.6695	1.6722	1.6999	1.7045	1.7163	1.72	1.7232	1.7262	1.7475	1.7495
1.763	1.764	1.7735	1.775	1.7985	1.8345	1.8435	1.869	1.8715	1.8739	1.876	1.882	1.8825
1.8892	1.8995	1.9005	1.909	1.911	1.9135	1.924	1.927	1.9312	1.9339	1.9479	1.949	1.956
1.9635	1.9635	1.9825	1.986	1.986	1.9945	1.9999	2.013	2.014	2.0215	2.022	2.0255	2.029
2.03	2.031	2.032	2.0339	2.037	2.0445	2.0449	2.051	2.0585	2.0615	2.0939	2.1055	2.1055
2.1087	2.141	2.1445	2.1589	2.1595	2.1595	2.1795	2.1825	2.184	2.19	2.1965	2.2	2.201
2.2035	2.218	2.2225	2.226	2.229	2.235	2.2388	2.2395	2.245	2.2515	2.257	2.2595	2.2735
2.2775	2.3215	2.329	2.3495	2.3495	2.356	2.3675	2.3699	2.376	2.3785	2.382	2.3845	2.3895
2.3895	2.3955	2.413	2.4225	2.4295	2.4345	2.452	2.4589	2.4695	2.478	2.4885	2.4895	2.495
2.4955	2.5	2.5045	2.5092	2.513	2.5135	2.5193	2.5215	2.5395	2.552	2.564	2.5645	2.57
2.57	2.5717	2.592	2.5935	2.594	2.5955	2.5995	2.6	2.606	2.6135	2.6189	2.6395	2.647
2.651	2.6545	2.656	2.665	2.686	2.691	2.693	2.696	2.699	2.6992	2.702	2.7145	2.72
2.7339	2.73	2.745	2.749	2.749	2.756	2.771	2.7905	2.793	2.7995	2.8345	2.837	2.8495
2.8495	2.8739	2.875	2.879	2.88	2.9282	2.9322	2.9345	2.938	2.944	2.9562	2.9595	2.967
2.9795	2.9865	2.9995	2.9995	3.0245	3.0295	3.0315	3.0492	3.0795	3.0835	3.086	3.0895	3.092
3.095	3.1045	3.1145	3.123	3.137	3.1545	3.1545	3.1745	3.184	3.1849	3.189	3.2235	3.2245
3.228	3.235	3.2415	3.2445	3.2455	3.2495	3.266	3.278	3.2845	3.3112	3.318	3.3195	3.326
3.3295	3.336	3.343	3.348	3.35	3.354	3.378	3.384	3.3895	3.3995	3.3995	3.439	3.448
3.4495	3.4495	3.456	3.4845	3.4895	3.5105	3.5145	3.5495	3.5515	3.5545	3.5695	3.5725	3.592
3.594	3.594	3.5995	3.61	3.6395	3.664	3.6895	3.6945	3.6995	3.7	3.7245	3.739	3.753
3.756	3.763	3.773	3.7885	3.7895	3.7995	3.838	3.883	3.9195	3.9235	3.925	3.9465	3.964
3.9995	4.0095	4.0235	4.032	4.034	4.0565	4.059	4.067	4.072	4.084	4.0845	4.101	4.1045
4.125	4.1465	4.1475	4.1815	4.1995	4.249	4.2565	4.2735	4.284	4.2845	4.2915	4.3175	4.3365
4.3365	4.3495	4.3755	4.3895	4.424	4.4295	4.4535	4.4925	4.4995	4.521	4.5445	4.57	4.5707
4.61	4.6265	4.647	4.6995	4.7955	4.804	4.817	4.8195	4.845	4.909	4.969	4.9995	4.9995
5.047	5.0595	5.067	5.1535	5.212	5.2195	5.2365	5.2545	5.2775	5.2795	5.28	5.2975	5.4765
5.4995	5.575	5.617	5.6595	5.6665	5.727	5.9995	6.067	6.312	6.32	6.48	6.5	6.8995
6.919	6.9195	6.9995	7.225	7.3195	7.432	7.4995	7.4995	7.5	7.62	7.6765	7.687	7.9165
8.1795	8.1995	8.4165	8.46	8.697	8.6995	8.9765	9.052	9.482	12.177	12.667	12.842	19.2465

Table 12 Descriptive statistics car price data set

Min	25th quartile	Median	Mean	75th quartile	Max	Skewness	Kurtosis
1.028	2.033	2.764	3.277	3.921	19.247	2.7883	16.7036

The estimated variance-covariance matrix for MO-Gom-EHL-LLoG model on car price data set is given by

$$\begin{bmatrix} 0.798463 & -1.375227 & 0.067669 & -0.395936 \\ -1.375227 & 2.841029 & -0.154745 & 0.893540 \\ 0.067669 & -0.154745 & 0.010320 & -0.052957 \\ -0.395936 & 0.893540 & -0.052957 & 0.298089 \end{bmatrix}$$

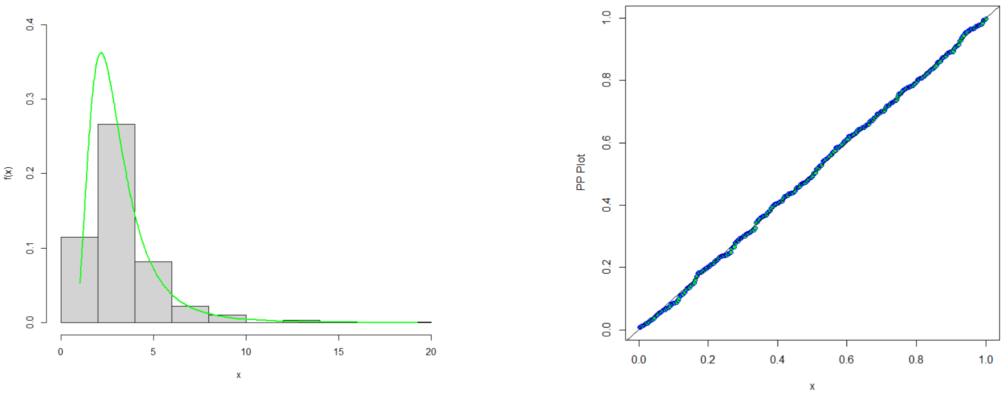
and the 95% confidence intervals for the model parameters are given by
$\alpha \in [4.0638 \pm 1.7514]$, $\delta \in [1.9621 \pm 3.3036]$, $\gamma \in [0.0575 \pm 0.1991]$ and $c \in [0.5460 \pm 1.0701]$.

Table 13 Parameter estimates for various models fitted for car price data set

Model	Estimates			
	α	δ	γ	c
MO-Gom-EHL-LLoG	4.0638	1.9621	0.0575	2.8173
	(0.8936)	(1.6855)	(0.1016)	(0.5460)
	b	β	δ	c
EHLOW-TL-LLoG	2.0306	1.0388	6.5367	0.4875
	(0.2314)	(0.3248)	(1.2990)	(0.1507)
	b	δ	θ	γ
MO-TL-HL-W	2.6632	0.0021	0.0163	1.3782
	(0.0961)	(0.2679)	(0.0001)	(0.0261)
	α	λ	a	b
OEHLBXII	0.6507	0.1001	2.4539	0.7186
	(0.0513)	(0.0217)	(0.2361)	(0.0840)
	γ	b	λ	
TL-Gom-W	1.9474×10^{-9}	5.2168	0.4191	-
	(0.0071)	(0.2573)	(0.0144)	
	α	β	δ	c
EHLLLoGW	0.0000	3.9190×10^{-5}	5.0573	2.6102
	(1.9888×10^{-8})	(0.3571)	(0.3419)	(0.08374)

Table 14 Goodness-of-fit statistics for various models fitted for car price data set

Model	Statistics							
	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	$P-value$
MO-Gom-EHL-LLoG	1489.2	1497.2	1497.3	1513.5	0.0221	0.2034	0.0218	0.9871
EHLOW-TL-LLoG	1517.5	1525.5	1525.6	1541.7	0.2414	1.7447	0.0451	0.3476
MO-TL-HL-W	1505.4	1513.4	1513.5	1529.7	0.0786	0.6271	0.0278	0.8963
OEHLBXII	1890.9	1899.0	1899.0	1915.1	1.9077	11.7362	0.1438	4.123×10^{-8}
TL-Gom-W	2177.0	2183.0	2183.0	2195.1	0.1935	1.4180	0.5231	2.2×10^{-16}
EHLLLoGW	1496.6	1504.6	1504.7	1520.9	0.1415	0.9047	0.0357	0.6472



(a) **(b)**

Figure 9 Fitted densities and probability plots for car price data

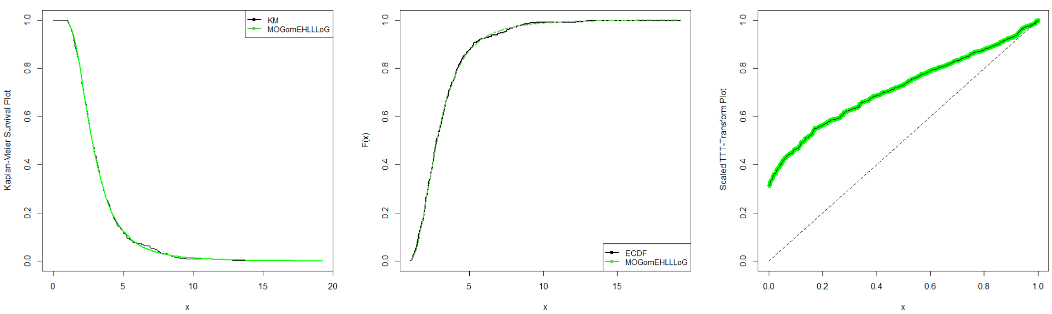


Figure 10 Kaplan-Meier survival, estimated cdf (ECDF) plots and the total time on test (TTT) plot of the MO-Gom-EHL-LLoG distribution for the car price data set.

From the results displayed in Table 13, we can conclude that the MO-Gom-EHL-G model performs better on car price data compared to the known competing non-nested models included in this paper. Furthermore, Figures 9(a) and 7(b) displays the flexibility enjoyed on fitting the car price data set using the MO-Gom-EHL-LLoG distribution. From Figure 10, we can see that the new proposed distribution follows the Kaplan-Meier and ECDF curves very closely. The TTT plot shows that the car price has increasing hrf which means that the MO-Gom-EHL-LLoG distribution can be used to model this data.

8. Concluding Remarks

We introduced a new family of distributions, referred to as the Marshall-Olkin-Gompertz -Exponentiated Half Logistic-G distribution. The new distribution can handle both heavy-tailed and symmetric data and also have non-monotonic hazard rate shapes. The proposed distribution can be expressed as an infinite linear combination of the Exp-G distributions. We applied a special case of the new family of distributions to three real data sets and our model perform better than the competing non-nested models as shown in Tables 6, 9 and 13.

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References

Aldahlan M, Afify AZ. The odd exponentiated half-logistic Burr XII distribution. Pakistan J Stat Oper Res. 2018; 14(2): 305-317.

Alizadeh M, Cordeiro GM, Pinho LGB, Ghosh I. The Gompertz-G family of distributions. J Stat Theory Pract. 2017; 11(1):179-207.

Chambers J, Cleveland W, Kleiner B, Tukey, P. Graphical methods of data analysis. Chapman and Hall; 1983.

Chen G, Balakrishnan N. A general purpose approximate goodness-of-fit test. J Qual Technol. 1985; 27(2): 154-161.

Chipepa F, Oluyede B, Makubate B. The Topp-Leone Marshall-Olkin-G family of distributions with applications. Int J Probab Stat. 2020; 9(4): https://doi.org/10.5539/ijsp.v9n4p15.

Chipepa F, Oluyede B, Wanduku, D. The Exponentiated Half Logistic odd Weibull-Topp-Leone-G: Model, properties and applications. J Stat Model Theory Appl. 2021; 2(1):15-38.

Chipepa F, Oluyede, B. The Marshall-Olkin-Gompertz-G family of distributions: properties and applications. J Nonlinear Sci Appl. 2021; 14: 60-65.

Elgarhy M, Arslan Nasir M, Jamal F, Gamze Ozel G. The type II Topp-Leone generated family of distributions: properties and applications. Int J Stat Manag Syst. 2018; 21: 1529-1551.

El-Gohary A, Alshamrani A, and Al-Otaibi A, The generalized Gompertz distribution. Appl Math Model. 2013; 37: 13-24.

Ieren TG, Kromtit FM, Agbhor BU, Eraikhuemen IB, Koleoso PO. A power Gompertz distribution: model, properties and application to bladder cancer data. Asian Res J Math. 2019; 15(2): 1-14.

Ghitany ME. Marshall-Olkin extended Pareto distribution and its application. Int J Appl Math. 2005; 18(1): 17-31.

Ghitany ME, Al-Awadhi FA, Alkhalfan LA. MarshallOlkin extended Lomax distribution and its application to censored data. Commun Stat. 2007; 36(10): 18551866.

Jafari AA, Tahmasebi S. The Gompertz power series distributions. Commun Stat. 2016; 45(13): 3761-3781.

Jafari AA, Tahmasebi S, Alizadeh M. The beta-Gompertz distribution. Rev Colomb Estad. 2014; 37(1): 144-158.

Makubate B, Chamunorwa S, Oluyede B, Chipepa F. The exponentiated half logistic-log-logistic Weibull distribution: model, properties and applications. Stat Model Theory Appl. 2021; 2(1): 101-120.

Marshall AN, Olkin I. A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families. Biometrika. 1997; 84: 641-652.

Oluyede B, Chipepa F, The Marshall-Olkin odd exponential half logistic-G family of distributions: properties and applications. Stat Optim Inf Comput. 2021; 11(2), 479-503.

Oluyede B, Chamunorwa S, Chipepa F, Alizadeh M. The Topp-Leone-Gompertz-G family of distributions with applications. Int J Stat Manag Syst. 2021; 25(6): 1399-1423.

Ristic MM, Jose KK, Ancy J. A MarshallOlkin gamma distribution and minification process. STARS: Stress and Anxiety Research Society. 2007; 11: 107117.

Ristic MM, Kundu D. Marshall-Olkin generalized exponential distribution. Metron. 2015; 73:317-333.

Rényi A, On measures of entropy and information. Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability. Califonia: The Regents of the University of California; 1961.

Santos Neto M, Bourguignon M, Zea L, Nascimento A, Cordeiro G. The Marshall-Olkin extended Weibull family of distributions. J Stat Distrib Appl. 2014; 1: 9.

Sengweni W, Oluyede B, Makubate B. The Marshall-Olkin-Topp-Leone half-logistic-G family of distributions. Stat Optim Inf Comput. 2023; 11(4): 1001-1026.

Shaked M, Shanthikumar JG. Stochastic Orders and Their Applications. Academic Press. University of Michigan; 1994.

Silva RV, Silva FG, Ramos MWA, Cordeiro GM. A new extended gamma generalized model. Int J Pure Appl Math. 2015; 100(2): 309-335.