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Programs for Performing the A Priori Procedure in Skew Normal Contexts

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Abstract

Despite being only a few years old, there is already a reasonably large literature on the a priori procedure, designed to aid researchers in determining sample sizes needed to ensure that sample statistics to be obtained provide good estimates of corresponding population parameters. One of the directions this literature has taken has been to expand the typical assumption of normally distributed data to include skew normally distributed data. In turn, the expansion allows the use of locations, or differences in locations; as opposed merely to means, or differences in means. Another direction the expansion has taken is to consider correlation coefficients. However, there is considerable mathematics associated with both expansions that might not be accessible to most substantive researchers. Therefore, the present goal is to provide links to free and user-friendly programs so that even mathematically unsophisticated substantive researchers can perform the calculations.

Keywords: A priori procedure, free and user-friendly programs, sample sizes, skew normally distributed data.

1. Introduction

Our present focus is on the a priori procedure (APP). The general idea is that researchers often wish to use sample statistics as estimates of corresponding population parameters. For example, a researcher might wish to use a sample mean as an estimate of a corresponding population mean. Or a researcher might wish to use a difference between sample means as an estimate of the difference between corresponding population means. To perform the APP, the researcher asks two questions bullet-listed below.

- Precision: How closely do I want the sample statistics of interest to approximate their corresponding population parameters?
- Confidence: With what probability do I want to attain the precision specification?

In turn, given specifications for precision and confidence, the APP informs the researcher of the sample size needed to meet the specifications. Thus, if the researcher collects the APP-informed sample size, or a larger one, she can be assured that her sample statistics will be within the desired

distance of the corresponding population parameters, at the desired level of confidence. For a simple example, suppose the researcher has one group and is willing to assume that the population from which he had sampled is normally distributed. In that case, the relevant APP equation is Equation (1) below:

$$n = \left(\frac{z_{(1-c)/2}}{f} \right)^2, \tag{1}$$

where n is the necessary sample size to meet specifications, f is the precision specification as a fraction of a standard deviation (0.1 in the example), and $z_{(1-c)/2}$ is the z-score that corresponds to the confidence specification (e.g., 1.96 for 95% confidence level in the example below). Suppose the researcher wishes to obtain a sample mean that has a 95% probability of being within a quarter of a standard deviation of the population mean. The calculation would run as follows:

$$n = \left(\frac{1.96}{0.25} \right)^2 = 61.47.$$

Rounding up to the nearest whole number, as participants do not come in fractions, implies that the researcher needs to collect 62 participants to have a 95% chance of obtaining a sample mean within a quarter of a standard deviation of the population mean. In contrast, if the researcher were to insist on having a 95% probability of being within one-tenth of a standard deviation of the population mean, 385 participants would be required. An advantage of the APP is that provided the researcher collects the requisite sample size, it is not necessary to perform the null hypothesis significance testing procedure as the researcher can be assured of obtaining sample statistics that meet specifications for precision and confidence. Equation (1) is so simple as not to require a program, though Li et al. (2020) nevertheless provided one. Li et al. also provided programs to handle differences in means for matched samples and differences in means for independent samples. Although these programs are useful, a limitation is that they all assume normal distributions, but most distributions are skewed (Blanca et al., 2013; Ho and Yu, 2015; Micceri, 1989). It would be an advantage to have programs that handle the difficult mathematics associated with skew normal distributions, which is the present objective.

2. Basic Properties of Skew Normal Distributions

In reality, the most data sets collected are not symmetric so that using the skew normal models to fit asymmetric these data sets are more suitable than normal models. Variables with skewed distributions are also commonly used in a variety of psychological and social research, Arnau et al. (2014) listed some of these variables: reaction times or response latency in cognitive studies, survival data from clinical trials, physical and verbal violence in couples, and labor income or health care costs in sociological studies, etc. To set up the present argument, it is necessary to review the family of skew normal distributions. As a generalization of the normal distribution, Azzalini (1985) formalized the skew normal distribution for modeling asymmetric data and provided the extension of the normal probability density function (pdf) as follows.

The random variable Z is said to have a standard skew normal distribution with skewness parameter $\alpha \in \Re$ if its pdf is given by

$$f_Z(z) = 2\phi(z)\Phi(\alpha z), \tag{2}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cumulative distribution function (cdf) of the standard normal distribution, respectively. The location-scale extension of the skew normal random variable $Y = \xi + \omega Z$ with $\xi \in \Re$ and $\omega \in \Re^+$, is said to have a skew normal distribution with location parameter ξ and scale parameter ω, denoted by $Y \sim SN((\xi,\ \omega^2,\ \alpha)$. The pdf of Y is

$$f_Y(y) = 2\phi(y;\ \xi,\ \omega^2)\Phi\left(\alpha \frac{y-\xi}{\omega}\right). \tag{3}$$

For the sampling distributions Y, see Azzalini and Capitanio (1999), Wang et al. (2016) and Zhu et al. (2019).

For the multivariate case we have the following definition.

Definition 1 (Azzalini and Dalla Valle (1996)) *A random vector* $\mathbf{X}$ *is said to have an* n*-dimensional multivariate skew normal distribution with vector of location parameters* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)' \in \Re^n$*, scale parameter of nonnegative definite matrix* Σ*, and the vector of skewness (shape) parameters* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)' \in \Re^n$*, denoted as* $\boldsymbol{X} \sim SN_n(\mu,\ \boldsymbol{\Sigma},\ \alpha)$*, if its pdf is given by*

$$f_{\mathbf{x}}(\mathbf{x}) = 2\phi_n(\mathbf{x}; \boldsymbol{\mu}, \Sigma)\Phi\left(\boldsymbol{\alpha}'\Sigma^{-1/2}(\mathbf{x} - \boldsymbol{\mu})\right), \tag{4}$$

where $\phi_n(\mathbf{x}; \boldsymbol{\mu}, \Sigma)$ *is the density of the* n*-dimensional multivariate normal distribution* $N_n(\boldsymbol{\mu},\ \Sigma)$ *with mean* $\boldsymbol{\mu}$ *and covariance matrix* Σ*, and* $\Phi(z)$ *is the cumulative distribution function (cdf) of the standard normal random variable* $Z \sim N(0, 1)$*.*

Note that the mean vector and the covariance matrix of $\mathbf{X}$ are given, respectively, by

$$\begin{aligned} E(\mathbf{X}) &= \boldsymbol{\mu} + \sqrt{\frac{2}{\pi}}\frac{\Sigma^{1/2}\boldsymbol{\alpha}}{(1 + \boldsymbol{\alpha}'\boldsymbol{\alpha})^{1/2}}, \\ Cov(\mathbf{X}) &= \Sigma^{1/2}\left(I_n - \frac{2\boldsymbol{\alpha}\boldsymbol{\alpha}'}{\pi(1 + \boldsymbol{\alpha}'\boldsymbol{\alpha})}\right)\Sigma^{1/2}, \end{aligned} \tag{5}$$

see Azzalini and Capitanio (1999) for more details.

The following lemma of Wang et al. (2020) will be used for establishing our APP programs.

Lemma 1 *Suppose that* $\mathbf{X} = (X_1, \ldots, X_n)' \sim SN_n(\boldsymbol{\mu}, \Sigma, \boldsymbol{\alpha})$ *with* $\boldsymbol{\mu} = \xi\mathbf{1}_n$*,* $\Sigma = \omega^2 I_n$ *and* $\boldsymbol{\alpha} = \lambda\mathbf{1}_n$*, where* $\xi,\ \lambda \in \Re$*,* $\omega > 0$*, and* $\mathbf{1_n} \in \Re^{\mathbf{n}}$ *is the vector with entries of 1's. Then the following results hold.*

(a) *The sample mean* $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ *has a skew normal distribution:*

$$\bar{X} \sim SN\left(\xi, \frac{\omega^2}{n}, \sqrt{n}\lambda\right).$$

(b) *Random variables* $X_1, \cdots, X_n$ *are identically skew normally distributed:*

$$X_i \sim SN\left(\xi,\ \omega^2,\ \lambda_*\right) \qquad i = 1, \ldots, n, \quad \lambda_* = \frac{\lambda}{\sqrt{1 + (n-1)\lambda^2}}.$$

(c) *The covariance of* X_i *and* X_j *is*

$$cov(X_i,\ X_j) = -\frac{2}{\pi}\delta_*^2\omega^2, \quad \text{for } i \neq j, \quad i,\ j = 1, \ldots, n.$$

(d) *The mean and variance of* $\bar{X}$ *are given by*

$$E(\bar{X}) = \xi + \sqrt{\frac{2}{\pi}}\omega\delta_* \quad \text{and} \quad V(\bar{X}) = \frac{\omega^2}{n}\left(1 - \frac{2}{\pi}\delta_*^2\right),$$

respectively, where $\delta_* = \frac{\lambda}{\sqrt{1+n\lambda^2}}$.

3. The APP's under Skew Normal Distribution — Single Sample Cases

Fortunately, APP skew normal equations already have been invented. Trafimow et al. (2019) performed the first work of this type with respect to a single group. One of the complexities involving skew normal distributions is that the usual normal distribution parameters of mean and standard deviation are not parameters of skew normal distributions. Specifically, the mean μ and standard deviation σ are replaced by the location ξ and the scale ω, respectively. In addition, there is a shape parameter α. When the shape parameter equals zero, the distribution is normal, and the location equals the mean and the scale equals the standard deviation. But when the shape parameter does not equal zero, the distribution is skew normal; the location does not equal the mean and the scale does not equal the standard deviation. In symbols, when $\alpha = 0$, $\xi = \mu$ and $\omega = \sigma$; but when $\alpha \neq 0$, $\xi \neq \mu$ and $\omega \neq \sigma$.

3.1. The APP for estimating location parameter

According to Trafimow et al. (2019), a researcher can determine the minimum sample size necessary to meet specifications for precision and confidence as follows when the goal is to obtain a sample location that is a good estimate of the population location. The necessary sample size n for known scale ω and shape α when given precision f and confidence c can be obtained by solving

$$\int_{L(n)}^{U(n)} 2\phi(z)\Phi\left(\sqrt{n}\alpha z\right) dz = c \tag{6}$$

so that $U(n) - L(n)$ is minimized and $\max\{|f_1|, f_2\} \leq f$, where

$$L(n) = \sqrt{n}\left(f_1\sqrt{1 - \frac{2}{\pi}\delta_*^2} + \sqrt{\frac{2}{\pi}}\delta_*\right), \quad U(n) = \sqrt{n}\left(f_2\sqrt{1 - \frac{2}{\pi}\delta_*^2} + \sqrt{\frac{2}{\pi}}\delta_*\right),$$

and $\delta_* = \alpha/\sqrt{1 + n\alpha^2}$. For more details, see Trafimow et al. (2019). We provide an easier alternative. It is merely necessary to click on the following link,

https://appforsn.shinyapps.io/nonesample/,

or paste it into one's browser, to access a user-friendly program. Simply enter the desired precision, desired confidence, and anticipated shape parameter into the boxes provided. After clicking "update", the program will return the required sample size to meet specifications. It is important to reiterate, however, that because the relevant central tendency parameter for skew normal distributions is the location, and not the mean, the program gives the sample size needed to meet specifications for precision and confidence with respect to the location and not with respect to the mean.

If the researcher is unsure about the shape parameter that applies, it is possible to be extremely cautious by using 0 for the shape parameter, which will maximize the minimum sample size needed to meet specifications. However, we do not necessarily recommend this option because it negates an important advantage of skewness, which is that it substantially reduces sample sizes needed to meet specifications. An alternative is to assume a shape parameter of 0.50, which is a minimal deviation from 0. Because the deviation is so minimal, it should underestimate the shape parameters of most populations while nevertheless availing the researcher of much of the benefit of skewness. We also recommend that researchers consult the Trafimow et al. (2019) article, or simply use the program trying different shape parameters, to gain an idea of the participants savings that can be enjoyed based on different shape parameters.

Of course, assumptions researchers make about distribution shapes can be checked against the data, once the data are collected. In contrast to study paradigms involving a single group, researchers more often compare two groups. These groups might be dependent, such as if there is a pretest, an intervention, and a posttest. Or these groups might be independent, such as when researchers randomly assign participants to experimental and control conditions. In skew normal contexts, the issue is the difference in locations rather than the difference in means that features in normal contexts.

3.2. The APP for estimating scale parameter

We have seen that APP thinking can be applied to finding the necessary sample size to meet specifications for precision and confidence for estimating a single location parameter. However, there may be occasions where researchers desire to estimate a scale parameter. Wang et al. (2019) derived the mathematics to facilitate this goal. When the precision f and confidence c are known, the necessary sample size n, for given shape α, can be obtained by

$$\int_{L}^{U} f(z)dz = c, \tag{7}$$

so that $U - L$ is minimized and $\max\{|f_1|, f_2\} \leq f$, where $f(z)$ is the probability density function of the chi-square distribution with $n-1$ degrees of freedom. Here, $L = (n-1)(f_1+1-\frac{2}{\pi}\delta^2)$, $U = (n-1)(f_2+1-\frac{2}{\pi}\delta^2)$ with $\delta = \dfrac{\alpha}{\sqrt{1+n\alpha^2}}$. The program link is given by

https://appforsn.shinyapps.io/nscale/.

It is merely necessary to click on the above link, or paste it into ones browser, to access a user-friendly program. Simply enter the desired precision, desired confidence, and anticipated shape parameter into the boxes provided. After clicking "update", the program will return the required sample size to meet specifications. It is important to reiterate, that the program gives the sample size needed to meet specifications for precision and confidence with respect to the scale and not with respect to the variance.

3.3. The APP for estimating shape parameter

In the foregoing APP procedures, the researcher must make an assumption about the shape parameter. However, the APP can be applied to determine the sample size needed to obtain a good estimate of the shape parameter, as is explained in this section. Let c be the confidence level and f be the precision. Then according to Wang et al. (2019), the required sample size n can be obtained by

$$\int_L^U f(t)dt = c, \tag{8}$$

so that $U - L$ is minimized and f_1 and f_2 are restricted by $\max(|f_1|, f_2) \leq f$, where $f(z)$ is the probability density function of the chi-square distribution with $n-1$ degrees of freedom, and

$$L = (n-1)(f_1+1), \qquad U = (n-1)(f_2+1).$$

The following link provides a easier way for obtaining the necessary n when we input f and c. It is merely necessary to click on the following link,

https://appforsn.shinyapps.io/nskewness/

or paste it into ones browser, to access a user-friendly program. Simply enter the desired precision, and desired confidence into the boxes provided. After clicking "update", the program will return the required sample size to meet specifications.

4. The APP for Estimating the Difference in Location Parameters

We will discuss the cases of estimating the difference of two location parameters for both the dependent (or matched) samples and independent samples, while the scale parameters are not necessarily equal.

4.1. The APP for estimating ξ_d with matched samples

The following presents the relevant mathematics for dependent groups (Wang et al., 2019). For given precision f and confidence c Then the required sample size related to the difference of the locations $\xi_d = \xi_1 - \xi_2$ for known shape α_d can be obtained by

$$\int_{L_*}^{U_*} f_T(t)dt = c \tag{9}$$

such that the length of the interval $U_* - L_*$ is minimum, where f_T is the pdf of T with $T \sim St_{n-1}(\sqrt{n}\alpha_d)$. Here

$$L_* = \sqrt{n}\left(f_1\sqrt{1-2\delta_d^2/\pi} + \sqrt{2/\pi}\delta_d\right), \quad U_* = \sqrt{n}\left(f_2\sqrt{1-2\delta_d^2/\pi} + \sqrt{2/\pi}\delta_d\right).$$

and $\max\{|f_1|, f_2\} \leq f$. Wang et al. (2019) provided the derivation. An easier way is to use our new user-friendly program that can be obtained at the following website:

https://appforsn.shinyapps.io/nmatch/.

You only need click on the link, or paste it into ones browser, to access a user-friendly program. Simply enter the desired precision, desired confidence, and anticipated shape parameter into the boxes provided. After clicking "update", the program will return the required sample size to meet specifications. It is important to reiterate, however, that because the relevant central tendency parameter for skew normal distributions is the location, and not the mean, the program gives the sample size needed to meet specifications for precision and confidence with respect to the difference of locations and not with respect to that of means.

4.2. The APP for estimation ξ_d with independent samples

The following presents the relevant mathematics for independent groups (Wang et al., 2019). Let c be the confidence level and f be the precision. Then the necessary sample size n for known scales ω_1, ω_2, and shapes α_1 and α_2 can be obtained by

$$\int_L^U 4\phi(v)E_Z[h(Z,\ v)]dv = c \tag{10}$$

such that the length of the interval $U - L$ is minimum and $\max\{|f_1|, f_2\} \leq f$ where $L = \sqrt{n}f_1 + \frac{\gamma}{\omega}$ and $U = \sqrt{n}f_2 + \frac{\gamma}{\omega}$ with $\gamma = \omega_1\delta_1 - \omega_2\delta_2$ and $Z \sim N(0,\ 1)$. Here

$$h(z, v) = \Phi(s_1v + t_1z)\Phi(s_2v + t_2z)$$

with $s_1 = \frac{\omega_2\alpha_1}{\omega}$, $s_2 = \frac{\omega_1\alpha_2}{\omega}$, $t_1 = \frac{\omega_1\alpha_1}{\omega}$ and $t_2 = -\frac{\omega_2\alpha_2}{\omega}$.

An easier way is to use our new user-friendly program that can be obtained at the following websites:

https://one-location.shinyapps.io/ninddiffloctn/.

Click on the link, or paste it into ones browser, and you will access a user-friendly program. You need enter the desired precision, desired confidence, and anticipated shape and scale parameters into the boxes provided. After clicking "update", the program will return the required sample size to meet specifications.

Remark 1 When the shape parameters are set equal to 0, the necessary sample size obtained by the foregoing link will be half that showed by the program featured in Li et al. (2020). This is because Li et al. used a single population standard deviation whereas the present program uses the square root of the sum of the two variances. Thus, the present program provides an improvement over the Li et al. program, even when normality is assumed.

5. The APP for Estimating Correlation Coefficient

It sometimes is not feasible to perform an experimental study. In such cases, researchers often opt for correlational studies. Whether the researcher is interested simply in a correlation coefficient, or in a more complex path analysis based on a set of underlying correlation coefficients, it is desirable to know the sample sizes needed to ensure that sample correlation coefficients meet precision and confidence specifications that render them good estimates of population correlation coefficients. As Wang et al. (2020) showed, this can be done mathematically, that is, the sample size n required for specific confidence c and precision f can be obtained by

$$\int_L^U f(z)dz = c \tag{11}$$

such that $U - L$ is minimized and f_1 and f_2 are restricted by $\max|f_1|,\ f_2 \leq f$, where $f(z)$ is the probability density function of the chi-squared distribution with $n-1$ degrees of freedom, and

$$L = (n-1)(1+f_1), \qquad U = (n-1)(1+f_2).$$

However, it is now possible to use our new user-friendly program that can be obtained at the following website:

https://appforsn.shinyapps.io/ncorr/.

Click on the link, or paste it into ones browser, and you will access a user-friendly program. You need enter the desired precision and desired confidence into the boxes provided. After clicking "update", the program will return the required sample size to meet specifications.

6. Discussion

Although the articles cited above have provided mathematics for performing the APP in paradigms where researchers are interested in estimating locations, or differences in locations, under the rubric of skew normal distributions, an important limitation is that the equations are difficult to use. The present article solves the problem by providing user-friendly computer programs to perform the operations. Thus, even researchers who are mathematically unsophisticated should be able to use the APP in skew normal contexts. In addition, we provide programs for correlational paradigms too. A limitation is that the programs take some time to run as the equations must be solved iteratively. Normally, this would be approximately 30 or fewer seconds, but it might take up to 60 seconds. However, this is not an important limitation. Finally, for the convenience of the reader, we present the programs that are applicable to single-group designs, dependent groups, independent groups, or correlational research in Table 1. The researcher who wishes to use a program can simply click on the link provided in Table 1 or paste the link into a browser. We hope and expect that the present programs will enhance the ability of researchers to perform the APP under a wider range of paradigms and distributional assumptions than was formerly practical.

For summery, a table of publicly available computer programs that can be used to find the required sample size for different researcher goals is listed below.

Application Type	Website
Location for single sample, known scale and shape	https://appforsn.shinyapps.io/nonesample/
Scale for single sample, known shape	https://appforsn.shinyapps.io/nscale/
shape for single sample	https://appforsn.shinyapps.io/nskewness/
Difference in locations for matched samples, known shapes	https://appforsn.shinyapps.io/nmatch/
Difference in locations for independent samples, known scales and unequal shapes	https://one-location.shinyapps.io/indp0nteq/
Difference in locations for independent samples, known scales and equal shapes	https://one-location.shinyapps.io/ninddiffloctn/
Correlation coefficient	https://appforsn.shinyapps.io/ncorr/

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References

Arnau J, Bendayan R, Blanca MJ, Bono R. Should we rely on the Kenward-Roger approximation when using linear mixed models if the groups have different distributions? Br J Math Stat Psychol. 2014; 67(3): 408-29.

Azzalini A. A class of distributions which includes the normal ones. Scand J Stat. 1985; 12(2): 171-178.

Azzalini A, Dalla Valle A. The multivariate skew-normal distribution, Biometrica. 1996; 83(4): 715-726.

Azzalini A, Capitanio A. Statistical application of the multivariate skew normal distribution. J Roy Stat Soc B. 1999; 61(3): 579-602.

Blanca MJ, Arnau J, Lpez-Montiel D, Bono R, Bendayan R. Skewness and kurtosis in real data samples. Methodology: European J Res Meth Behav Soc Sci. 2013; 9(2): 78-84.

Ho AD, Yu CC. Descriptive statistics for modern test score distributions: Skewness, kurtosis, discreteness, and ceiling effects. Educ Psychol Meas. 2015; 75(3): 365-388.

Li H, Trafimow D, Wang T, Wang C, Hu L. User-friendly computer programs so econometricians can run the a priori procedure. Front Manag Bus. 2020; 1(1): 2-6.

Micceri T. The unicorn, the normal curve, and other improbable creatures. Psychological Bulletin. 1989; 105(1): 156-166.

Trafimow D, Wang T, Wang C. From a sampling precision perspective, skewness is a friend and not an enemy!, Educ Psychol Meas. 2019; 79(1): 129-150.

Wang C, Wang T, Trafimow D, Chen J. Extending a priori procedure to two independent samples under skew normal setting. Asian J Econ Bank. 2019; 3(2): 29-40.

Wang C, Wang T, Trafimow D, Li H, Hu L, Rodriguez A. Extending a priori procedure (APP) to address correlation coefficients. In: Ngoc TN, Kreinovich V, Trung ND. (eds.) Data Science for Financial Econometrics. Studies in Computational Intelligence, vol 898. Springer, Cham; 2021.

Wang C, Wang T, Trafimow T, Talordphop K. Estimating the location parameter under skew normal settings: is violating the independence assumption good or bad? Soft Comput. 2021; 25: 7795-7802.

Wang C, Wang T, Trafimow D, Myuz HA. Desired sample size for estimating the skewness parameter under skew normal settings. In: Kreinovich V, Sriboonchitta S (eds.) Structural Changes and their Economic Modeling. Springer-Verlag, Switzerland; 2019. p. 152-162.

Wang C, Wang T, Trafimow D, Myuz HA. Necessary sample size for specified closeness and confidence of matched data under the skew normal setting. Commun Stat Simulat. 2022; 51(5) : 2083-2094.

Wang C, Wang T, Trafimow D, Zhang X. Necessary Sample Size for Estimating the Scale Parameter With Specified Closeness and Confidence. Int J Intel Technol Appl Stat. 2019; 12(1): 17-29.

Wang Z, Wang C, Wang T. Estimation of location parameter on the skew normal setting with known coefficient of variation and skewness. Int J Intel Technol Appl Stat. 2016; 9(3): 45-63.

Zhu X, Li B, Wang T, Gupta A. Sampling distributions of skew normal populations associated with closed skew normal distributions. Random Oper Stoch. 2019; 27(2): 75-87.