Thailand Statistician
January 2026; 24(1): 34-62
http://statassoc.or.th
Contributed paper

Modeling Heavy-Tailed Data via the New Topp-Leone Heavy-Tailed Gompertz-G Family of Distributions

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Received: 12 August 2024
Revised: 11 November 2024
Accepted: 4 August 2025

Abstract

In this article, we introduce a new distribution named, Topp-Leone Heavy-Tailed Gompertz-G (TL-HT-Gom-G) family of distributions. Several mathematical and statistical properties of the new TL-HT-Gom-G family of distributions including quantile function, moments, moment generating function, Reńyi entropy, distribution of order statistics, stochastic orderings are derived. Risk measures and a numerical simulation study are also presented for this family of distributions. To estimate the model parameters, we use six different estimation methods, namely, maximum likelihood, Anderson-Darling, Right-Tail Anderson-Darling, Ordinary Least Squares, Weighted Least Squares, and Cramér-von Mises. A simulation study is further performed to assess these estimation techniques and finally we demonstrate the applicability of the new family of distributions using applications to three real data sets.

Keywords: Topp-Leone-G, heavy-tailed, Gompertz-G, estimation methods, simulations, applications.

1. Introduction

As the world around us continues to evolve, unforeseen challenges will undoubtedly arise. With the availability of data, one of the efficient ways to tackle these never-ending problems is to make data-driven decisions. However, the sprawling amount of data presents its own set of challenges, one of which is its complexity. With the saturation of complex data, classical distributions such as normal, Poisson, beta, Weibull, gamma, and exponential often fail to fit real data well enough. Therefore, there is need for some modifications and generalizations of the well-known distributions to improve their flexibility. Some well-known generalized distributions proposed in the literature involving Topp-Leone, heavy-tailed and Gompertz transformations, respectively are Topp-Leone Cauchy family of distributions by Atchad et al. (2023), Topp-Leone-Harris-G family of distributions by Oluyede et al. (2023), Topp-Leone exponentiated Pareto distribution by Correa et al. (2024), type II Topp-Leone Burr XII distribution by Ogunde and Adeniji (2022), type II Topp-Leone Frechet distribution by Shanker and Rahman (2021), Topp-Leone Gompertz distribution by Nzei et al. (2020), heavy-tailed generalized Topp-Leone-G distribution by Moakofi et al. (2024), heavy-tailed log-logistic distribution by Teamah et al. (2021), the type I heavy-tailed odd power generalized Weibull-G family of distributions by Moakofi and Oluyede (2023), a new heavy tailed Weibull distribution by Ahmad et al. (2022), a new heavy-tailed exponentiated generalised-G family of distributions by Lekono et al. (2024), Gompertz flexible Weibull distribution by Khaleel et al. (2020), Gompertz ToppLeone inverse Rayleigh distribution by Khaleel and Hammed (2023), Gompertz Topp-Leone inverse exponential distribution by Hammed and Khaleel (2023), to mention a few.

Al-Shomrani et al. (2016) proposed the Topp-Leone-G (TL-G) family of distributions with the cumulative distribution function (cdf) and probability density function (pdf) given by

$$F_{TL-G}(x;b,\xi) = \left[1-\overline{G}^2(x;\xi)\right]^b, \quad (1)$$

and

$$f_{TL-G}(x;b,\xi) = 2b\left[1-\overline{G}^2(x;\xi)\right]^{b-1}\overline{G}(x;\xi)g(x;\xi),$$

respectively, for $b > 0$ and parameter vector ξ. Note that b is a shape parameter. $\overline{G}(x;\xi) = 1 - G(x;\xi)$, where $G(x;\xi)$ is the cdf of any baseline distribution with the parameter vector ξ.

The cdf and pdf of the type I heavy-tailed (TI-HT) family of distributions introduced by Zhao et al. (2020) are given by

$$F_{TI-HT-G}(x;\theta,\varphi) = 1-\left(\frac{1-G(x;\varphi)}{1-(1-\theta)G(x;\varphi)}\right)^\theta \quad (2)$$

and

$$f_{TI-HT-G}(x;\theta,\varphi) = \frac{\theta^2 g(x;\varphi)\left(1-G(x;\varphi)\right)^{\theta-1}}{\left(1-(1-\theta)G(x;\varphi)\right)^{\theta+1}}, \quad (3)$$

respectively, for $\theta > 0$, $x \in \mathbb{R}$ and parameter vector φ, where $G(x;\varphi)$ is the cdf of the baseline distribution.

Alizadeh et al. (2017) developed the Gompertz-G family of distributions with the cdf and pdf given by

$$F(x;\gamma,\delta,\psi) = 1-\exp\left(\frac{\delta}{\gamma}\left(1-[1-G(x;\psi)]^{-\gamma}\right)\right), \quad (4)$$

and

$$f(x;\gamma,\delta,\psi) = \delta\,[1-G(x;\psi)]^{-\gamma-1}\exp\left(\frac{\delta}{\gamma}\left(1-[1-G(x;\psi)]^{-\gamma}\right)\right)g(x;\psi),$$

respectively, for $\gamma,\delta > 0$ and parameter vector ψ. In this paper, we let $\delta = 1$.

The basic motivations for developing the Topp-Leone Heavy-Tailed Gompertz-G (TL-HT-Gom-G) family of distributions in practice include the following:

- to produce skewness for symmetrical models;
- to define special models with different shapes of hazard rate function;
- to construct heavy-tailed distributions for modeling real data;
- to provide consistently better fits than other generalized distributions with the same underlying model;
- to generalize some existing models in the literature.

The rest of the paper is outlined as follows. The proposed family and some of its statistical properties are discussed in Section 2. Mathematical and statistical properties of the new family of distributions are given in Section 3. In Section 4, some special cases of the new family of distributions are provided. Risk measures and numerical study are presented in Section 5. In Section 6, estimation of the model parameters is carried out using different estimation methods. Section 7 presents a simulation study. Applications with real-life data are given in Section 8 to demonstrate the usefulness of the proposed new family of distributions. Finally, in Section 9 we give some concluding remarks.

2. The New Family and Properties

In this section, we derive a new distribution, the Topp-Leone Heavy-Tailed Gompertz-G (TL-HT-Gom-G) family of distributions. Inserting Equations (2) and (4) into Equation (1) we obtain the cdf and hence pdf of the TL-HT-Gom-G family of distributions as

$$F(x;b,\theta,\gamma,\psi) = \left[1-\left(\frac{\exp\left[\frac{1}{\gamma}\left(1-\left[\bar{G}(x,\psi)\right]^{-\gamma}\right)\right]}{1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-\left[\bar{G}(x,\psi)\right]^{-\gamma}\right)\right]\right)}\right)^{2\theta}\right]^{b} \quad (5)$$

and

$$\begin{aligned} f(x;b,\theta,\gamma,\psi) &= 2\theta^2 b g(x,\psi)[\bar{G}(x,\psi)]^{-\gamma-1}\exp\left[\frac{2\theta}{\gamma}\left(1-\left[\bar{G}(x,\psi)\right]^{-\gamma}\right)\right] \\ &\times \left[1-\left(\frac{\exp\left[\frac{1}{\gamma}\left(1-\left[\bar{G}(x,\psi)\right]^{-\gamma}\right)\right]}{1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-\left[\bar{G}(x,\psi)\right]^{-\gamma}\right)\right]\right)}\right)^{2\theta}\right]^{b-1} \\ &\times \left(1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-\left[\bar{G}(x,\psi)\right]^{-\gamma}\right)\right]\right)\right)^{-(2\theta+1)}, \end{aligned} \quad (6)$$

for $b,\theta,\gamma>0$ and parameter vector ψ.

The hazard function (hrf) of the TL-HT-Gom-G family of distributions is given by

$$\begin{aligned} f(x;b,\theta,\gamma,\psi) &= 2\theta^2 b g(x,\psi)[\bar{G}(x,\psi)]^{-\gamma-1}\exp\left[\frac{2\theta}{\gamma}\left(1-\left[\bar{G}(x,\psi)\right]^{-\gamma}\right)\right] \\ &\times \left[1-\left(\frac{\exp\left[\frac{1}{\gamma}\left(1-\left[\bar{G}(x,\psi)\right]^{-\gamma}\right)\right]}{1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-\left[\bar{G}(x,\psi)\right]^{-\gamma}\right)\right]\right)}\right)^{2\theta}\right]^{b-1} \\ &\times \left(1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-\left[\bar{G}(x,\psi)\right]^{-\gamma}\right)\right]\right)\right)^{-(2\theta+1)} \\ &\times \left(1-\left[1-\left(\frac{\exp\left[\frac{1}{\gamma}\left(1-\left[\bar{G}(x,\psi)\right]^{-\gamma}\right)\right]}{1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-\left[\bar{G}(x,\psi)\right]^{-\gamma}\right)\right]\right)}\right)^{2\theta}\right]^{b}\right)^{-1}, \end{aligned}$$

for $b,\theta,\gamma>0$ and parameter vector ψ.

2.1. Sub-families

This sub-section presents several sub-families of the TL-HT-Gom-G family of distributions.

- When $b=1$, we obtain the new family of distributions family of distributions with the cdf

$$F(x;\theta,\gamma,\psi) = \left[1-\left(\frac{\exp\left[\frac{1}{\gamma}\left(1-\left[\bar{G}(x,\psi)\right]^{-\gamma}\right)\right]}{1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-\left[\bar{G}(x,\psi)\right]^{-\gamma}\right)\right]\right)}\right)^{2\theta}\right],$$

  for $\theta,\gamma>0$ and parameter vector ψ.

- Setting $\theta=1$, we obtain the Topp-Leone Gompertz-G (TL-Gom-G) family of distributions with the cdf

$$F(x;b,\gamma,\psi) = \left[1-\left(\exp\left[\frac{1}{\gamma}\left(1-\left[\bar{G}(x,\psi)\right]^{-\gamma}\right)\right]\right)^{2}\right]^{b},$$

  for $b,\gamma>0$ and parameter vector ψ. (See Oluyede et al. (2022)).

- When $\gamma = 1$, we obtain the new family of distributions with the cdf

$$F(x;b,\theta,\psi) = \left[1 - \left(\frac{\exp\left[\left(1-[\bar{G}(x,\psi)]^{-1}\right)\right]}{1-(1-\theta)\left(1-\exp\left[\left(1-[\bar{G}(x,\psi)]^{-1}\right)\right]\right)}\right)^{2\theta}\right]^{b},$$

for $b, \theta > 0$ and parameter vector ψ.

- When $b = \theta = 1$, we obtain the new family of distributions family of distributions with the cdf

$$F(x;\gamma,\psi) = \left[1 - \left(\exp\left[\frac{1}{\gamma}\left(1-[\bar{G}(x,\psi)]^{-\gamma}\right)\right]\right)^{2}\right],$$

for $\gamma > 0$ and parameter vector ψ.

- When $b = \gamma = 1$, we obtain the new family of distributions family of distributions with the cdf

$$F(x;\theta,\psi) = \left[1 - \left(\frac{\exp\left[\left(1-[\bar{G}(x,\psi)]^{-1}\right)\right]}{1-(1-\theta)\left(1-\exp\left[\left(1-[\bar{G}(x,\psi)]^{-1}\right)\right]\right)}\right)^{2\theta}\right],$$

for $\theta > 0$ and parameter vector ψ.

- Setting $\theta = \gamma = 1$, we find the new family of distributions with the cdf

$$F(x;b,\psi) = \left[1 - \left(\exp\left[\left(1-[\bar{G}(x,\psi)]^{-1}\right)\right]\right)^{2}\right]^{b},$$

for $b > 0$ and parameter vector ψ.

- When $b = \theta = \gamma = 1$, we obtain the new family of distributions with the cdf

$$F(x;\psi) = \left[1 - \left(\exp\left[\left(1-[\bar{G}(x,\psi)]^{-1}\right)\right]\right)^{2}\right],$$

for parameter vector ψ.

2.2. Expansion of the density function

In this section, we express the pdf of the TL-HT-Gom-G family of distributions as an infinite linear combination of the pdf of exponentiated-G (Exp-G) family of distributions. Note that after utilizing the following series expansions,

$$e^{z} = \sum_{k=0}^{\infty}\frac{z^k}{k!},\quad (1-z)^{k-1} = \sum_{j=0}^{\infty}(-1)^j\binom{k-1}{j}z^j,\quad \text{for } |z|<1$$

and

$$(1+z)^{-(k+1)} = \sum_{j=0}^{\infty}(-1)^j\binom{k+j}{j}z^j,\quad \text{for } |z|<1 \text{ and } k>0,$$

the pdf in Equation (5) can be written as

$$f(x;b,\theta,\gamma,\psi) = \sum_{p=0}^{\infty}\omega_{p+1}g_{p+1}(x;\psi), \tag{7}$$

where

$$\begin{aligned}\omega_{p+1} &= \sum_{i,j,k,l,m=0}^{\infty} 2\theta^2 b\binom{b-1}{i}\binom{2\theta(i+1)+j}{j}\binom{j}{k}(1-\theta)^j \\ &\times \binom{l}{m}\binom{\gamma(m+1)+p}{p}\frac{\left(\frac{2\theta(i+1)+k}{\gamma}\right)^l}{l!}\frac{(-1)^{i+k+m}}{(p+1)},\end{aligned} \tag{8}$$

and $g_{p+1}(x;\psi) = (p+1)G^p(x;\psi)g(x;\psi)$ is the exponentiated-G (Exp-G) pdf with power parameter $(p+1)$. (See Appendix for derivations). Thus, using the tractability property, we can obtain the properties of the TL-HT-Gom-G family of distributions from those of the Exp-G family of distributions.

2.3. Quantile function

In the sub-section, we obtain the quantile function of the TL-HT-Gom-G family of distributions by solving the non-linear equation:

$$F(x;b,\theta,\gamma,\psi) = \left[1 - \left(\frac{\exp\left[\frac{1}{\gamma}\left(1-\left[\bar{G}(x,\psi)\right]^{-\gamma}\right)\right]}{1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-\left[\bar{G}(x,\psi)\right]^{-\gamma}\right)\right]\right)}\right)^{2\theta}\right]^{b} = v,$$

for $0 \leq v \leq 1$, that is,

$$G(x;\psi) = \left(1 - \left[1 - \gamma \log\left(\theta\left[\left(1-v^{\frac{1}{b}}\right)^{\frac{-1}{2\theta}} - (1-\theta)\right]^{-1}\right)\right]^{\frac{-1}{\gamma}}\right).$$

Consequently, the quantile function of the TL-HT-Gom-G family of distributions is given by

$$Q(v) = G^{-1}\left(1 - \left[1 - \gamma \log\left(\theta\left[\left(1-v^{\frac{1}{b}}\right)^{\frac{-1}{2\theta}} - (1-\theta)\right]^{-1}\right)\right]^{\frac{-1}{\gamma}}\right). \quad (9)$$

Thus, random numbers can be obtained from the TL-HT-Gom-G family of distributions using Equation (9), for specified cdf G.

3. Mathematical and Statistical Properties

This section deals with some important properties of the TL-HT-Gom-G family of distributions. The properties include moments and generating function, Reńyi entropy, distribution of order statistics, and stochastic orderings. Let $f(x;b,\theta,\gamma,\psi) = f(x)$.

3.1. Moments and generating functions

The n^{th} moment of the TL-HT-Gom-G family of distributions can be obtained as:

$$E(X^n) = \int_{-\infty}^{\infty} x^n f(x)dx = \sum_{p=0}^{\infty} \omega_{p+1} E(Y_{p+1}^n),$$

where $E(Y_{p+1}^n)$ is the n^{th} moment of the Exp-G distribution with power parameter $(p+1)$, and ω_{p+1} is given in Equation (8). Other measures such as coefficient of skewness and coefficient of kurtosis can be readily obtained. Furthermore, the moment generating function (mgf) for $t < 1$ is

$$M_X(t) = \sum_{p=0}^{\infty} \omega_{p+1} M_{p+1}(t),$$

where $M_{p+1}(t)$ is the mgf of Y_{p+1}, and ω_{p+1} is defined in Equation (8).

3.2. Reńyi Entropy

Reńyi entropy plays an important role in information theory as a measure of uncertainity of a random variable. By letting $f(x;b,\theta,\gamma,\psi) = f(x)$, the Reńyi entropy of the TL-HT-Gom-G family of distributions is given by

$$I_R(s) = \frac{1}{1-s}\log\left(\int_0^{\infty} f^s(x;b,\theta,\gamma,\psi)dx\right), \quad s > 0 \text{ and } s \neq 1.$$

Note that using series expansion in Section 2.2, we have

$$f^s(x) = \left(2\theta^2 b\right)^s \sum_{i,j,k,l,m,p=0}^{\infty} \binom{s(b-1)}{i} \binom{2\theta(i+s)+s+j-1}{j} (-1)^{i+k+m} \times \binom{j}{k}\binom{l}{m}\binom{\gamma(m+s)+s+p-1}{p}(1-\theta)^j \frac{\left(\frac{2\theta(i+s)+k}{\gamma}\right)^l}{l!} G^p(x,\psi) g^s(x,\psi). \quad (10)$$

Consequently, the Reńyi entropy of the TL-HT-Gom-G family of distributions can be written as

$$\begin{aligned} I_R(s) &= \frac{1}{1-s}\log\Bigg[\sum_{i,j,k,l,m,p=0}^{\infty} \binom{s(b-1)}{i}\binom{2\theta(i+s)+s+j-1}{j}(-1)^{i+k+m} \\ &\times \binom{j}{k}\binom{l}{m}\binom{\gamma(m+s)+s+p-1}{p}(1-\theta)^j \\ &\times \frac{\left(2\theta^2 b\right)^s}{\left[1+\frac{p}{s}\right]^s}\int_0^{\infty}\left(\left[1+\frac{p}{s}\right] g(x;\psi) G^{\frac{p}{s}}(x;\psi)\right)^s dx\Bigg] \\ &= (1-s)^{-1}\log\left(\sum_{p=0}^{\infty} C_{p+1} e^{(1-s)I_{REG}}\right), \end{aligned} \quad (11)$$

where $I_{REG} = (1-s)^{-1}\log\left[\int_0^{\infty}\left[\left(1+\frac{p}{s}\right)G^{\frac{p}{s}}(x;\psi)g(x;\psi)\right]^s dx\right]$ is the Reńyi entropy of Exp-G distribution with power parameter $\left(1+\frac{p}{s}\right)$, and

$$C_{p+1} = \sum_{i,j,k,l,m=0}^{\infty}\binom{s(b-1)}{i}\binom{2\theta(i+s)+s+j-1}{j}(-1)^{i+k+m} \times \binom{j}{k}\binom{l}{m}\binom{\gamma(m+s)+s+p-1}{p}(1-\theta)^j \frac{\left(2\theta^2 b\right)^s}{\left[1+\frac{p}{s}\right]^s}.$$

Consequently, Reńyi entropy of the TL-HT-Gom-G family of distributions can be readily obtained from the exponentiated-G family of distributions.

3.3. Stochastic orderings

In this sub-section, we presents some stochastic orders for the TL-HT-Gom-G family of distributions. These include stochastic order, hazard rate order, and likelihood ratio order.

Suppose $F_X(t)$ and $F_Y(t)$ are the cdfs of two random variables X and Y, and define $\bar{F}_X(t) = 1-F_X(t)$ and $\bar{F}_Y(t) = 1-F_Y(t)$ as the corresponding survival functions. Then, the random variable X is said to be stochastically smaller than Y if, for all t, $\bar{F}_X(t) \le \bar{F}_Y(t)$ (or $F_X(t) \ge F_Y(t)$). It is represented by $X <_{st} Y$ or $X \preceq Y$. Moreover, if $\bar{F}_X(t) < \bar{F}_Y(t)$ for some t, then X is stochastically strictly less than Y and denoted as $X \prec Y$. In the case of hazard rate order, denoted by $X \preceq_{hr} Y$, $h_X(t) \ge h_Y(t)$ for all t. Similarly, X is said to be smaller than Y in the likelihood ratio order denoted by $X \preceq_{lr} Y$ if $\frac{f_X(t)}{f_Y(t)}$ is decreasing in t. It has been shown that $X \preceq_{lr} Y \implies X \preceq_{hr} Y \implies X \preceq Y$ (Szekli (2012)). It is also well know that if $X_1 \preceq_{hr} X_2$, then $X_1 \preceq_{mrl} X_2$, that is $m_{X_1}(x) \ge m_{X_2}(x)$ for all x, where $m_X(x) = E\left[(X-x) \mid X > x\right] = \frac{1}{\bar{F}(x)}\int_x^{\infty} yf(y)dx - x$.

Theorem 1 *Consider two independent random variables, denoted by X_1 and X_2, which follow the TL-HT-Gom-G family of distributions, that is, $X_1 \sim f_1(x; b_1, \theta, \gamma, \psi)$ and $X_2 \sim f_2(x; b_2, \theta, \gamma, \psi)$. If $b_2 > b_1$, then X_1 and X_2 are stochastically ordered, and $X_1 \preceq_{mrl} X_2$.*

Proof: Note that we can write the pdf's of X_1 and X_2 as follows:

$$\begin{aligned} f_1(x) = f_1(x; b_1, \theta, \gamma, \psi) &= 2\theta^2 b_1 g(x,\psi)[\bar{G}(x,\psi)]^{-\gamma-1} \exp\left[\frac{2\theta}{\gamma}\left(1 - [\bar{G}(x,\psi)]^{-\gamma}\right)\right] \\ &\times \left[1 - \left(\frac{\exp\left[\frac{1}{\gamma}\left(1-[\bar{G}(x,\psi)]^{-\gamma}\right)\right]}{1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-[\bar{G}(x,\psi)]^{-\gamma}\right)\right]\right)}\right)^{2\theta}\right]^{b_1-1} \\ &\times \left(1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-[\bar{G}(x,\psi)]^{-\gamma}\right)\right]\right)\right)^{-(2\theta+1)}, \end{aligned}$$

and

$$\begin{aligned} f_2(x) = f_2(x; b_2, \theta, \gamma, \psi) &= 2\theta^2 b_2 g(x,\psi)[\bar{G}(x,\psi)]^{-\gamma-1} \exp\left[\frac{2\theta}{\gamma}\left(1 - [\bar{G}(x,\psi)]^{-\gamma}\right)\right] \\ &\times \left[1 - \left(\frac{\exp\left[\frac{1}{\gamma}\left(1-[\bar{G}(x,\psi)]^{-\gamma}\right)\right]}{1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-[\bar{G}(x,\psi)]^{-\gamma}\right)\right]\right)}\right)^{2\theta}\right]^{b_2-1} \\ &\times \left(1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-[\bar{G}(x,\psi)]^{-\gamma}\right)\right]\right)\right)^{-(2\theta+1)}, \end{aligned}$$

respectively. Then, the ratio

$$\frac{f_1(x)}{f_2(x)} = \frac{b_1}{b_2}\left[1 - \left(\frac{\exp\left[\frac{1}{\gamma}\left(1-[\bar{G}(x,\psi)]^{-\gamma}\right)\right]}{1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-[\bar{G}(x,\psi)]^{-\gamma}\right)\right]\right)}\right)^{2\theta}\right]^{b_1-b_2}. \quad (12)$$

Differentiating Equation (12) with respect to x yields

$$\begin{aligned} \frac{d}{dx}\left(\frac{f_1(x)}{f_2(x)}\right) = &\frac{b_1(b_1-b_2)}{b_2}\left[1 - \left(\frac{\exp\left[\frac{1}{\gamma}\left(1-[\bar{G}(x,\psi)]^{-\gamma}\right)\right]}{1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-[\bar{G}(x,\psi)]^{-\gamma}\right)\right]\right)}\right)^{2\theta}\right]^{b_1-b_2-1} \\ &\times 2\theta^2 g(x,\psi)[\bar{G}(x,\psi)]^{-\gamma-1}\exp\left[\frac{2\theta}{\gamma}\left(1-[\bar{G}(x,\psi)]^{-\gamma}\right)\right] \\ &\times \left(1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-[\bar{G}(x,\psi)]^{-\gamma}\right)\right]\right)\right)^{-(2\theta+1)}, \end{aligned} \quad (13)$$

which is negative if $b_2 > b_1$. Therefore, likelihood ratio order $X \preceq_{lr} Y$ exists, and we can conclude that the random variables X_1 and X_2 are stochastically ordered, and $X_1 \preceq_{mrl} X_2$.

3.4. Order statistics

Order statistics are useful in survival analysis, reliability theory, probability and statistics. Let $X_1, X_2, ..., X_n$ be independent and identically distributed random variables from the TL-HT-Gom-G family of distributions. Then, the pdf of the r^{th} order statistic from the TL-HT-Gom-G family of distributions is given by

$$f_{r:n}(x) = \frac{n! f(x)}{(r-1)!(n-r)!}\sum_{q=0}^{n-r}(-1)^q\binom{n-r}{q}[F(x)]^{q+r-1}. \quad (14)$$

Note that

$$f(x)[F(x)]^{q+r-1} = 2\theta^2 b g(x,\psi)[\bar{G}(x,\psi)]^{-\gamma-1}\exp\left[\frac{2\theta}{\gamma}\left(1-[\bar{G}(x,\psi)]^{-\gamma}\right)\right]$$

$$\times \left[1-\left(\frac{\exp\left[\frac{1}{\gamma}\left(1-\left[\bar{G}(x,\psi)\right]^{-\gamma}\right)\right]}{1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-\left[\bar{G}(x,\psi)\right]^{-\gamma}\right)\right]\right)}\right)^{2\theta}\right]^{b(q+r)-1}$$
$$\times \left(1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-\left[\bar{G}(x,\psi)\right]^{-\gamma}\right)\right]\right)\right)^{-(2\theta+1)}.$$

Now following the same steps leading to Equation (7), we obtain

$$f(x)[F(x)]^{q+p-1} = \sum_{p=0}^{\infty}\omega^*_{p+1}g_{p+1}(x;\psi), \tag{15}$$

where
$$\omega^*_{p+1} = \sum_{i,j,k,l,m=0}^{\infty} 2\theta^2 b\binom{b(q+r)-1}{i}\binom{2\theta(i+1)+j}{j}\binom{j}{k}(1-\theta)^j \times \binom{l}{m}\binom{\gamma(m+1)+p}{p}\frac{\left(\frac{2\theta(i+1)+k}{\gamma}\right)^l}{l!}\frac{(-1)^{i+k+m}}{(p+1)},$$

and $g_{p+1}(x;\psi) = (p+1)G^p(x;\psi)g(x;\psi)$ is the Exp-G pdf with power parameter $(p+1)$.

Thus, by substituting Equation (15) into Equation (14), the pdf of the r^{th} order statistic for the TL-HT-Gom-G family of distributions can be written as

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}\sum_{q=0}^{n-r}\sum_{p=0}^{\infty}(-1)^q\binom{n-r}{q}\omega^*_{p+1}g_{p+1}(x;\psi).$$

4. Some Special Cases

This section contain some special cases of the TL-HT-Gom-G family of distributions when the baseline distribution is specified. We consider the cases when the baseline distributions are log-logistic, Weibull and Burr III distributions.

4.1. Topp-Leone Heavy-Tailed Gompertz-Log-Logistic (TL-HT-Gom-LLoG) distribution

Consider the log-logistic distribution as the baseline distribution with parameter $c > 0$ having cdf and pdf $G(x;c) = 1-(1+x^c)^{-1}$ and $g(x;c) = cx^{c-1}(1+x^c)^{-2}$, respectively. Then, the cdf and pdf of TL-HT-Gom-LLoG distribution are given by

$$F(x;b,\theta,\gamma,c) = \left[1-\left(\frac{\exp\left[\frac{1}{\gamma}\left(1-\left[(1+x^c)^{-1}\right]^{-\gamma}\right)\right]}{1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-[(1+x^c)^{-1}]^{-\gamma}\right)\right]\right)}\right)^{2\theta}\right]^{b}$$

and

$$f(x;b,\theta,\gamma,c) = 2\theta^2 bcx^{c-1}(1+x^c)^{-2}[(1+x^c)^{-1}]^{-\gamma-1}\exp\left[\frac{2\theta}{\gamma}\left(1-\left[(1+x^c)^{-1}\right]^{-\gamma}\right)\right]$$
$$\times \left[1-\left(\frac{\exp\left[\frac{1}{\gamma}\left(1-\left[(1+x^c)^{-1}\right]^{-\gamma}\right)\right]}{1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-[(1+x^c)^{-1}]^{-\gamma}\right)\right]\right)}\right)^{2\theta}\right]^{b-1}$$
$$\times \left(1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-\left[(1+x^c)^{-1}\right]^{-\gamma}\right)\right]\right)\right)^{-(2\theta+1)},$$

for $b,\theta,\gamma,c > 0$.

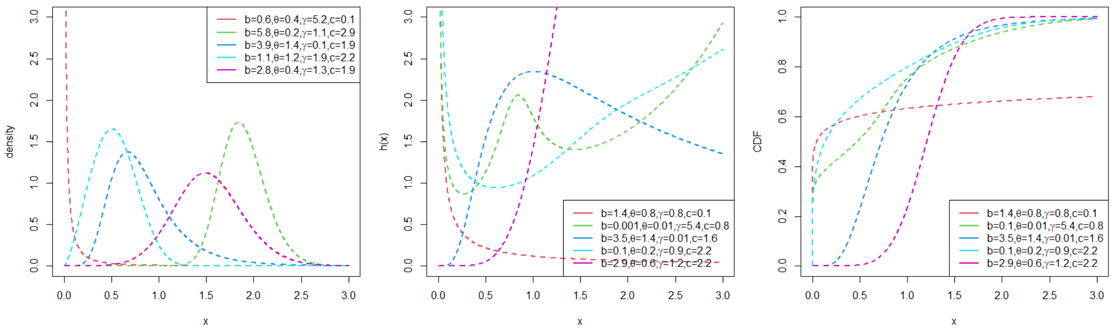

Figure 1 Plots of the pdf, hrf and cdf for TL-HT-Gom-LLoG distribution

Figure 1 illustrates the flexibility of the TL-HT-Gom-LLoG distribution. The pdf of the TL-HT-Gom-LLoG distribution can take various shapes that include reverse-J, almost symmetric, left-skewed and right-skewed. The hrf of the TL-HT-Gom-LLoG distribution exhibit decreasing, increasing, bathtub, upside down bathtub and bathtub followed by upside down bathtub followed by bathtub shapes.

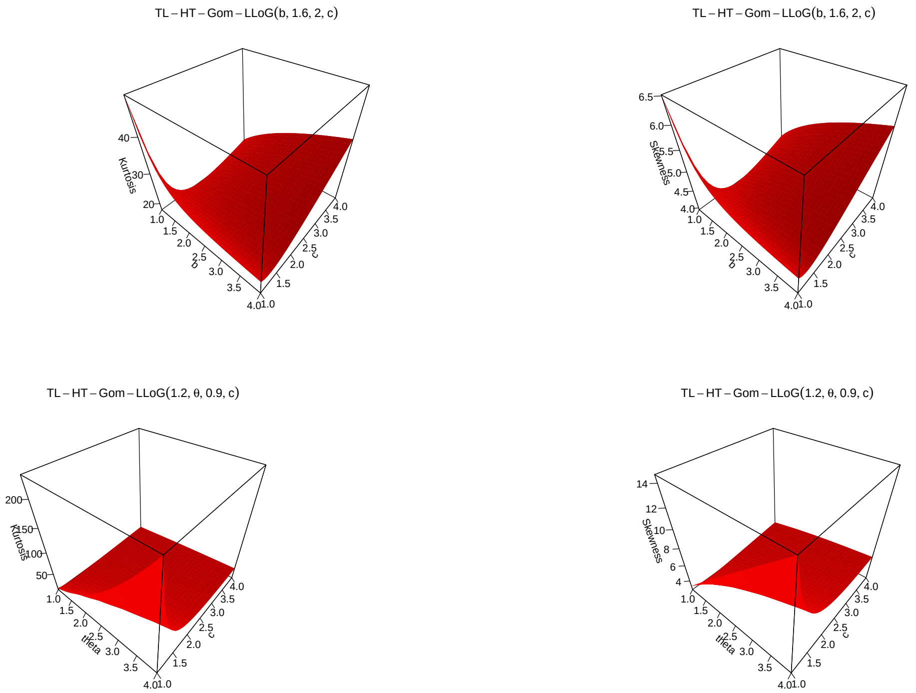

Figure 2 3D-Plots of the skewness and kurtosis for TL-HT-Gom-LLoG distribution

Figure 2 shows plots of skewness and kurtosis for the TL-HT-Gom-LLoG distribution. We can see that for fixed value of θ and γ, skewness and kurtosis decreases and increases when b and θ changes. On another note, when we fix b and γ, skewness becomes positive and kurtosis is leptokurtic when θ and c increases.

4.2. Topp-Leone Heavy-Tailed Gompertz-Weibull (TL-HT-Gom-W) distribution

Let the one parameter Weibull distribution be the baseline distribution with pdf and cdf given by $g(x;\lambda) = \lambda x^{\lambda-1}\exp(-x^{\lambda})$ and $G(x;\lambda) = 1-\exp(-x^{\lambda})$, for $\lambda > 0$, respectively. Then, the cdf and pdf of TL-HT-Gom-W distribution are given by

$$F(x;b,\theta,\gamma,\lambda) = \left[1-\left(\frac{\exp\left[\frac{1}{\gamma}\left(1-\left[\exp(-x^{\lambda})\right]^{-\gamma}\right)\right]}{1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-\left[\exp(-x^{\lambda})\right]^{-\gamma}\right)\right]\right)}\right)^{2\theta}\right]^{b}$$

and

$$\begin{aligned} f(x;b,\theta,\gamma,\lambda) &= 2\theta^2 b\lambda x^{\lambda-1}\exp(-x^{\lambda})[\exp(-x^{\lambda})]^{-\gamma-1}\exp\left[\frac{2\theta}{\gamma}\left(1-\left[\exp(-x^{\lambda})\right]^{-\gamma}\right)\right] \\ &\times \left[1-\left(\frac{\exp\left[\frac{1}{\gamma}\left(1-\left[\exp(-x^{\lambda})\right]^{-\gamma}\right)\right]}{1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-\left[\exp(-x^{\lambda})\right]^{-\gamma}\right)\right]\right)}\right)^{2\theta}\right]^{b-1} \\ &\times \left(1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-\left[\exp(-x^{\lambda})\right]^{-\gamma}\right)\right]\right)\right)^{-(2\theta+1)}, \end{aligned}$$

for $b,\theta,\gamma,\lambda > 0$.

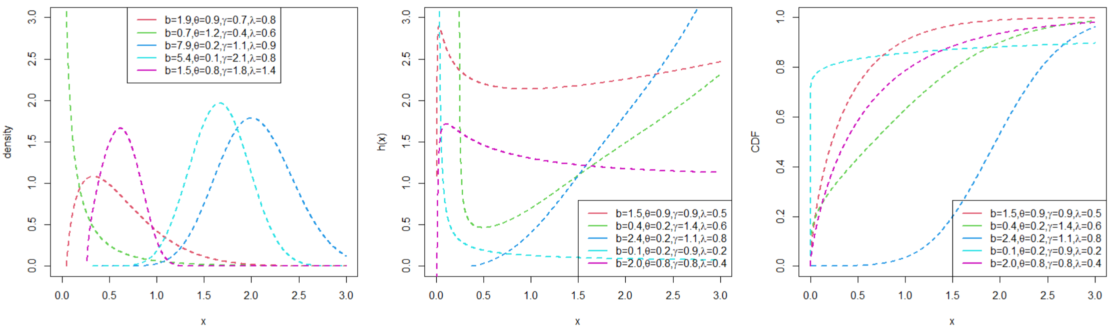

Figure 3 Plots of the pdf, hrf and cdf for TL-HT-Gom-W distribution

Figure 3 shows the flexibility of the pdf and hrf of the TL-HT-Gom-W distribution. The pdf can take several shapes including almost symmetric, reverse-J, left-skewed and right-skewed. Furthermore, plots of the hrf for the TL-HT-Gom-W distribution display increasing, decreasing, bathtub, upside-down bathtub, and upside-down bathtub followed by bathtub shapes.

4.3. Topp-Leone Heavy-Tailed Gompertz-Burr III (TL-HT-Gom-BIII) distribution

Consider the Burr III distribution as the baseline distribution with parameter $c,\lambda > 0$ having cdf and pdf $G(x;c,\lambda) = (1+x^{-c})^{-\lambda}$ and $g(x;c,\lambda) = c\lambda x^{-(c+1)}(1+x^{-c})^{-(\lambda+1)}$, respectively. Then, the cdf and pdf of TL-HT-Gom-BIII distribution are given by

$$F(x;b,\theta,\gamma,c,\lambda) = \left[1-\left(\frac{\exp\left[\frac{1}{\gamma}\left(1-\left[1-(1+x^{-c})^{-\lambda}\right]^{-\gamma}\right)\right]}{1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-\left[1-(1+x^{-c})^{-\lambda}\right]^{-\gamma}\right)\right]\right)}\right)^{2\theta}\right]^{b}$$

and

$$\begin{aligned} f(x;b,\theta,\gamma,c,\lambda) &= 2\theta^2 bc\lambda x^{-(c+1)}(1+x^{-c})^{-(\lambda+1)}[1-(1+x^{-c})^{-\lambda}]^{-\gamma-1} \\ &\times \exp\left[\frac{2\theta}{\gamma}\left(1-\left[1-(1+x^{-c})^{-\lambda}\right]^{-\gamma}\right)\right] \\ &\times \left[1-\left(\frac{\exp\left[\frac{1}{\gamma}\left(1-\left[1-(1+x^{-c})^{-\lambda}\right]^{-\gamma}\right)\right]}{1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-[1-(1+x^{-c})^{-\lambda}]^{-\gamma}\right)\right]\right)}\right)^{2\theta}\right]^{b-1} \\ &\times \left(1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-\left[1-(1+x^{-c})^{-\lambda}\right]^{-\gamma}\right)\right]\right)\right)^{-(2\theta+1)}, \end{aligned}$$

for $b, \theta, \gamma, c, \lambda > 0$.

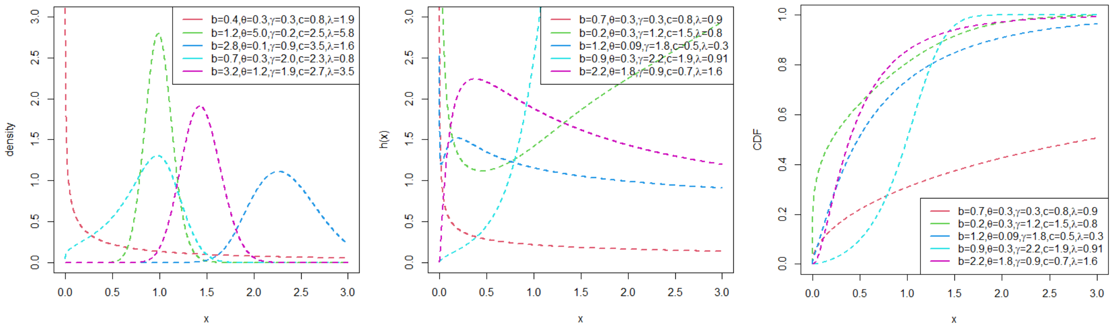

Figure 4 Plots of the pdf, hrf and cdf for TL-HT-Gom-BIII distribution

Figure 4 show the plots of the pdf and hrf of the TL-HT-Gom-BIII distribution. The pdf can take various shapes that include almost symmetric, reverse-J, left or right-skewed. Furthermore, the graphs of the hrf for the TL-HT-Gom-BIII distribution exhibit increasing, decreasing, bathtub, upside-down bathtub and bathtub followed by upside-down bathtub shapes.

5. Estimation Methods

In this section, we estimate the parameters of the TL-HT-Gom-G family of distributions by utilizing different estimation methods. These methods include Maximum Likelihood (ML), Anderson-Darling (AD), Right-Tail Anderson-Darling (RAD), Ordinary Least Squares (OLS), Weighted Least Squares (WLS) and Cramér-von Mises (CVM).

5.1. Maximum likelihood estimation

Let $X \sim$ TL-HT-Gom-G$(b, \theta, \gamma, \psi)$ and $\tau = (b, \theta, \gamma, \psi)^T$ be the vector of model parameters, then the log-likelihood function $\ell_n = \ell_n(\tau)$ based on a random sample of size n from the TL-HT-Gom-G family of distributions is given by

$$\begin{aligned} \ell(\tau) &= (n)\ln(2\theta^2 b) - (\gamma+1)\sum_{i=1}^{n}\ln[\bar{G}(x_i,\psi)] + \sum_{i=1}^{n}\ln[g(x_i,\psi)] \\ &+ (b-1)\sum_{i=1}^{n}\ln\left[1-\left(\frac{\exp\left[\frac{1}{\gamma}\left(1-[\bar{G}(x_i,\psi)]^{-\gamma}\right)\right]}{1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-[\bar{G}(x_i,\psi)]^{-\gamma}\right)\right]\right)}\right)^{2\theta}\right] \\ &- (2\theta+1)\sum_{i=1}^{n}\ln\left(1-(1-\theta)\left(1-\exp\left[\frac{1}{\gamma}\left(1-[\bar{G}(x_i,\psi)]^{-\gamma}\right)\right]\right)\right) \end{aligned}$$

$$+ \quad \sum_{i=1}^{n}\left[\frac{2\theta}{\gamma}\left(1-\left[\bar{G}(x_i,\psi)\right]^{-\gamma}\right)\right].$$

In order to obtain the maximum likelihood estimates (MLEs) of the unknown parameters from the TL-HT-Gom-G family of distributions, we solve $U = (\frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \theta}, \frac{\partial \ell_n}{\partial \gamma}, \frac{\partial \ell_n}{\partial \psi_k})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The elements of the score vector U are given in the appendix.

5.2. Anderson-Darling estimation

Suppose $x_{(1)}, x_{(2)}, ..., x_{(n)}$ are the order statistics of a random sample of size n from the TL-HT-Gom-G family of distributions. Then, the Anderson-Darling estimates (ADEs) of the TL-HT-Gom-G family of distributions are obtained by minimizing the function

$$A\left(b,\theta,\gamma,\psi\right) = -n - \frac{1}{n}\sum_{i=1}^{n}\left(2i-1\right)\left[\log\left(F(x_{(i)};b,\theta,\gamma,\psi)\right) + \log\left(S(x_{(i)};b,\theta,\gamma,\psi)\right)\right],$$

where $F(x_{(i)};b,\theta,\gamma,\psi)$ and $S(x_{(i)};b,\theta,\gamma,\psi)$ be the cdf and survival function of the i^{th} order statistic from the TL-HT-Gom-G family of distributions.

The ADEs can also be derived by solving the non-linear equations:

$$\sum_{i=1}^{n}\left(2i-1\right)\left[\frac{\vartheta_z\left(x_{(i)};b,\theta,\gamma,\psi\right)}{F(x_{(i)};b,\theta,\gamma,\psi)} - \frac{\vartheta_z\left(x_{(n+1-i)};b,\theta,\gamma,\psi\right)}{S\left(x_{(n+1-i)};b,\theta,\gamma,\psi\right)}\right] = 0, z = 1,2,3,4,$$

$$\begin{aligned} \text{where} \quad \vartheta_1\left(x_{(i)};b,\theta,\gamma,\psi\right) &= \frac{\partial F\left(x_{(i)};b,\theta,\gamma,\psi\right)}{\partial b}, \\ \vartheta_2\left(x_{(i)};b,\theta,\gamma,\psi\right) &= \frac{\partial F\left(x_{(i)};b,\theta,\gamma,\psi\right)}{\partial \theta}, \\ \vartheta_3\left(x_{(i)};b,\theta,\gamma,\psi\right) &= \frac{\partial F\left(x_{(i)};b,\theta,\gamma,\psi\right)}{\partial \gamma}, \\ \text{and} \quad \vartheta_4\left(x_{(i)};b,\theta,\gamma,\psi\right) &= \frac{\partial F\left(x_{(i)};b,\theta,\gamma,\psi\right)}{\partial \psi_k}. \end{aligned} \tag{16}$$

5.3. Right-Tail Anderson-Darling estimation

Right-Tail Anderson-Darling estimates (RADEs) of the TL-HT-Gom-G family of distributions are determined by minimizing

$$R(b,\alpha,\varphi) = \frac{n}{2} - 2\sum_{i=1}^{n} F(x_{(i)};b,\theta,\gamma,\psi) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S(x_{(n-i+1)};b,\theta,\gamma,\psi).$$

The RADEs may also be obtained by solving the non-linear equation

$$-2\sum_{i=1}^{n}\vartheta_z\left(x_{(i)};b,\theta,\gamma,\psi\right) + \frac{1}{n}\sum_{i=1}^{n}\left(2i-1\right)\frac{\vartheta_z\left(x_{(i)};b,\theta,\gamma,\psi\right)}{S(x_{n+1-i:n};b,\theta,\gamma,\psi)} = 0,$$

where $\vartheta_z\left(x_{(i)};b,\theta,\gamma,\psi\right)$ are defined in Equation (16).

5.4. Ordinary least squares estimation

The Ordinary Least Squares estimates (OLSEs) of the parameters of the TL-HT-Gom-G family of distributions are obtained by minimizing the function

$$V\left(b,\theta,\gamma,\psi\right) = \sum_{i=1}^{n}\left[F\left(x_{(i)};b,\theta,\gamma,\psi\right) - \frac{i}{n+1}\right]^2.$$

The OLSEs can be obtained by solving the non-linear equations:

$$\sum_{i=1}^{n}\left[F\left(x_{(i)};b,\theta,\gamma,\psi\right)-\frac{i}{n+1}\right]\vartheta_z\left(x_{(i)};b,\theta,\gamma,\psi\right)=0, z=1,2,3,4,$$

where $\vartheta_z\left(x_{(i)};b,\theta,\gamma,\psi\right)$ are defined in Equation (16).

5.5. Weighted least squares estimation

The Weighted Least Squares estimates (WLSEs) of the parameters of the TL-HT-Gom-G family of distributions are obtained by minimizing the function

$$W\left(b,\theta,\gamma,\psi\right)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-1+1)}\left[F\left(x_{(i)};b,\theta,\gamma,\psi\right)-\frac{i}{n+1}\right]^2,$$

with respect to b,θ,γ and parameter vector ψ. The WLSEs can be obtained by solving the non-linear equations:

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-1+1)}\left[F\left(x_{(i)};b,\theta,\gamma,\psi\right)-\frac{i}{n+1}\right]\vartheta_z\left(x_{(i)};b,\theta,\gamma,\psi\right)=0, z=1,2,3,4,$$

where $\vartheta_z\left(x_{(i)};b,\theta,\gamma,\psi\right)$ are defined in Equation (16).

5.6. Cramér-von Mises estimation

The Cramér-von Mises estimates (CVMEs) of the parameters of the TL-HT-Gom-G family of distributions parameters are obtained through the minimization of the function

$$C\left(b,\theta,\gamma,\psi\right)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{(i)};b,\theta,\gamma,\psi\right)-\frac{2i-1}{2n}\right]^2,$$

with respect to b,θ,γ and parameter vector ψ. The CVMEs can also be obtained by solving the non-linear equations

$$\sum_{i=1}^{n}\left[F\left(x_{(i)};b,\theta,\gamma,\psi\right)-\frac{2i-1}{2n}\right]\vartheta_z\left(x_{(i)};b,\theta,\gamma,\psi\right)=0, z=1,2,3,4,$$

where $\vartheta_z\left(x_{(i)};b,\theta,\gamma,\psi\right)$ are defined in Equation (16).

6. Simulation

In this section, a Monte Carlo simulation study is employed to assess the consistency property of six estimation methods in estimating the parameters of the TL-HT-Gom-LLoG distribution. Random samples of sizes $n = 25, 50, 100, 200, 400,$ and 800, were generated from the TL-HT-Gom-LLoG distribution and repeated 3000 times.

The average bias (ABias) and root mean square error (RMSE) are computed to assess the efficiency of the different estimation methods. The ABias and RMSE for the estimated parameter, say, $\hat{\lambda}$, are given by:

$$ABias(\hat{\lambda})=\frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}_i-\lambda), \quad \text{and} \quad RMSE(\hat{\lambda})=\sqrt{\frac{\sum_{i=1}^{N}(\hat{\lambda}_i-\lambda)^2}{N}},$$

respectively.

Table 1 Simulation Results for Different Estimation Methods for $b = 0.4, \theta = 0.4, \gamma = 1.6, c = 2.3$

		MLE		LS		WLS		RADE		CVME		ADE	
n	Parameter	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE
25	b	$0.0194^{\{1\}}$	$0.2132^{\{1\}}$	$-0.8667^{\{6\}}$	$0.9716^{\{6\}}$	$-0.5283^{\{2\}}$	$0.5359^{\{2\}}$	$-0.5347^{\{3\}}$	$0.8568^{\{5\}}$	$-0.5600^{\{4\}}$	$0.5651^{\{3\}}$	$-0.6747^{\{5\}}$	$0.7169^{\{4\}}$
	θ	$-0.0429^{\{1\}}$	$0.1632^{\{1\}}$	$-0.9250^{\{6\}}$	$1.0024^{\{6\}}$	$-0.8250^{\{4\}}$	$0.9376^{\{5\}}$	$-0.5006^{\{3\}}$	$1.4953^{\{2\}}$	$-0.9175^{\{5\}}$	$0.9920^{\{4\}}$	$-0.4182^{\{2\}}$	$0.8610^{\{2\}}$
	γ	$0.1627^{\{1\}}$	$1.0804^{\{3\}}$	$-0.9098^{\{3\}}$	$1.2763^{\{4\}}$	$-0.5618^{\{2\}}$	$0.6466^{\{1\}}$	$0.9557^{\{5\}}$	$3.4254^{\{6\}}$	$-0.9160^{\{4\}}$	$0.9311^{\{2\}}$	$-1.5565^{\{6\}}$	$1.5571^{\{5\}}$
	c	$-0.0511^{\{1\}}$	$0.2024^{\{1\}}$	$-0.3232^{\{4\}}$	$0.3447^{\{3\}}$	$-0.423^{\{5\}}$	$0.4969^{\{4\}}$	$0.1436^{\{2\}}$	$1.1868^{\{6\}}$	$-0.5300^{\{6\}}$	$0.6530^{\{5\}}$	$-0.2819^{\{3\}}$	$0.2820^{\{2\}}$
$\sum$ ranks		10		38		25		32		33		29	
50	b	$-0.0381^{\{1\}}$	$0.1152^{\{1\}}$	$-0.6539^{\{6\}}$	$0.6569^{\{4\}}$	$-0.4330^{\{2\}}$	$0.4407^{\{2\}}$	$-0.4910^{\{3\}}$	$0.8525^{\{6\}}$	$-0.5248^{\{4\}}$	$0.5555^{\{3\}}$	$-0.5923^{\{5\}}$	$0.6863^{\{5\}}$
	θ	$0.1527^{\{1\}}$	$0.8922^{\{5\}}$	$-0.4774^{\{5\}}$	$0.5441^{\{3\}}$	$-0.6228^{\{6\}}$	$0.6402^{\{4\}}$	$-0.4019^{\{3\}}$	$1.3564^{\{6\}}$	$-0.4751^{\{4\}}$	$0.5416^{\{2\}}$	$-0.3439^{\{2\}}$	$0.5399^{\{1\}}$
	γ	$-0.0283^{\{1\}}$	$0.1506^{\{1\}}$	$-0.6652^{\{4\}}$	$0.9896^{\{4\}}$	$-0.4017^{\{2\}}$	$0.6089^{\{2\}}$	$0.6043^{\{3\}}$	$3.0331^{\{4\}}$	$-0.6683^{\{5\}}$	$0.9015^{\{5\}}$	$-0.9954^{\{6\}}$	$1.0930^{\{6\}}$
	c	$0.0165^{\{1\}}$	$0.1760^{\{1\}}$	$-0.2177^{\{4\}}$	$0.2731^{\{2\}}$	$-0.3344^{\{5\}}$	$0.3793^{\{4\}}$	$0.1085^{\{2\}}$	$1.0874^{\{6\}}$	$-0.4249^{\{6\}}$	$0.4754^{\{5\}}$	$-0.1792^{\{3\}}$	$0.2961^{\{3\}}$
$\sum$ ranks		12		32		27		33		34		31	
100	b	$-0.0353^{\{1\}}$	$0.0824^{\{1\}}$	$-0.4351^{\{6\}}$	$0.5446^{\{5\}}$	$-0.3992^{\{5\}}$	$0.4134^{\{2\}}$	$-0.3045^{\{3\}}$	$0.8031^{\{6\}}$	$-0.3800^{\{4\}}$	$0.4391^{\{3\}}$	$-0.3014^{\{2\}}$	$0.5202^{\{4\}}$
	θ	$0.0756^{\{1\}}$	$0.7122^{\{5\}}$	$-0.3601^{\{5\}}$	$0.3615^{\{1\}}$	$-0.5717^{\{6\}}$	$0.6360^{\{4\}}$	$-0.3200^{\{3\}}$	$1.2997^{\{6\}}$	$-0.3600^{\{4\}}$	$0.3735^{\{2\}}$	$-0.2076^{\{2\}}$	$0.4083^{\{3\}}$
	γ	$-0.0138^{\{1\}}$	$0.1047^{\{1\}}$	$-0.5083^{\{5\}}$	$0.7093^{\{3\}}$	$-0.3583^{\{2\}}$	$0.5971^{\{2\}}$	$0.4950^{\{3\}}$	$2.0760^{\{6\}}$	$-0.5075^{\{4\}}$	$0.8086^{\{5\}}$	$-0.5414^{\{6\}}$	$0.7425^{\{4\}}$
	c	$-0.0122^{\{1\}}$	$0.1379^{\{1\}}$	$-0.1533^{\{4\}}$	$0.1928^{\{3\}}$	$-0.2351^{\{5\}}$	$0.3670^{\{5\}}$	$0.0844^{\{3\}}$	$0.4135^{\{6\}}$	$-0.2842^{\{6\}}$	$0.2924^{\{4\}}$	$-0.0769^{\{2\}}$	$0.1770^{\{2\}}$
$\sum$ ranks		12		32		31		36		32		25	
200	b	$-0.0261^{\{1\}}$	$0.0587^{\{1\}}$	$-0.3550^{\{6\}}$	$0.3649^{\{4\}}$	$-0.2789^{\{4\}}$	$0.3851^{\{5\}}$	$-0.2060^{\{2\}}$	$0.7835^{\{6\}}$	$-0.3330^{\{5\}}$	$0.3424^{\{3\}}$	$-0.2199^{\{3\}}$	$0.3220^{\{2\}}$
	θ	$0.0145^{\{1\}}$	$0.5732^{\{5\}}$	$-0.2033^{\{4\}}$	$0.3368^{\{3\}}$	$-0.2979^{\{6\}}$	$0.3568^{\{4\}}$	$-0.1073^{\{2\}}$	$0.7405^{\{6\}}$	$-0.2632^{\{5\}}$	$0.3288^{\{2\}}$	$-0.1261^{\{3\}}$	$0.3018^{\{1\}}$
	γ	$-0.0081^{\{1\}}$	$0.0667^{\{1\}}$	$-0.4020^{\{6\}}$	$0.6340^{\{4\}}$	$-0.2676^{\{2\}}$	$0.5866^{\{3\}}$	$-0.3564^{\{5\}}$	$1.6679^{\{6\}}$	$-0.3392^{\{4\}}$	$0.5339^{\{2\}}$	$-0.3243^{\{3\}}$	$0.6491^{\{5\}}$
	c	$-0.0081^{\{1\}}$	$0.1165^{\{2\}}$	$-0.0832^{\{4\}}$	$0.1734^{\{3\}}$	$-0.1875^{\{5\}}$	$0.2223^{\{4\}}$	$0.0758^{\{3\}}$	$0.3745^{\{6\}}$	$-0.2783^{\{6\}}$	$0.2831^{\{5\}}$	$-0.0744^{\{2\}}$	$0.0845^{\{1\}}$
$\sum$ ranks		13		34		33		36		32		20	
400	b	$-0.0168^{\{1\}}$	$0.0419^{\{1\}}$	$-0.2435^{\{6\}}$	$0.2486^{\{4\}}$	$-0.2002^{\{4\}}$	$0.2627^{\{5\}}$	$-0.1959^{\{3\}}$	$0.5835^{\{6\}}$	$-0.2396^{\{5\}}$	$0.2449^{\{3\}}$	$-0.1333^{\{2\}}$	$0.2365^{\{2\}}$
	θ	$-0.0037^{\{1\}}$	$0.4517^{\{5\}}$	$-0.1033^{\{4\}}$	$0.2031^{\{2\}}$	$-0.2243^{\{6\}}$	$0.2530^{\{3\}}$	$-0.0936^{\{2\}}$	$0.5787^{\{6\}}$	$-0.2034^{\{5\}}$	$0.3034^{\{4\}}$	$-0.0948^{\{3\}}$	$0.1990^{\{1\}}$
	γ	$-0.0073^{\{1\}}$	$0.0499^{\{1\}}$	$-0.3416^{\{6\}}$	$0.5341^{\{5\}}$	$-0.1964^{\{2\}}$	$0.4731^{\{2\}}$	$-0.2800^{\{4\}}$	$0.8666^{\{6\}}$	$-0.2821^{\{5\}}$	$0.5003^{\{3\}}$	$-0.2304^{\{3\}}$	$0.5232^{\{4\}}$
	c	$-0.0079^{\{1\}}$	$0.0886^{\{2\}}$	$-0.0785^{\{4\}}$	$0.0966^{\{3\}}$	$-0.1450^{\{5\}}$	$0.1462^{\{4\}}$	$0.0530^{\{3\}}$	$0.1495^{\{5\}}$	$-0.1786^{\{6\}}$	$0.2086^{\{6\}}$	$-0.0272^{\{2\}}$	$0.0759^{\{1\}}$
$\sum$ ranks		13		34		31		35		37		18	
800	b	$-0.0083^{\{1\}}$	$0.0283^{\{1\}}$	$-0.1822^{\{6\}}$	$0.2022^{\{6\}}$	$-0.1378^{\{5\}}$	$0.1831^{\{4\}}$	$-0.0972^{\{3\}}$	$0.2004^{\{5\}}$	$-0.1018^{\{4\}}$	$0.1804^{\{3\}}$	$-0.0724^{\{2\}}$	$0.1202^{\{2\}}$
	θ	$-0.0011^{\{1\}}$	$0.3335^{\{5\}}$	$-0.0421^{\{2\}}$	$0.1042^{\{2\}}$	$-0.1887^{\{6\}}$	$0.2026^{\{3\}}$	$-0.0508^{\{3\}}$	$0.3679^{\{6\}}$	$-0.1042^{\{5\}}$	$0.2043^{\{4\}}$	$-0.0931^{\{4\}}$	$0.1015^{\{1\}}$
	γ	$-0.0053^{\{1\}}$	$0.0344^{\{1\}}$	$-0.2693^{\{6\}}$	$0.4528^{\{5\}}$	$-0.0937^{\{2\}}$	$0.3953^{\{2\}}$	$-0.1843^{\{5\}}$	$0.7761^{\{6\}}$	$-0.1353^{\{4\}}$	$0.4353^{\{4\}}$	$-0.1336^{\{3\}}$	$0.4147^{\{3\}}$
	c	$-0.0045^{\{1\}}$	$0.0669^{\{1\}}$	$-0.0278^{\{4\}}$	$0.0796^{\{3\}}$	$-0.1415^{\{6\}}$	$0.1433^{\{5\}}$	$0.0233^{\{3\}}$	$0.0984^{\{4\}}$	$-0.0902^{\{5\}}$	$0.1793^{\{6\}}$	$-0.0180^{\{2\}}$	$0.0681^{\{2\}}$
$\sum$ ranks		12		34		33		35		35		19	

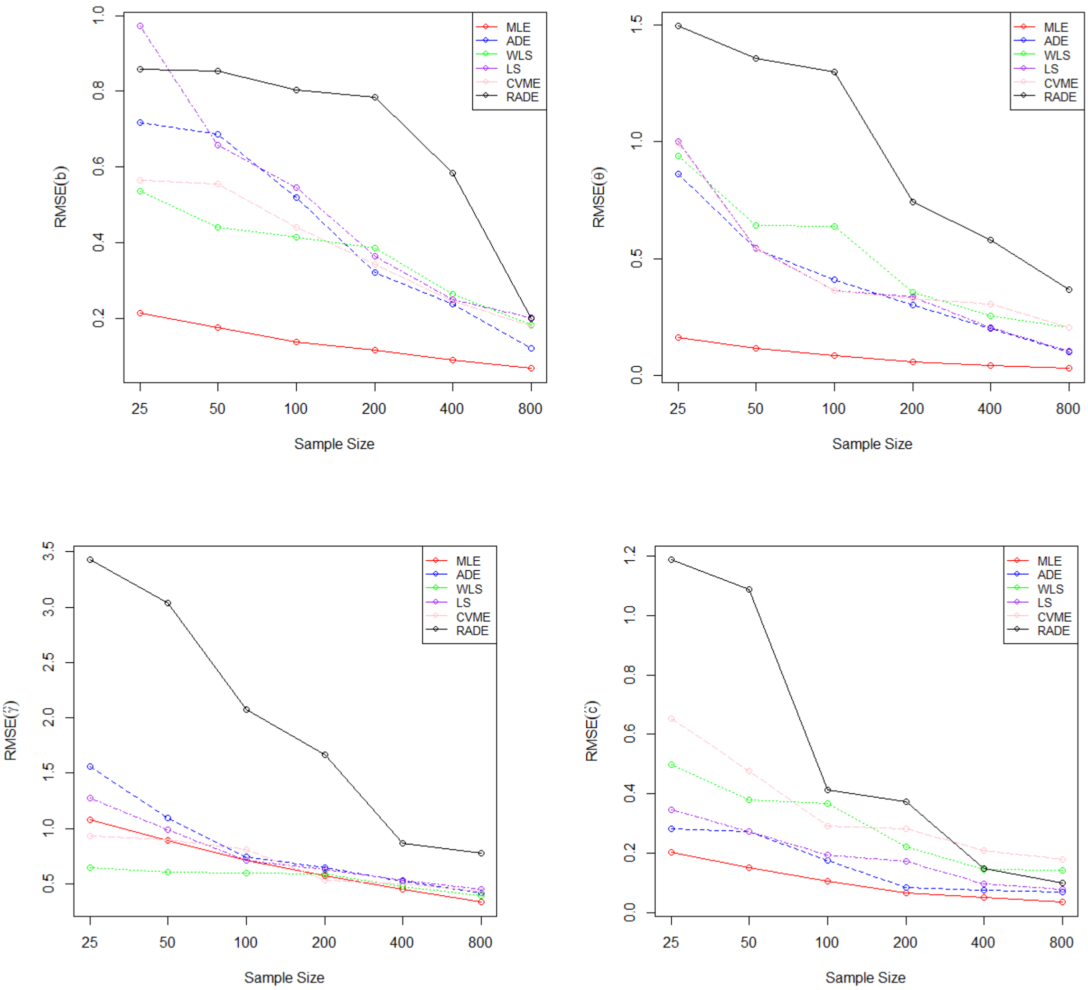

Figure 5 Plots of RMSEs of parameters in Table 1

Table 2 Simulation Results for Different Estimation Methods for $b = 1.6, \theta = 0.4, \gamma = 0.8, c = 0.4$

		MLE		LS		WLS		RADE		CVME		ADE	
n	Parameter	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE
25	b	0.1338{1}	0.9476{2}	0.7609{3}	1.6285{6}	−0.5601{2}	0.5614{1}	1.3002{6}	2.1775{2}	0.9564{4}	1.1124{3}	1.1393{5}	1.1683{5}
	θ	0.1700{1}	0.4511{2}	0.1995{2}	0.4888{3}	−0.3352{4}	0.3356{1}	0.7488{6}	1.0382{6}	0.2932{3}	0.4927{4}	−0.6893{5}	0.8633{5}
	γ	−0.02918{1}	0.2562{1}	1.2352{4}	3.0748{4}	−0.6981{2}	0.9354{3}	4.6626{6}	5.3063{6}	1.7422{5}	3.7278{5}	−0.8808{3}	0.9176{2}
	c	0.0372{1}	0.2682{1}	0.5011{3}	0.9852{5}	−0.8302{6}	0.8362{4}	−0.3810{2}	0.3855{2}	0.5154{4}	0.9860{6}	0.7429{5}	0.7509{3}
$\sum$ ranks		10		30		23		36		34		33	
50	b	−0.0791{1}	0.2776{1}	0.6859{4}	1.2525{5}	−0.5224{2}	0.5235{2}	1.0935{6}	2.1047{6}	0.8348{5}	1.0107{4}	0.5406{3}	0.7454{3}
	θ	0.1240{1}	0.2914{1}	0.1480{2}	0.3059{2}	−0.3218{4}	0.3223{4}	0.7195{6}	0.8352{6}	0.2761{3}	0.3156{3}	−0.5796{5}	0.6893{5}
	γ	0.0116{1}	0.1826{1}	1.0975{4}	3.0243{4}	−0.4097{2}	0.2956{2}	2.5940{6}	3.9439{6}	1.6337{5}	3.5691{5}	−0.4854{3}	0.5334{3}
	c	−0.0113{1}	0.1428{1}	0.4864{3}	0.7923{4}	−0.8280{6}	0.8282{6}	−0.2877{2}	0.3027{2}	0.5029{4}	0.8028{5}	0.5510{5}	0.7511{3}
$\sum$ ranks		8		28		28		40		34		30	
100	b	−0.0750{1}	0.2666{1}	0.4994{4}	1.0390{5}	−0.4070{2}	0.5041{2}	0.9923{6}	1.0885{6}	0.7950{5}	0.8073{4}	0.4608{3}	0.5673{3}
	θ	0.0767{1}	0.2194{2}	0.1076{2}	0.2123{1}	−0.2102{3}	0.3033{3}	0.4306{6}	0.6783{6}	0.2148{4}	0.3081{4}	−0.3910{5}	0.4073{5}
	γ	0.0100{1}	0.1396{1}	1.0022{4}	2.3891{5}	−0.1622{2}	0.2086{3}	1.7991{6}	2.0602{2}	1.3806{5}	3.4062{6}	−0.4270{3}	0.4329{4}
	c	−0.0108{1}	0.1055{1}	0.2276{2}	0.4959{4}	−0.8077{6}	0.8079{6}	−0.2301{3}	0.2876{2}	0.3971{5}	0.5144{5}	0.2936{4}	0.4046{3}
$\sum$ ranks		9		27		27		37		38		30	
200	b	−0.0500{1}	0.2460{1}	0.4034{4}	0.7823{4}	−0.3067{3}	0.4696{2}	0.8351{6}	0.8869{6}	0.5282{5}	0.7853{5}	0.2162{2}	0.4778{3}
	θ	0.0640{1}	0.1858{1}	0.0822{2}	0.1986{2}	−0.1273{4}	0.2771{5}	0.1209{3}	0.3027{6}	0.1538{5}	0.2111{3}	−0.2089{6}	0.2204{4}
	γ	0.0084{1}	0.0738{1}	0.9882{6}	1.2329{5}	−0.0366{2}	0.1041{2}	0.8878{4}	1.1370{3}	0.8933{5}	2.8466{6}	−0.2612{3}	0.2965{4}
	c	−0.0099{1}	0.0821{1}	0.1860{3}	0.3341{3}	−0.7819{6}	0.7861{6}	−0.1893{4}	0.2093{2}	0.2428{5}	0.4698{5}	0.1587{2}	0.3982{4}
$\sum$ ranks		8		29		30		34		39		28	
400	b	−0.0459{1}	0.2359{1}	0.3454{4}	0.5278{5}	−0.2377{3}	0.3248{3}	0.3685{6}	0.5848{6}	0.3501{5}	0.5041{4}	0.1823{2}	0.2868{2}
	θ	0.0551{1}	0.1568{1}	0.0766{2}	0.1838{3}	−0.0914{4}	0.1918{4}	−0.1150{5}	0.2055{6}	0.0887{3}	0.1794{2}	−0.1994{6}	0.2036{5}
	γ	0.0028{1}	0.0567{1}	0.4353{4}	1.0678{5}	−0.0112{2}	0.0726{2}	0.7756{6}	0.8571{4}	0.5307{5}	1.5007{6}	−0.1818{3}	0.2050{3}
	c	−0.0047{1}	0.0653{1}	0.0875{2}	0.2363{3}	−0.5455{6}	0.7549{6}	−0.0949{3}	0.1896{2}	0.1957{5}	0.3857{5}	0.1568{4}	0.2579{4}
$\sum$ ranks		8		28		30		38		35		29	
800	b	−0.0377{1}	0.1888{1}	0.1090{3}	0.3619{6}	−0.2001{4}	0.2497{3}	0.2500{6}	0.3576{5}	0.2064{5}	0.3172{4}	0.1007{2}	0.1920{2}
	θ	0.0332{1}	0.1263{1}	0.0583{3}	0.1536{5}	−0.0363{2}	0.1641{6}	0.0947{6}	0.1296{2}	0.0722{4}	0.1525{4}	−0.0750{5}	0.1477{3}
	γ	0.0025{1}	0.0415{1}	0.1068{4}	0.9831{6}	−0.0083{2}	0.0651{2}	0.3039{5}	0.4867{4}	0.4324{6}	0.9132{5}	−0.0940{3}	0.1116{3}
	c	−0.0041{1}	0.0547{1}	0.0498{2}	0.1362{3}	−0.3184{6}	0.5573{6}	−0.0899{3}	0.1299{2}	0.1027{4}	0.2873{5}	0.1058{5}	0.1590{4}
$\sum$ ranks		8		32		31		33		37		27	

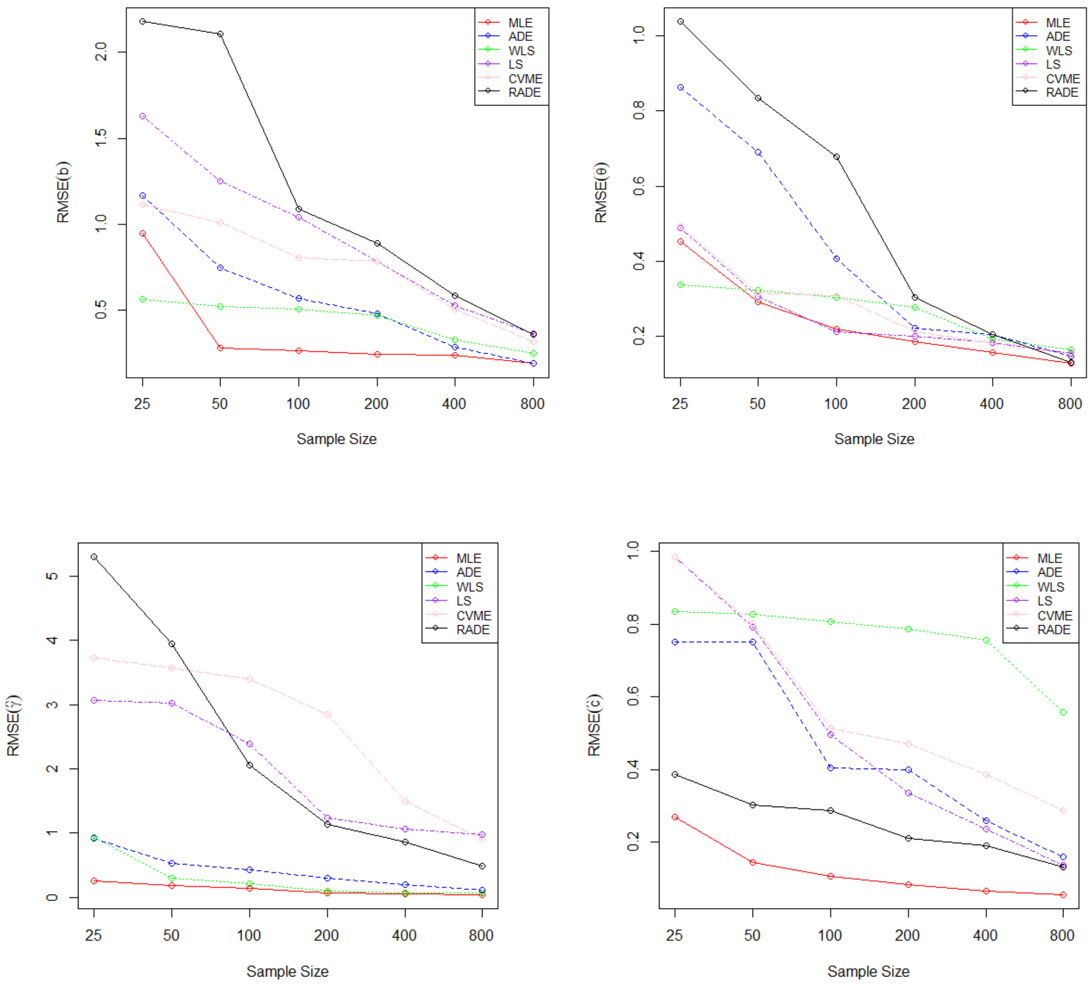

Figure 6 Plots of RMSEs of parameters in Table 2

Table 3 Partial and Overall Ranks of all Estimation Methods of TL-HT-Gom-LLoG Distribution by Various Model Parameter Values

Parameters	n	MLE	LS	WLS	RADE	CVME	ADE
	25	1	6	2	4	5	3
	50	1	4	2	5	6	3
$b = 0.4, \theta = 0.4, \gamma = 1.6, c = 2.3$	100	1	4.5	3	6	4.5	2
	200	1	5	4	6	3	2
	400	1	4	3	5	6	2
	800	1	4	3	5.5	5.5	2
	25	1	3	2	6	5	4
	50	1	2.5	2.5	6	5	4
$b = 1.6, \theta = 0.4, \gamma = 0.8, c = 0.4$	100	1	2.5	2.5	5	6	4
	200	1	3	4	5	6	2
	400	1	2	4	6	5	3
	800	1	4	3	5	6	2
$\sum$ ranks		12.0	44.5	35	64.5	63	33
Overall rank		1	4	3	6	5	2

In Tables 1 and 2, the row indicating $\sum$ Ranks represents the partial sum of the ranks. Among all the estimators for a given metric, the superscript indicates their rank. Table 1 presents, for example, the ABIAS of $\hat{b}$ obtained via MLE method as $0.0194^{\{1\}}$ for $n = 25$. This indicates that the ABIAS of $\hat{b}$ obtained using the MLE method ranks first among all other estimators.

Table 3 shows the partial and overall ranks of all the estimation methods of TL-HT-Gom-LLoG distribution by various model parameter values. Based on the results in Tables 1 and 2, with increasing sample size, the ABIAS and RMSE decreases across all estimation methods. In general, all estimation methods are consistent and efficient. Table 3 shows that MLE method allows us to obtain better estimates of TL-HT-Gom-LLoG parameters, followed by ADE, WLS, LS, CVME and then WLS methods.

7. Risk Measures

In this section, risk measures including: value at risk (VaR), tail value at risk (TVaR), tail variance (TV), and tail variance premium (TVP) commonly used by financial and actuarial professionals to assess the exposure to market risk in a portfolio of instruments are discussed.

7.1. Value at risk

VaR is an actuarial measure that is often used to assess risk in the financial markets. It is referred to as the quantile risk measure or the quantile premium principle, and it is always provided with a stated degree of confidence, such as 90%, 95%, or 99%. The VaR of the TL-HT-Gom-G family of distributions is given by

$$VaR_q = G^{-1}\left(1 - \left[1 - \gamma \log\left(\theta\left[\left(1 - q^{\frac{1}{b}}\right)^{\frac{-1}{2\theta}} - (1-\theta)\right]^{-1}\right)\right]^{\frac{-1}{\gamma}}\right), \tag{17}$$

where $q \in (0, 1)$ is a specified level of significance.

7.2. Tail value at risk

TVaR is used to express the expected value of loss in the case that an event beyond the predetermined probability threshold has actually occurred. The TVaR of the TL-HT-Gom-G family of distributions is given as

$$\begin{aligned} TVaR_q &= E(X \mid X > x_q) = \frac{1}{1-q}\int_{VaR_q}^{\infty} xf(x)dx \\ &= \frac{1}{1-q}\sum_{p=0}^{\infty}\int_{VaR_q}^{\infty} x\omega_{p+1}g_{p+1}(x;\psi)dx, \end{aligned} \tag{18}$$

where ω_{p+1} is given in Equation (8) and $g_{p+1}(x;\psi) = (p+1)G^p(x;\psi)g(x;\psi)$ is the pdf of Exp-G distribution with the power parameter $(p+1)$.

7.3. Tail variance

TV examines variation outside of the VaR. The TV of the TL-HT-Gom-G family of distributions is given by

$$\begin{aligned} TV_q &= E(X^2 \mid X > x_q) - (TVaR_q)^2 \\ &= \frac{1}{1-q}\int_{VaR_q}^{\infty} x^2 f(x)dx - (TVaR_q)^2 \\ &= \frac{1}{1-q}\sum_{p=0}^{\infty}\omega_{p+1}\int_{VaR_q}^{\infty} x^2 g_{p+1}(x;\psi)dx - (TVaR_q)^2. \end{aligned} \tag{19}$$

Thus, TV of TL-HT-Gom-G family of distributions can be obtained from those of Exp-G distribution.

7.4. Tail variance premium

The TVP is a significant risk measure that is crucial to the study of insurance. The TVP of the TL-HT-Gom-G family distributions is given by

$$TVP_q = TVaR_q + \delta(TV_q), \tag{20}$$

where $0 < \delta < 1$. The TVP of the TL-HT-Gom-G family of distributions can be obtained by substituting Eqns. (18) and (19) into Equation (20).

7.5. Numerical study for the risk measures

Here, we examine the suitability of the Topp-Leone heavy-tailed Gompertz-log-logistic (TL-HT-Gom-LLoG) distribution in modelling heavy tailed data by performing a numerical simulation of the risk measures. The obtained results are compared to those of the sub-models, and the equi-parameter models: Topp-Leone odd Burr III log-logistic (TL-OBIII-LLoG) by Moakofi et al. (2022) and alpha power exponentiated log-logistic (APExLLD) distribution by Teamah et al. (2021). Simulation results are obtained as follows:

1. Random samples of size $n = 100$ are generated from each one of the used distributions and parameters have been estimated via maximum likelihood method.

2. 1000 repetitions are made to calculate the VaR, TVaR, TV and TVP for these distributions.

Tables 4 shows the numerical findings of VaR, TVaR, TV and TVP for the six compared distributions. A model with higher values of VaR, TVaR, TV and TVP is said to have a heavier tail. From the figures in Table 4, we conclude that the TL-HT-Gom-LLoG distribution have a heavier tail than its sub-models, and the non-nested equi-parameter TL-OBIII-LLoG and APExLLD distributions, hence it is suitable for modelling heavy-tailed data.

Table 4 Simulation Results of VaR, TVaR, TV and TVP

Significance level		0.7	0.75	0.8	0.85	0.9	0.95	0.99
TL-HT-Gom-LLoG	VaR	0.7619	0.9124	1.0426	1.1683	1.2959	1.4298	1.5705
$(b = 1.7, \theta = 2.7, \gamma = 0.9, c = 1.9)$	TVaR	1.3029	1.6612	2.2253	3.1967	5.1513	10.8745	18.7321
	TV	2.4667	4.2563	7.8332	16.2323	42.1406	190.8545	460.3752
	TVP	0.4237	1.5310	4.0412	10.6008	32.7752	170.4372	370.5104
TL-HT-Gom-LLoG	VaR	0.1405	0.2193	0.2913	0.3616	0.4324	0.5053	0.5890
$(b = 1, \theta = 2.7, \gamma = 0.9, c = 1.9)$	TVaR	0.4193	0.5273	0.6972	1.0018	1.6700	3.8629	6.9304
	TV	0.5431	1.3366	2.9618	6.8454	19.1130	92.6773	250.1842
	TVP	0.0390	0.4751	1.6722	4.8167	15.5317	84.1806	180.5402
TL-HT-Gom-LLoG	VaR	0.4219	0.5386	0.6458	0.7524	0.8622	0.9778	1.1053
$(b = 1.7, \theta = 1, \gamma = 0.9, c = 1.9)$	TVaR	0.4757	0.6474	0.9361	1.4704	2.6502	6.5754	12.5436
	TV	0.8456	1.5801	3.1246	6.9700	19.7611	101.9116	250.9420
	TVP	0.1162	0.5375	1.5636	4.4541	15.1347	90.2406	180.7923
TL-HT-Gom-LLoG	VaR	0.5081	0.6284	0.9282	1.1138	1.2132	1.3068	1.4106
$(b = 1, \theta = 1, \gamma = 1, c = 1.9)$	TVaR	0.7904	0.8147	1.5533	1.6104	2.0485	5.6043	11.3409
	TV	0.9391	0.9515	1.2434	1.4711	1.7376	25.1845	80.5403
	TVP	0.3468	0.5784	0.9081	1.2409	0.4846	18.3209	42.0512
TL-OBIII-LLoG	VaR	0.3580	0.4191	0.4941	0.5898	0.7201	0.9239	1.1014
$(b = 2.0, \delta = 0.5, \gamma = 0.9, \lambda = 1.9)$	TVaR	0.6643	0.7197	0.7858	0.8678	0.9761	1.1397	1.3086
	TV	0.0454	0.0511	0.0580	0.0639	0.0687	0.0726	0.0782
	TVP	0.5152	0.7712	0.8369	0.9172	1.0222	1.1829	1.3154
APExLLD	VaR	0.6057	0.6241	0.6451	0.6702	0.7028	0.7537	0.8129
$(\alpha = 0.3, a = 1.6, b = 1.9, c = 1.9)$	TVaR	0.6141	0.6410	0.6890	0.6928	0.7465	0.8672	0.9843
	TV	0.1037	0.1243	0.1550	0.2052	0.2999	0.5267	0.7841
	TVP	0.3867	0.7073	0.7330	0.7673	0.8164	0.8676	0.9102

8. Applications

In this section, the flexibility of the TL-HT-Gom-LLoG distribution is demonstrated via applications to three real datasets. The goodness-of-fit of the TL-HT-Gom-LLoG distribution are compared to that of the Topp-Leone-Marshall-Olkin-Weibull (TL-MO-W) distribution by Chipepa et al. (2020), alpha power exponentiated log-logistic distribution (APExLLD) by Teamah et al. (2021), the Marshall-Olkin odd Burr III log-logistic (MOO-BIII-LLoG) distribution by Afify et al. (2020), Topp-Leone odd Burr III log-logistic (TL-OBIII-LLoG) by Moakofi et al. (2022), the logistic Burr XII (LBXII) distribution by Guerra et al. (2023), the Weibull-Burr XII (WBXII) distribution by Guerra et al. (2021), and the Marshall-Olkin generalized Burr XII (MOGBXII) distribution by Afify and Abdellatif (2020). The probability density functions (pdfs) of these distributions are given in the appendix.

The goodness-of-fit is assessed using the following statistics: -2log-likelihood $(-2\ln(L))$, Akaike Information Criterion ($AIC = 2p - 2\ln(L)$), Consistent Akaike Information Criterion ($CAIC = AIC + 2\frac{p(p+1)}{n-p-1}$), Bayesian Information Criterion ($BIC = p\ln(n) - 2\ln(L)$), (n is the number of observations, and p is the number of estimated parameters), Cramér-von Mises statistic (W^*), Anderson-Darling statistic (A^*) (Chen and Balakrishnan (1995)) and Kolmogorov-Smirnov (K-S) statistic. The model with the smallest values of the goodness-of-fit statistics is regarded as the best model.

Probability plots with sum of squares (SS) from the plots were also used to evaluate the fit. In addition, fitted densities, empirical cumulative distribution function (ECDF), Kaplan-Meier (K-M) survival curve, total time on test (TTT) plots and hrf plots are presented.

8.1. Failure times data

The first real data set is from a test that involved accelerated life for 59 conductors. Electromigration, or the movement of atoms within the conductors of a circuit, is a cause of failures in microcircuits. The data was analyzed by Atchad et al. (2023). Failure times are given in hours. The data are:

$6.545, 9.289, 7.543, 6.956, 6.492, 5.459, 8.120, 4.706, 8.687, 2.997, 8.591, 6.129, 11.038,$
$5.381, 6.958, 4.288, 6.522, 4.137, 7.459, 7.495, 6.573, 6.538, 5.589, 6.087, 5.807, 6.725,$
$8.532, 9.663, 6.369, 7.024, 8.336, 9.218, 7.945, 6.869, 6.352, 4.700, 6.948, 9.254, 5.009,$
$7.489, 7.398, 6.033, 10.092, 7.496, 4.531, 7.974, 8.799, 7.683, 7.224, 7.365, 6.923, 5.640,$

$5.434, 7.937, 6.515, 6.476, 6.071, 10.491, 5.923.$

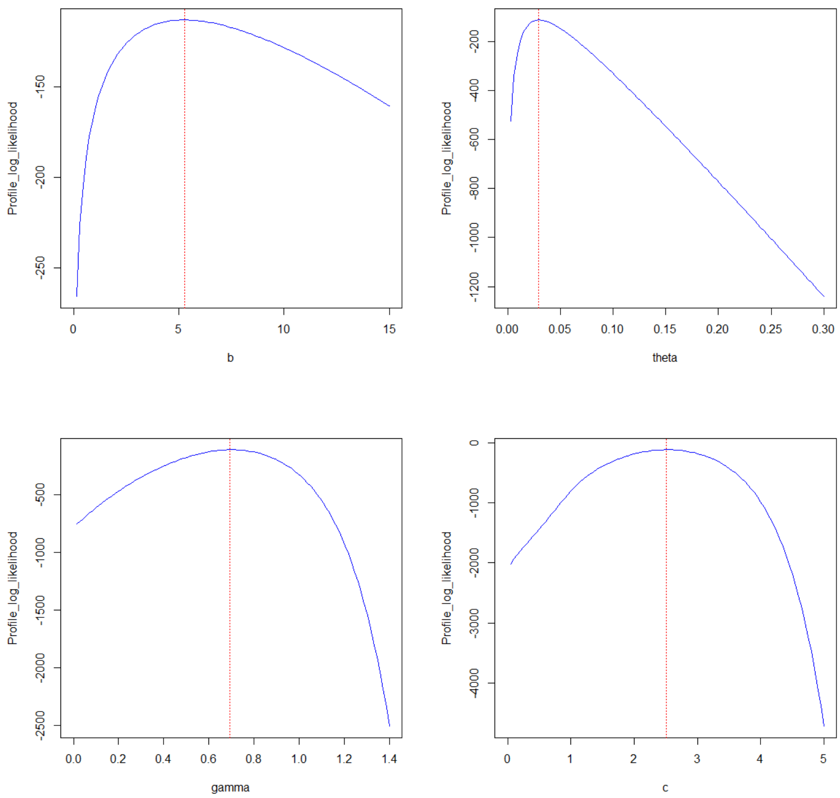

Figure 7 Profile likelihood function plots for parameters of TL-HT-Gom-LLoG distribution on the failure times data set

Figure 7 displays the profile likelihood plots for the parameters of the TL-HT-Gom-LLoG distribution applied to the failure times data set. The plots demonstrate that the maximum likelihood estimates (MLEs) for the TL-HT-Gom-LLoG distribution are unique, indicating that the parameters are identifiable.

Table 5 displays the maximum likelihood estimates (MLEs) of the fitted distributions together with the standard errors (in parenthesis) and the values of goodness-of-fit statistics for the failure times data. From Table 5, the selection criteria shows that TL-HT-Gom-LLoG distribution can be considered the best fitting model to represent the failure times data. Figure 8 shows the fitted densities and probability plots for the competing models applied to the considered data sets. From Figure 8, we conclude that the TL-HT-Gom-LLoG distribution has the best overall fit and can therefore be selected as the most appropriate model to explain the data.

Table 5 MLEs and goodness-of-fit statistics

Model	Estimates				Statistics						
	b	θ	γ	c	$-2\log(L)$	AIC	$CAIC$	BIC	W^*	A^*	K-S
TL-HT-Gom-LLoG	5.2909	0.0291	0.6913	2.5052	226.3343	234.3344	235.0751	242.6445	0.0597	0.3646	0.0767
	(1.5022)	(0.0122)	(0.1174)	(0.3246)							
	b	δ	γ	λ							
TL-MO-W	49.8703	0.3641	1.3630	0.0981	230.2458	238.2458	238.9866	246.556	0.1106	0.6764	0.1085
	(0.0010)	(0.2159)	(0.2394)	(0.0739)							
	b	δ	γ	λ							
TL-OBIII-LLoG	0.0974	365.8400	0.6969	29.4560	237.9898	245.9898	246.7306	254.3000	0.1951	1.1832	0.1287
	(0.0045)	(0.0002)	(0.2178)	(1.5195×10^{-05})							
	α	a	b	c							
APExLLD	9.6754×10^{03}	4.5342	0.7711	1.4412×10^{03}	237.6232	245.6232	246.3639	253.9333	0.2160	1.3044	0.1073
	(3.2751×10^{-06})	(0.4575)	(0.1609)	(1.9221×10^{-05})							
	δ	α	β	λ							
MOO-BIII-LLoG	37.0421	0.5045	411.2542	10.0417	234.6833	242.6833	243.4241	250.9935	0.0583	0.3732	0.1287
	(23.1290)	(0.4884)	(0.4837)	(9.7206)							
	λ	d	c	s							
LBXII	0.2863	36.8371	30.3190	7.9546	248.5148	256.5148	257.2555	264.8249	0.1264	0.9226	0.1557
	(0.2472)	(27.1532)	(27.4128)	(0.5359)							
	α	β	d	c							
WBXII	0.0066	5.3689	0.2058	3.0351	223.6743	231.6743	232.4151	239.9845	0.0614	0.3427	0.0875
	(0.0103)	(0.8868)	(0.0858)	(1.2704)							
	α	β	δ	a							
MOGBXII	2.4100	37.5230	1.0980×10^{04}	0.0540	238.2812	246.2814	247.0221	254.5915	0.0433	0.2779	0.1423
	(0.6921)	(0.0079)	(1.2610×10^{-06})	(0.0153)							

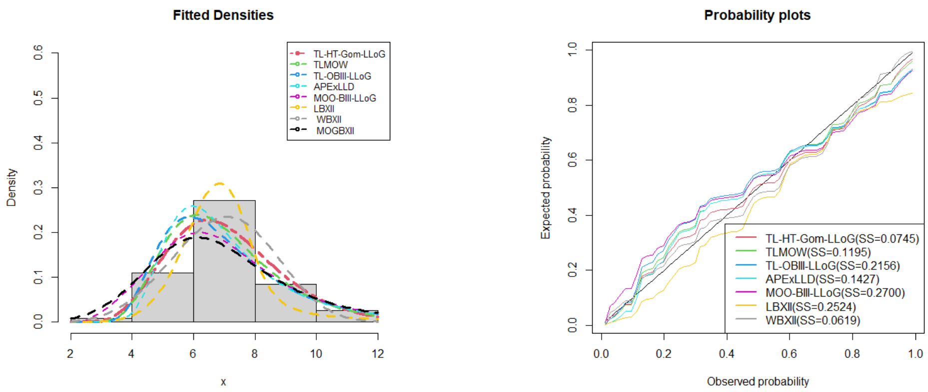

Figure 8 Histogram superposed by fitted density (left) and observed vs expected probability plots (right) for the failure times data

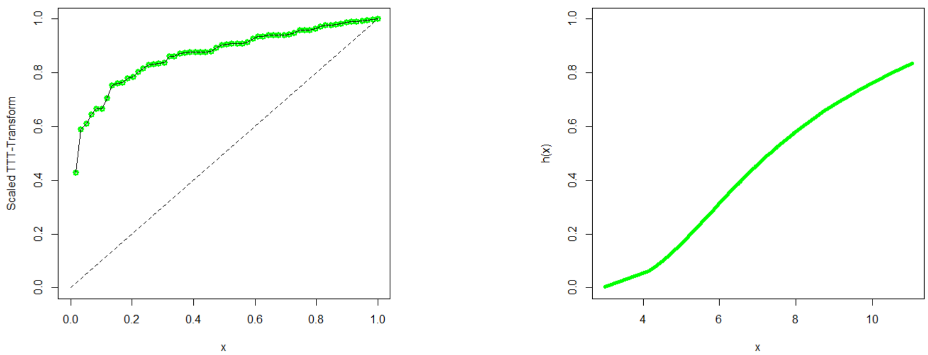

Figure 9 TTT and hrf plots

The total test on time (TTT) scaled plot and the estimated hazard rate function (hrf) plot are displayed in Figure 9. These are plotted to check the compatibility between the new distribution and the dataset. We can observe the hrf of the data is increasing as shown by the TTT scaled plot. The estimated hrf of the TL-HT-Gom-LLoG distribution on the failure times data is also increasing. Hence, we conclude that the data and the TL-HT-Gom-LLoG distribution are compatible.

8.2. Environmental data

The second data measures the acidity of rainfalls for forty days in the state of Minnesota. This data set was analyzed by Elbatal et al. (2022). The data are

$3.71, 4.23, 4.16, 2.98, 3.23, 4.67, 3.99, 5.04, 4.55, 3.24, 2.80, 3.44, 3.27, 2.66, 2.95,$
$4.70, 5.12, 3.77, 3.12, 2.38, 4.57, 3.88, 2.97, 3.70, 2.53, 2.67, 4.12, 4.80, 3.55, 3.86,$
$2.51, 3.33, 3.85, 2.35, 3.12, 4.39, 5.09, 3.38, 2.73, 3.07.$

Table 6 MLEs and goodness-of-fit statistics

	Estimates				Statistics						
Model	b	θ	γ	c	$-2\log(L)$	AIC	$CAIC$	BIC	W^*	A^*	K-S
TL-HT-Gom-LLoG	13.6200	0.0161	6.6857	0.3904	93.70089	101.7009	102.8437	108.4564	0.0397	0.3033	0.0768
	(3.0096×10^{-04})	(3.5744×10^{-03})	(1.6284×10^{-03})	(3.9301×10^{-02})							
	b	δ	γ	λ							
TL-MO-W	53.999	1.0381×10^{-03}	2.6609×10^{-04}	2.7749	96.27976	104.2797	105.4226	111.0353	0.0621	0.4625	0.0987
	(1.6708×10^{-05})	(2.0315×10^{-02})	(5.1996×10^{-03})	(2.2998×10^{-03})							
	b	δ	γ	λ							
TL-OBIII-LLoG	149.3400	228.8100	0.4673	0.0252	94.88592	102.8859	104.0288	109.6414	0.0485	0.3728	0.0888
	(5.2548×10^{-06})	(2.6969×10^{-04})	(1.7375×10^{-01})	(1.7448×10^{-03})							
	α	a	b	c							
APExLLD	1.2186×10^{05}	5.1532	1.0563	25.9980	96.91563	104.9156	106.0585	111.6711	0.0676	0.4984	0.1021
	(1.5726×10^{-07})	(0.6272)	(0.1376)	(1.3377×10^{-03})							
	δ	α	β	λ							
MOO-BIII-LLoG	13.3171	0.3188	28.0343	15.2031	104.6461	112.6461	113.789	119.4017	0.0443	0.3375	0.1206
	(9.1878)	(1.0318)	(13.9274)	(49.1741)							
	λ	d	c	s							
LBXII	0.1965	218.8128	51.5909	3.7995	132.6436	140.6436	141.7864	147.3991	0.2657	1.7084	0.1685
	(0.0813)	(2.3894)	(24.3740)	(0.1532)							
	α	β	d	c							
WBXII	0.0868	4.1334	0.1283	5.8980	94.3715	102.3715	103.5144	109.127	0.0617	0.4287	0.0926
	(0.1058)	(0.6631)	(0.0492)	(2.2449)							
	α	β	δ	a							
MOGBXII	3.4770	5.1920	1.4168×10^{03}	0.3210	101.0151	109.0150	110.1579	115.7706	0.0505	0.3752	0.0983
	(0.9423)	(1.6358)	(3.9306×10^{-04})	(0.1011)							

Figure 10 illustrates that the maximum likelihood estimates (MLEs) of the TL-HT-Gom-LLoG parameters for the environmental data are identifiable.

The data analysis results for environmental data are presented in Table 6. This table shows that the TL-HT-Gom-LLoG distribution has the lowest values of the $-2\ln(L), AIC, CAIC, BIC, W^*, A^*$, and K-S statistic compared to other fitted distributions. Therefore, the TL-HT-Gom-LLoG distribution is considered the best model to characterize the environmental data. Figure 11 supports these findings visually.

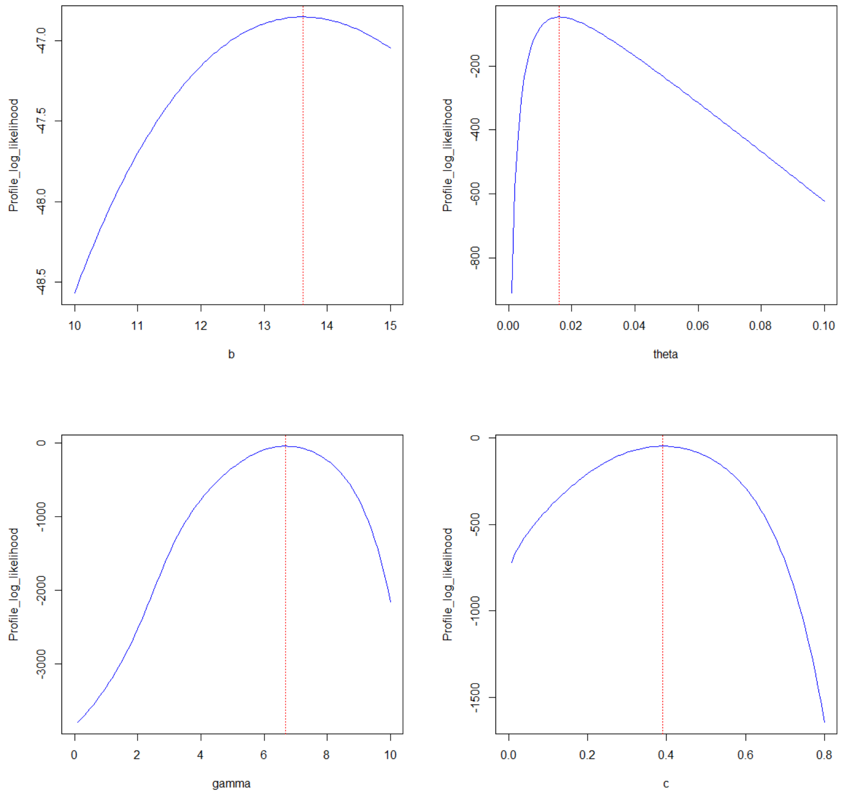

Figure 10 Profile likelihood function plots for parameters of TL-HT-Gom-LLoG distribution on the environmental data Set

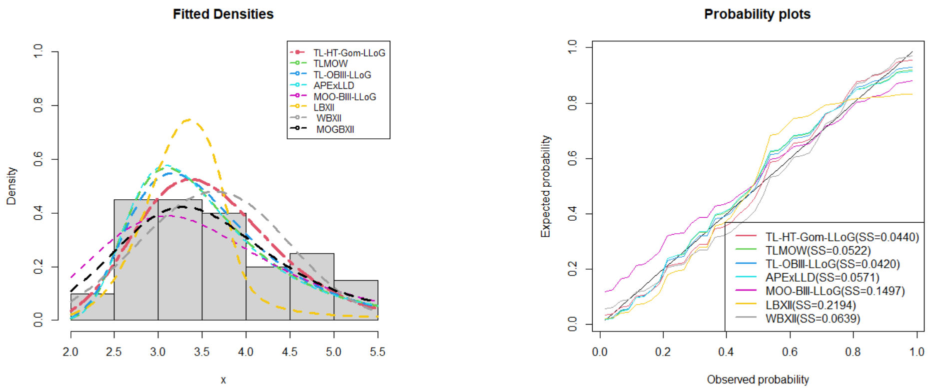

Figure 11 Histogram superposed by fitted density (left) and observed vs expected probability plots (right) for the environmental data

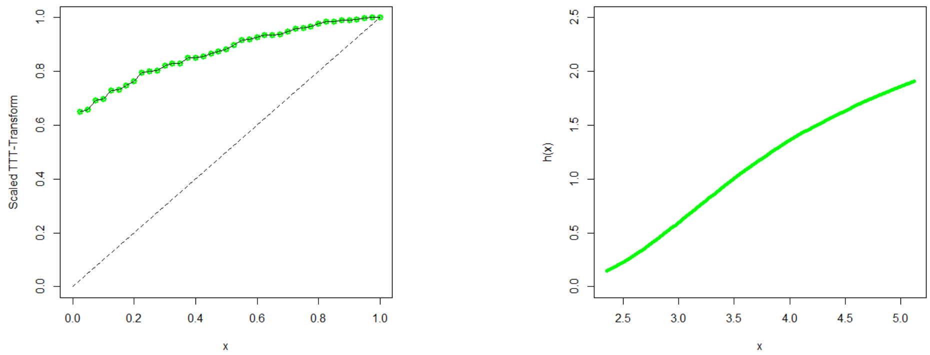

Figure 12 TTT and hrf plots

Figure 12 shows the TTT scaled plot, and hrf plot. The TTT scaled plot shows an increasing hrf. Furthermore, the estimated hrf in is agreement with the TTT scaled plot as it also displays an increasing shape for environmental data set.

8.3. Kevlar Epoxy data

The third data set relates to the stress-rupture life of Kevlar 49/epoxy strands that were continuously compressed at a 90% stress level until they all failed (Andrews and Herzberg 2012). The data are:

0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09,
0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35,
0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72,
0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01,
1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43,
1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81,
2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89.

The profile likelihood plots for the TL-HT-Gom-LLoG distribution parameters serve as a tool for assessing parameter identifiability. From the plots in Figure 13, it is evident that the maximum likelihood estimates (MLEs) for the TL-HT-Gom-LLoG distribution are distinct, leading to the conclusion that the parameters are identifiable.

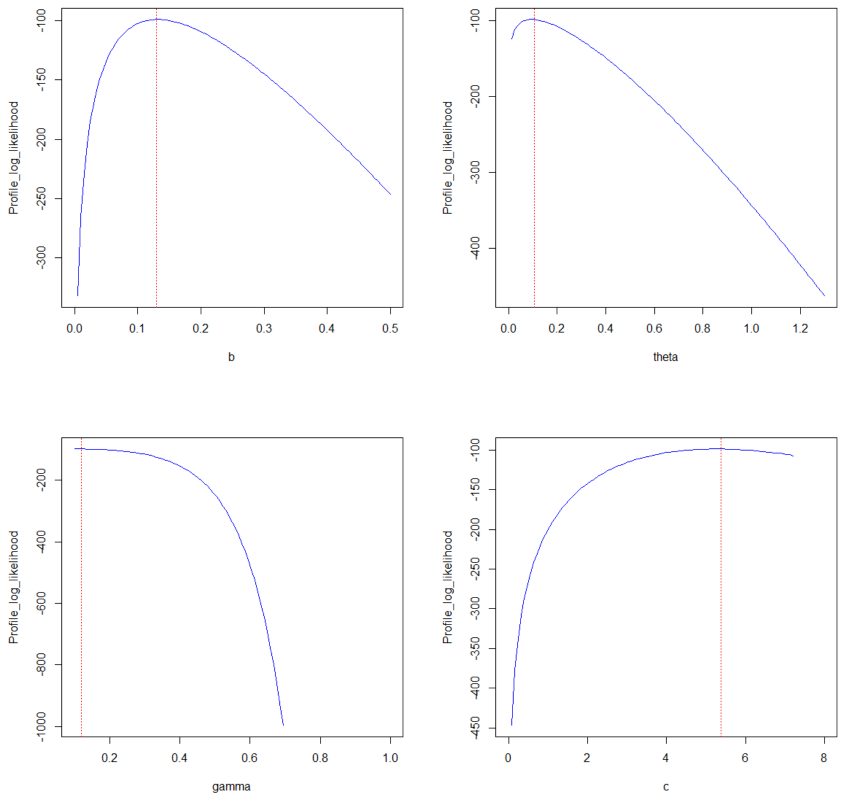

Figure 13 Profile likelihood function plots for parameters of TL-HT-Gom-LLoG distribution on the Kevlar 49/epoxy data Set

Table 7 MLEs and goodness-of-fit statistics

Model	Estimates				Statistics						
	b	θ	γ	c	$-2\log(L)$	AIC	$CAIC$	BIC	W^*	A^*	K-S
TL-HT-Gom-LLoG	0.1298	0.1081	0.1199	5.3876	197.8368	205.8368	206.2535	216.2973	0.0499	0.3666	0.0605
	(0.0282)	(0.0451)	(0.0402)	(1.0713)							
	b	δ	γ	λ							
TL-MO-W	49.8703	0.3641	1.3630	0.0981	230.2458	238.2458	238.9866	246.556	0.1106	0.6764	0.1085
	(0.0010)	(0.2159)	(0.2394)	(0.0739)							
	b	δ	γ	λ							
TL-OBIII-LLoG	0.9175	3.5113	0.2141	1.4786	211.1044	219.1044	219.521	229.5649	0.3199	1.7422	0.1339
	(1.4688)	(1.6230)	(0.1281)	(2.3671)							
	α	a	b	c							
APExLLD	1.4541	3.3711	1.8022	0.2015	200.0885	208.0885	208.5052	218.549	0.0656	0.4631	0.0653
	(1.5263)	(0.6884)	(0.3462)	(0.0602)							
	δ	α	β	λ							
MOO-BIII-LLoG	3.6073	4.5429	0.2194	0.5323	204.5787	212.5787	212.9953	223.0391	0.1245	0.7888	0.0839
	(1.7422)	(0.0078)	(0.0855)	(0.0670)							
	λ	d	c	s							
LBXII	24.7201	1.3669	0.0713	0.2151	227.0805	235.0805	235.4972	245.5410	0.5946	3.2280	0.1143
	(18.6591)	(0.4160)	(0.0527)	(1.3105)							
	α	β	d	c							
WBXII	0.8821	0.1518	1.2884	5.4832	204.9981	212.9981	213.4148	223.4586	0.1359	0.8243	0.0818
	(0.1273)	(0.1832)	(0.2634)	(6.4138)							
	α	β	δ	a							
MOGBXII	0.7904	1.7437	7.5800	2.1961	207.5178	215.5178	215.9344	225.9782	0.2267	1.2733	0.0907
	(0.1671)	(181.5521)	(6.1039)	(228.6481)							

The values presented in Table 7 demonstrate that the TL-HT-Gom-LLoG distribution provides a superior fit to the data compared to other fitted distributions. This is because it is associated with lower values of the goodness-of-fit statistics: $-2\ln(L)$, AIC, $CAIC$, BIC, W^{*}, A^{*}, and K-S statistic. These results are supported graphically by the plots in Figure 14.

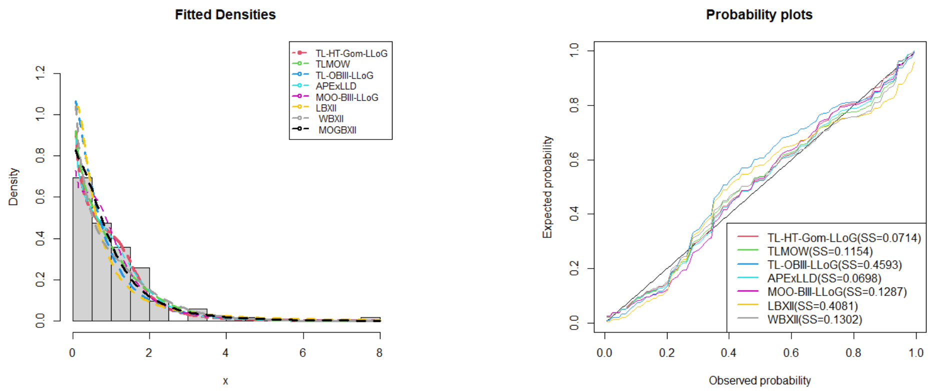

Figure 14 Histogram superposed by fitted density (left) and observed vs expected probability plots (right) for the kevlar epoxy data

From Figure 15, we see that the TL-HT-Gom-LLoG distribution is suitable for modeling the kevlar epoxy data data as both the TTT scaled and hrf plots are in agreement as they estimate the hrf of the data to be bathtub followed by upside-down bathtub.

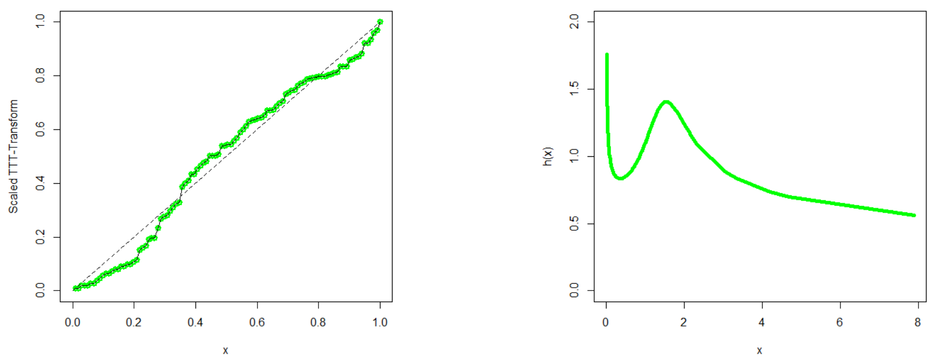

Figure 15 TTT and hrf plots

9. Concluding Remarks

We have proposed a new family of distributions called the Topp-Leone Heavy-Tailed Gompertz-G (TL-HT-Gom-G) family of distributions. Some of its statistical properties such as quantile function, linear representation, moments, moment generating function, Reńyi entropy, distribution of order statistics, and stochastic orderings are derived. Risk measures for this distribution were also presented, and the results revealed that the TL-HT-Gom-LLoG distribution is heavy-tailed. The unknown parameters of the new distribution are estimated using different estimation methods and evaluated via a simulation study. The TL-HT-Gom-LLoG distribution, as a special case of this family, demonstrated robustness and applicability with three real data sets, highlighting its usefulness in fields requiring heavy-tailed modeling, such as finance, insurance, and environmental studies. However, the distribution also has some disadvantages. Its complexity can make deriving statistical properties challenging, and it is not suitable for discrete data sets, limiting its applications to continuous data. Despite these constraints, the TL-HT-Gom-LLoG distribution remains a versatile tool for analyzing heavy-tailed data across various fields.

Appendix

Click on the link below for results in the appendix.
https://drive.google.com/file/d/18gnwzN3t0QJw3cJ1qIROKc9H1KDs2n6L/view?usp=sharing

Acknowledgements

We would like to thank the referees for their valuable comments and suggestions on the manuscript.

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