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Order Statistics from Inverse Lomax Log-Logistic (IL-LL) Distribution

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Abstract

Random variables with order have found extensive use across various fields, including sports, seismology, reliability, quality control, actuarial science, and more. This study focuses on the inverse Lomax log-logistic distribution. Inverse Lomax log-logistic distribution is a flexible and robust model that exhibits several noteworthy statistical properties. Its probability density function is typically right-skewed and decreasing, making it suitable for modeling positively skewed data. We establish precise formulations for moments, providing valuable tools for determining diverse statistical properties of this distribution. We compute moments at various parameter values to enhance our understanding of the distribution. Further, these moments are used to determine the best linear unbiased estimators.

Keywords: Order statistics, inverse Lomax log-logistic distribution, single moment, product moments, hypergeometric functions, best linear unbiased estimator.

1. Introduction

Ordered random variables prove advantageous in numerous practical scenarios. Scholars have dedicated considerable attention to these variables due to their extensive applications. For example, one might employ ordered random variables to arrange the pricing of goods or organize a list of students based on their final test scores. In games, ordered random variables come into play when managing records, and they offer a logical approach for addressing significant events like earthquakes. Arranging data in ascending or descending order leads to the creation of order statistics, providing insights into various data characteristics such as the maximum or minimum values within the data range. Statisticians traditionally aim to collect n independent random variables. Consider a set of random variables $X_1, X_2, \ldots, X_n$ arranged in ascending order of magnitude as $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$. Here, $X_{r:n}$ or $X_{(r)}$ denotes the r^{th} order statistics in a sample of size n. The well-established theory of ordered random variables proves to be a vital tool for handling observations $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ and various statistics derived from $X_{r:n}$.

Order statistics concern the properties and applications of these ordered random variables and their associated functions, as highlighted by Arnold et al. (2008). Explicit expressions for moments of order statistics of some distributions were determined by Nadarajah (2008). Joshi and Balakrishnan (1982) established the recurrence relations and found identities for the product moments of order statistics. Balakrishnan et al. (1988) established recurrence relations and identities for moments of order statistics for some specific continuous distributions.

For a comprehensive exploration of the moments of order statistics, refer to the work of Balakrishnan and Sultan (1998). Several authors have extensively investigated moments of order statistics. Childs et al. (2000) provided precise and explicit expressions for single, product, triple, and quadruple moments of order statistics derived from the Pareto distribution. They also established recurrence relations for these moments.

Adeyemi (2002) developed recurrence relations for both single and product moments of order statistics originating from a symmetric generalized log-logistic distribution. Zghoul (2010) obtained expressions for moments of order statistics from a family of J-shaped distributions. Gen (2012) obtained the exact expression for single and product moments of order statistics from the Topp-Leone distribution. Nagaraja (2013) derived moments and L-moments for the symmetric triangular distribution. Furthermore, Sultan and Al-Thubyani (2016) provided exact and explicit expressions for single, product, triple, and quadruple moments of order statistics, along with deriving the best linear unbiased estimator for scale and location parameters from the Lindley distribution. Kumar and Goyal (2019) presented explicit expressions for single and product moments of order statistics and derived the best linear unbiased estimators (*BLUEs*) for scale and location parameters from the power Lindley distribution. Akhter et al. (2020) established exact and explicit expressions for single and product moments of order statistics from the length-biased exponential distribution and derived the *BLUEs* for scale and location parameters. Gul and Mohsin (2021) derived recurrence relations for both single and product moments of order statistics from the half logistic-truncated exponential distribution. Anwar and Khan (2021) provided explicit expressions for single and product moments of order statistics from Ishita distribution and derived *BLUEs* for location and scale parameters. Nayabuddin and Khan (2022) obtained the explicit form of moments of order statistics from the exponentiated Burr XII distribution. Anwar et al. (2023) presented explicit expressions for single, product, triple and quadruple moments of order statistics from Pareto-Weibull distribution. For estimation methods, one can see: Khan et al. (2025) and Anwar et al. (2025a). Anwar et al. (2025b), this paper investigates the moment properties of the Benktander Type II distribution using generalized order statistics, deriving explicit moment formulas, recurrence relations, and characterization results.

The probability density function (*pdf*) of r^{th} order statistics is given by

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x); \quad -\infty < x < \infty. \tag{1}$$

The joint *pdf* of r^{th} and s^{th} order statistics $X_{(r)}$ and $X_{(s)}$ is given by

$$\begin{aligned} f_{r,s:n}(x,\ y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}&[F(x)]^{r-1}f(x)[F(y)-F(x)]^{s-r-1} \\ &\times[1-F(y)]^{n-s}f(y); \quad x < y. \end{aligned} \tag{2}$$

The log-logistic distribution is a well-established probability distribution with applications spanning across diverse academic disciplines such as survival analysis, hydrology, and economics. The log-logistic distribution has been thoroughly investigated by numerous researchers and authors. Lloyd (1952) originally introduced estimations of location and scale parameters based on order statistics. Bennett (1983) considered the log-logistic regression model for survival data, highlighting its usefulness as an alternative to the Weibull distribution for parametric modeling, especially in cases where the hazard rate is non-monotonic.

Ragab and Green (1984) derived some properties of order statistics from the log-logistic distribution, while Ali and Khan (1987) derived recurrence relations for the moments of order statistics. Varoon (1987) characterized the log-logistic distribution based on extreme-related stability with random sample size. Shoukri et al. (1988) demonstrated its suitability in fitting precipitation data from diverse Canadian regions. Collett (1994) applied the log-logistic distribution for modeling survival data in medical research. Additionally, Ashkar and Mahdi (2003) compared the log-logistic model with the two-parameter lognormal, two-parameter Weibull, and extreme value type I distributions for fitting maximum annual stream flow data. The log-logistic distribution is recognized for providing a good approximation to both normal and log-normal distributions. Falgore and Doguwa (2023) provide an extension of the inverse Lomax (IL) distribution with the log-logistic distribution called inverse Lomax log-logistic (IL-LL) distribution is considered and studied various statistical properties.

Over the last few decades, the log-logistic (LL) distribution, also known as the Risk distribution in economics, has found widespread use, particularly in survival and reliability analysis. The log-logistic distribution is considered an alternative to the log-normal distribution due to its provision of an initially increasing and decreasing fault rate function.

The random variable X follows the IL-LL distribution with *pdf*

$$f(x) = \alpha\beta\lambda x^{-\lambda-1}\left(1+\beta x^{-\lambda}\right)^{-\alpha-1};\ \ x>0,\ \ \alpha,\ \beta,\ \lambda>0 \tag{3}$$

and the corresponding cumulative distribution function (*cdf*) is given by

$$F(x) = \left(1+\beta x^{-\lambda}\right)^{-\alpha};\ \ x>0,\ \ \alpha,\ \beta,\ \lambda>0 \tag{4}$$

where $\alpha,\ \lambda$ are the shape parameters and β is the scale parameter.

Using (3) and (4), we get the relation between *cdf* and *pdf*

$$f(x) = \frac{\alpha\beta\lambda x^{-\lambda-1}}{(1+\beta x^{-\lambda})}F(x). \tag{5}$$

The reliability function for the IL-LL distribution is

$$R(t) = 1-F(t) = 1-\left(1+\beta t^{-\lambda}\right)^{-\alpha}.$$

The Hazard function for the IL-LL distribution is

$$h(t) = \frac{f(t)}{R(t)} = \frac{\alpha\beta\lambda t^{-\lambda-1}\left(1+\beta t^{-\lambda}\right)^{-\alpha-1}}{1-(1+\beta t^{-\lambda})^{-\alpha}}.$$

In Figure 1(a) and (b), various forms of density and distribution functions for the IL-LL distribution are presented. The illustrated distribution in the figure demonstrates its capability to capture diverse datasets behaviors. Figures 1(c) and (d) depict the reliability and hazard rate functions of the IL-LL distribution. The hazard rate function displays both increasing and decreasing behavior, as indicated in the figure.

The papers structure is as follows: Section 2 presents the derivation of the precise and explicit expression for the single moment from the IL-LL distribution. In Section 3, we have shown the product moments from the IL-LL distribution. In Section 4, the results of Sections 2 and 3, are utilized to compute the means, variances and covariances of order statistics from samples of sizes up to 10 and for different values of the parameters $\alpha,\ \beta,\ \lambda$. In Section 5, the moment's application is discussed by presenting the *BLUEs* of the IL-LL distribution for location and scale parameters, followed by parameter estimation and explanation. Finally, in Section 6, we draw conclusion for the paper.

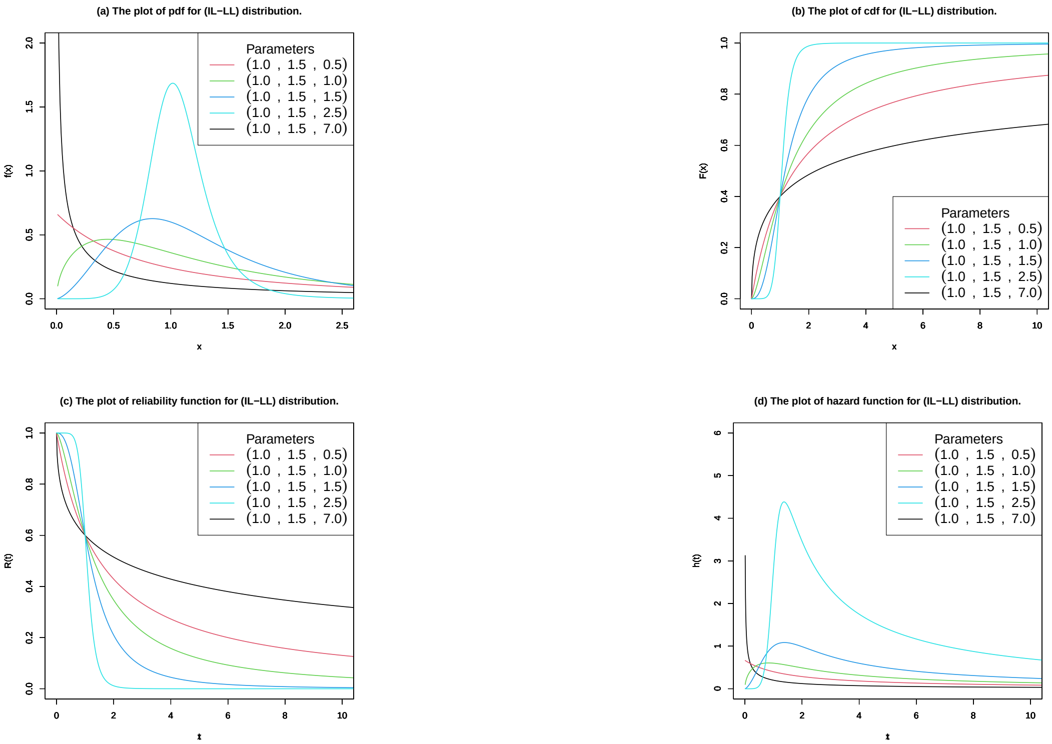

Figure 1 p.d.f., c.d.f., reliability, and hazard function graph at different parameter values

2. Single Moment

Theorem 1 *For the IL-LL distribution as given in (3) and* $1 \leq r \leq n,\ \alpha,\ \beta,\ \lambda\ > 0.$

$$E[X_{r:n}^{j}] = \mu_{r:n}^{j} = C_{r:n}\sum_{i=0}^{n-r}(-1)^{i}\binom{n-r}{i}\alpha\beta^{\frac{j}{\lambda}}B\left(1-\frac{j}{\lambda},\ \alpha(r+i)+\frac{j}{\lambda}\right). \tag{6}$$

Proof: Using binomial expansion in Equation (1), then after that in view of Equation (3) and (4), we get

$$E[X_{r:n}^{j}] = C_{r:n}\sum_{i=0}^{n-r}(-1)^{i}\binom{n-r}{i}\alpha\beta\lambda\int_{0}^{\infty}\frac{x^{j-\lambda-1}}{(1+\beta x^{-\lambda})^{\alpha(r+i)+1}}dx.$$

Substituting $t=\beta x^{-\lambda}$, we get

$$E[X_{r:n}^{j}] = C_{r:n}\sum_{i=0}^{n-r}(-1)^{i}\binom{n-r}{i}\alpha\beta^{\frac{j}{\lambda}}\int_{0}^{\infty}\frac{t^{\frac{-j}{\lambda}}}{(1+t)^{\alpha(r+i)+1}}dt.$$

Now using the beta function $B\left(c,\ d\right) = \int_{0}^{\infty}\frac{x^{c-1}}{(1+x)^{c+d}}dx$, we get the expression as given in (6).

Special Cases:

a) For $r = 1$ in Equation (6), we obtain an exact expression for single moments of the first order statistics, which is also denoted as sample minimum

$$E[X_{1:n}^{j}] = n\alpha\beta^{\frac{j}{\lambda}}\sum_{i=0}^{n-1}(-1)^{i}\binom{n-1}{i}B\left(1-\frac{j}{\lambda},\ \alpha(i+1)+\frac{j}{\lambda}\right).$$

b) In the other case, for sample maximum, putting $r = n$ in (6), we obtain the exact expression for single moments of the largest order statistics

$$E[X_{n:n}^{j}] = n\alpha\beta^{\frac{j}{\lambda}} B\left(1 - \frac{j}{\lambda},\ \alpha(n+i) + \frac{j}{\lambda}\right).$$

c) If $r = n = j = 1$, we get

$$E(X) = \alpha\beta^{\frac{1}{\lambda}} B\left(1 - \frac{1}{\lambda},\ \alpha(i+1) + \frac{1}{\lambda}\right).$$

It is possible to verify the accuracy of the single moments of order statistics in Equation (6) by applying $\sum_{r=1}^{n} \mu_{r:n} = nE(X)$, given by (David and Nagaraja, 2003).

3. Product Moments

Theorem 2 *For the IL-LL distribution as given in (3) with $n \in N,\ 1 \leq r < s \leq n$,*

$$\begin{aligned} E[X_{r:n}^{l} Y_{s:n}^{m}] =& \mu_{r,s:n}^{l,\,m} = C_{r,s:n} \sum_{i=0}^{s-r-1} \sum_{j=0}^{n-s} (-1)^{i+j} \binom{s-r-1}{i} \binom{n-s}{j} \alpha^2 \beta^{\frac{l+m}{\lambda}} \left(1 - \frac{m}{\lambda}\right)^{-1} \\ & \times B\left(2 - \left(\frac{l+m}{\lambda}\right), \frac{l}{\lambda} + \alpha(r+i)\right) {}_3F_2 \left[1 - \frac{m}{\lambda} - \alpha(s-r-i+j),\right. \\ & \left. 1 - \frac{m}{\lambda}, -\left(\frac{l+m}{\theta}\right) + 2; 2 - \frac{m}{\lambda}, -\frac{m}{\lambda} + \alpha(r+i) + 2; 1\right]. \end{aligned} \tag{7}$$

Proof: Using Equations (2), (3) and expanding binomially, we get

$$E[X_{r:n}^{l} Y_{s:n}^{m}] = C_{r,s:n} \sum_{i=0}^{s-r-1} \sum_{j=0}^{n-s} (-1)^{i+j} \binom{s-r-1}{i} \binom{n-s}{j} \alpha^2 I(x,\ y) \tag{8}$$

where

$$\begin{aligned} I(x,\ y) &= \lambda\beta \int_0^\infty x^{l-\lambda-1} \left(1 + \beta x^{-\lambda}\right)^{-\alpha(r+i)-1} I(y) dx, \\ I(y) &= \lambda\beta \int_x^\infty y^{m-\lambda-1} \left(1 + \beta y^{-\lambda}\right)^{-\alpha(s-r-i+j)-1} dy. \end{aligned} \tag{9}$$

Setting $t = \beta y^{-\lambda}$, we get

$$I(y) = \int_0^{\beta x^{-\lambda}} t^{\frac{-m}{\lambda}} (1+t)^{-\alpha(s-r-i+j)-1} dt.$$

Using the result, Gradshteyn et al. (2007), [Page-315, ET I 310(20)].

$$\int_0^u \frac{x^{\mu-1}}{(1+\beta x)^v} dx = \frac{u^\mu}{\mu} {}_2F_1(v,\ \mu;\ 1+\mu;\ -\beta u); \quad Re \quad \mu \geq 0, \quad |arg(1+\beta u)| \leq \pi,$$

and Pfaff transformation

$${}_2F_1(a,\ b;\ c;\ z) = (1-z)^{-b} {}_2F_1(c-a,\ b;\ c;\ \frac{z}{1-z}),$$

after simplification, we get

$$= \beta \left(1-\frac{m}{\lambda}\right)^{-1} (1+\beta x^{-\lambda})^{\frac{m}{\lambda}-1} {}_2F_1\left(1-\frac{m}{\lambda}-\alpha(s-r-i+j), 1-\frac{m}{\lambda}; 2-\frac{m}{\lambda}; \frac{\beta x^{-\lambda}}{1+\beta x^{-\lambda}}\right).$$

Substituting $I(y)$ in (9) and setting $t=\beta x^{-\lambda}$, we get

$$\begin{aligned} I(x,\ y) =& \beta^{\frac{l+m}{\lambda}} \left(1-\frac{m}{\lambda}\right)^{-1} \int_0^{\infty} t^{-\left(\frac{l+m}{\lambda}\right)+1} (1+t)^{-\alpha(r+i)+\frac{m}{\lambda}-2} \\ & \times {}_2F_1\left(1-\frac{m}{\lambda}-\alpha(s-r-i+j)\ ,\ 1-\frac{m}{\lambda};\ 2-\frac{m}{\lambda};\ \frac{t}{1+t}\right) dt. \end{aligned}$$

Substituting $u=\frac{t}{1+t}$ and using the identity

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1} {}_2F_1(\alpha,\ \beta,\ \gamma;\ x)\, dx = B(\rho,\ \sigma)\, {}_3F_2(\alpha,\ \beta,\ \rho;\ \gamma,\ \rho+\sigma,\ 1).$$

See, Gradshteyn et al. (2007), [Page-813, ET II 399(5)], $I(x,\ y)$ reduces to

$$\begin{aligned} I(x,\ y) =& \beta^{\frac{l+m}{\lambda}} \left(1-\frac{m}{\lambda}\right)^{-1} B\left(2-\frac{l}{\lambda}-\frac{m}{\lambda}, \frac{l}{\lambda}+\alpha(r+i)\right) {}_3F_2\left(1-\frac{m}{\lambda}-\alpha(s-r-i+j),\right. \\ & \left. 1-\frac{m}{\lambda},\ 2-\left(\frac{l+m}{\lambda}\right);\ 2-\frac{m}{\lambda},\ -\frac{m}{\lambda}+\alpha(r+i)+2;\ 1\right). \end{aligned}$$

Substituting $I(x,\ y)$ in (8) we get the required result as given in (7).

Using the following identity, the accuracy of the product moments of order statistics in Equation (7) may be verified

$$\sum_{r=1}^{n}\sum_{s=1}^{n} \sigma_{r,s:n} = n\sigma^2$$

where, $\sigma_{r,s:n} = \mathrm{Cov}\left(X_{r:n},\ X_{s:n}\right)$ and $\sigma^2 = \mathrm{Var}(X)$.

4. Computations of Means, Variances and Covariances

In this section, we have utilized the results established in Sections 2 and 3 to compute the means, variances and covariances of order statistics from the IL-LL distribution. By setting $j=1$ in (6), the means of the order statistics have been computed for $n=1(1)10$ and for different selected values of parameters $\alpha,\ \beta,\ \lambda$. The means values to five decimal places are reported in Table 1.

Here, it can be seen that the condition $\sum_{r=1}^{n} \mu_{r:n} = nE(X)$, David and Nagaraja (2003) is satisfied. By setting $j=1$ and $j=2$ in (6), respectively, the first two moments of the form $\mu_{r:n}^{(1)}$ and $\mu_{r:n}^{(2)}$ for all order statistics and for any sample sizes can be calculated in a systematic way, and then, using these two moments, the variances of order statistics may easily be computed.

It is clear from the expression (6) that all the product moments $\mu_{r,s:n}^{(1,1)} = \mu_{r,s:n},\ r<s$, can also be computed. Thus, setting $l=m=1$ in (7), all the product moments have been computed first and using these product moments with already computed first two moments, the covariances of all order statistics have been computed in a simple way. The covariances have been computed for $n=2(1)10$ and several choices of the parameters $\alpha,\ \beta,\ \lambda$. The variances and covariances to five decimal places are presented in Table 2. `MATLAB` has been used for computation of the moments.

Here, it may be seen that the condition $\sum_{r=1}^{n}\sum_{s=1}^{n} \sigma_{r,s:n} = n\sigma^2$, David and Nagaraja (2003), is satisfied, where $\sigma_{r,s:n} = \mathrm{Cov}\left(X_{r:n},\ X_{s:n}\right)$ and $\sigma^2 = \mathrm{Var}(X)$.

Table 1 Moments ($\mu_{r:n}$) of order statistics for the IL-LL distribution.

		α=0.1, β=0.2			
n	r	λ=3	λ=4	λ=5	λ=6
1	1	0.17018	0.21932	0.26534	0.30729
2	1	0.05324	0.08733	0.12298	0.15840
	2	0.28713	0.35132	0.40770	0.45619
3	1	0.02437	0.04671	0.07285	0.10091
	2	0.11097	0.16855	0.22323	0.27339
	3	0.37520	0.44271	0.49994	0.54759
4	1	0.01316	0.02859	0.04841	0.07106
	2	0.05801	0.10108	0.14619	0.19043
	3	0.16393	0.23602	0.30026	0.35634
	4	0.44563	0.51160	0.56650	0.61134
5	1	0.00787	0.01903	0.03454	0.05326
	2	0.03433	0.06686	0.10388	0.14228
	3	0.09352	0.15241	0.20966	0.26265
	4	0.21087	0.29176	0.36066	0.41880
	5	0.50432	0.56657	0.61796	0.65948
6	1	0.00505	0.01342	0.02589	0.04167
	2	0.02195	0.04706	0.07776	0.11121
	3	0.05909	0.10646	0.15610	0.20444
	4	0.12795	0.19836	0.26322	0.32087
	5	0.25233	0.33846	0.40939	0.46776
	6	0.55471	0.61219	0.65968	0.69782
7	1	0.00342	0.00989	0.02014	0.03365
	2	0.01484	0.03463	0.06044	0.08977
	3	0.03974	0.07812	0.12108	0.16479
	4	0.08488	0.14424	0.20281	0.25731
	5	0.16026	0.23895	0.30852	0.36854
	6	0.28916	0.37826	0.44974	0.50745
	7	0.59897	0.65118	0.69467	0.72955
8	1	0.00241	0.00753	0.01611	0.02785
	2	0.01046	0.02637	0.04833	0.07428
	3	0.02796	0.05941	0.09675	0.13626
	4	0.05938	0.10930	0.16163	0.21233
	5	0.11038	0.17919	0.24399	0.30228
	6	0.19019	0.27481	0.34724	0.40830
	7	0.32215	0.41274	0.48390	0.54051
	8	0.63852	0.68524	0.72478	0.75655
9	1	0.00176	0.00589	0.01318	0.02350
	2	0.00763	0.02063	0.03954	0.06267
	3	0.02037	0.04646	0.07911	0.11492
	4	0.04314	0.08532	0.13201	0.17893
	5	0.07967	0.13926	0.19864	0.25409
	6	0.13495	0.21112	0.28026	0.34083
	7	0.21781	0.30665	0.38073	0.44203
	8	0.35196	0.44306	0.51338	0.56864
	9	0.67434	0.71551	0.75121	0.78004
10	1	0.00132	0.00471	0.01098	0.02014
	2	0.00572	0.0165	0.03295	0.05371
	3	0.01527	0.03715	0.06591	0.09849
	4	0.03229	0.06817	0.10992	0.15327
	5	0.05943	0.11104	0.16515	0.21741
	6	0.0999	0.16749	0.23214	0.29077
	7	0.15831	0.24021	0.31235	0.37421
	8	0.24331	0.33513	0.41004	0.47109
	9	0.37912	0.47004	0.53921	0.59303
	10	0.70714	0.74278	0.77476	0.80082

Table 2 Product moments of order statistics for $n = 2, 3, \ldots, 10$.

			α= 0.1, β=0.2			
			λ=3	λ=4	λ=5	λ=6
s	r	n	$\mu_{r,s:n}$	$\mu_{r,s:n}$	$\mu_{r,s:n}$	$\mu_{r,s:n}$
2	1	2	0.02896	0.04810	0.07041	0.09443
		3	0.00657	0.01488	0.02639	0.04044
		4	0.00226	0.00629	0.01277	0.02154
		5	0.00094	0.00309	0.00701	0.01282
		6	0.00044	0.00167	0.00419	0.00822
		7	0.00022	0.00097	0.00266	0.00556
		8	0.00012	0.00060	0.00177	0.00392
		9	0.00007	0.00038	0.00123	0.00286
		10	0.00004	0.00026	0.00087	0.00215
3	1	3	0.01403	0.02770	0.04512	0.06514
		4	0.00398	0.01029	0.01984	0.03219
		5	0.00158	0.00490	0.01066	0.01884
		6	0.00072	0.00263	0.00632	0.01201
		7	0.00036	0.00152	0.00400	0.00810
		8	0.00020	0.00093	0.00266	0.00571
		9	0.00011	0.00060	0.00184	0.00416
		10	0.00007	0.00040	0.00131	0.00312
	2	3	0.06628	0.10173	0.13971	0.17770
		4	0.01779	0.03665	0.06017	0.08649
		5	0.00693	0.01729	0.03213	0.05040
		6	0.00316	0.00922	0.01900	0.03206
		7	0.00159	0.00533	0.01201	0.02162
		8	0.00086	0.00326	0.00799	0.01523
		9	0.00049	0.00209	0.00552	0.01111
		10	0.00029	0.00139	0.00394	0.00832
4	1	4	0.00809	0.01810	0.03195	0.04876
		5	0.00262	0.00754	0.01554	0.02637
		6	0.00115	0.00391	0.00899	0.01649
		7	0.00057	0.00223	0.00564	0.01106
		8	0.00030	0.00136	0.00374	0.00777
		9	0.00017	0.00087	0.00258	0.00566
		10	0.00010	0.00058	0.00184	0.00424
	2	4	0.03598	0.06428	0.09674	0.13087
		5	0.01149	0.02656	0.04679	0.07051
		6	0.00499	0.01373	0.02703	0.04404
		7	0.00247	0.00783	0.01695	0.02951
		8	0.00132	0.00477	0.01122	0.02073
		9	0.00075	0.00305	0.00774	0.01509
		10	0.00045	0.00203	0.00552	0.01131
	3	4	0.10567	0.15300	0.20096	0.24673
		5	0.03181	0.06104	0.09490	0.13058
		6	0.01355	0.03118	0.05439	0.08109
		7	0.00664	0.01771	0.03399	0.05421
		8	0.00354	0.01077	0.02248	0.03805
		9	0.00201	0.00688	0.01550	0.02769
		10	0.00120	0.00457	0.01105	0.02074

Table 2 (Continued)

			α= 0.1, β=0.2			
			λ=3	λ=4	λ=5	λ=6
s	r	n	$\mu_{r,s:n}$	$\mu_{r,s:n}$	$\mu_{r,s:n}$	$\mu_{r,s:n}$
5	1	5	0.00513	0.01273	0.02400	0.03833
		6	0.00183	0.00575	0.01253	0.02209
		7	0.00086	0.00318	0.00767	0.01452
		8	0.00045	0.00191	0.00503	0.01013
		9	0.00026	0.00122	0.00346	0.00735
		10	0.00015	0.00081	0.00246	0.00550
	2	5	0.02244	0.04479	0.07223	0.10244
		6	0.00795	0.02018	0.03765	0.05897
		7	0.00374	0.01113	0.02302	0.03873
		8	0.00196	0.00670	0.01510	0.02701
		9	0.00111	0.00426	0.01038	0.01960
		10	0.00066	0.00282	0.00738	0.01466
	3	5	0.06177	0.10263	0.14622	0.18948
		6	0.02152	0.04578	0.07570	0.10852
		7	0.01004	0.02515	0.04615	0.07114
		8	0.00526	0.01510	0.03024	0.04956
		9	0.00296	0.00959	0.02077	0.03595
		10	0.00176	0.00635	0.01477	0.02689
	4	5	0.14493	0.20044	0.25457	0.30453
		6	0.04740	0.08604	0.12831	0.17092
		7	0.02164	0.04666	0.07752	0.11128
		8	0.01122	0.02785	0.05060	0.07731
6	1	6	0.00346	0.00941	0.01877	0.03117
		7	0.00133	0.00452	0.01034	0.01884
		8	0.00066	0.00262	0.00660	0.01286
		9	0.00037	0.00165	0.00449	0.00926
		10	0.00022	0.00108	0.00318	0.00690
	2	6	0.01508	0.03300	0.05638	0.08320
		7	0.00576	0.01583	0.03103	0.05026
		8	0.00287	0.00919	0.01982	0.03431
		9	0.00159	0.00576	0.01348	0.02470
		10	0.00093	0.00379	0.00955	0.01840
	3	6	0.04073	0.07479	0.11330	0.15305
		7	0.01546	0.03575	0.06221	0.09228
		8	0.00769	0.02071	0.03969	0.06295
		9	0.00424	0.01297	0.02698	0.04530
		10	0.00249	0.00854	0.01910	0.03375
	4	6	0.08921	0.14015	0.19168	0.24075
		7	0.03322	0.06621	0.10439	0.14427
		8	0.01639	0.03817	0.06638	0.09816
		9	0.00900	0.02386	0.04505	0.07055
		10	0.00528	0.01568	0.03187	0.05254
	5	6	0.18324	0.24411	0.30185	0.35388
		7	0.06384	0.11069	0.15969	0.20740
		8	0.03076	0.06289	0.10050	0.14003
		9	0.01672	0.03905	0.06790	0.10031
		10	0.00975	0.02559	0.04793	0.07457

Table 2 (Continued)

			α= 0.1, β=0.2			
			λ=3	λ=4	λ=5	λ=6
s	r	n	$\mu_{r,s:n}$	$\mu_{r,s:n}$	$\mu_{r,s:n}$	$\mu_{r,s:n}$
7	1	7	0.00245	0.00721	0.01513	0.02600
		8	0.00099	0.00363	0.00869	0.01630
		9	0.00052	0.00220	0.00575	0.01148
		10	0.00030	0.00143	0.00402	0.00848
	2	7	0.01065	0.02527	0.04541	0.06936
		8	0.00431	0.01273	0.02606	0.04348
		9	0.00226	0.00769	0.01724	0.03061
		10	0.00130	0.00499	0.01207	0.02261
	3	7	0.02858	0.05704	0.09100	0.12735
		8	0.01154	0.02868	0.05218	0.07976
		9	0.00603	0.01732	0.0345	0.05613
		10	0.00347	0.01123	0.02416	0.04146
	4	7	0.06130	0.10555	0.15262	0.19900
		8	0.02456	0.05283	0.08725	0.12435
		9	0.01280	0.03184	0.05759	0.08742
		10	0.00734	0.02063	0.04030	0.06453
	5	7	0.11716	0.17594	0.23304	0.28574
		8	0.04597	0.08692	0.13197	0.17728
		9	0.02373	0.05207	0.08677	0.12424
		10	0.01355	0.03364	0.06060	0.09158
	6	7	0.22033	0.28443	0.34404	0.39680
		8	0.08064	0.13457	0.18891	0.24038
		9	0.04059	0.07935	0.12281	0.16701
		10	0.02292	0.0509	0.08533	0.12264
8	1	8	0.00180	0.00569	0.01248	0.02212
		9	0.00076	0.00298	0.00741	0.01428
		10	0.00042	0.00186	0.00504	0.01031
	2	8	0.00782	0.01993	0.03745	0.05899
		9	0.00332	0.01043	0.02223	0.03808
		10	0.00181	0.00652	0.01513	0.02749
	3	8	0.02091	0.0449	0.07497	0.10823
		9	0.00886	0.02349	0.04448	0.06984
		10	0.00482	0.01467	0.03026	0.05040
	4	8	0.04449	0.08268	0.12532	0.16870
		9	0.01878	0.04318	0.07425	0.10875
		10	0.01021	0.02694	0.05048	0.07845
	5	8	0.08309	0.13587	0.18944	0.24039
		9	0.03478	0.07058	0.11182	0.15452
		10	0.01883	0.04392	0.07589	0.11131
	6	8	0.14503	0.20976	0.27069	0.32557
		9	0.05934	0.10739	0.27069	0.20759
		10	0.03179	0.06639	0.10682	0.14901
	7	8	0.25617	0.32185	0.38213	0.43474
		9	0.09754	0.15750	0.21608	0.27030
		10	0.05089	0.09574	0.14421	0.19220

Table 2 (Continued)

			α= 0.1, β=0.2			
			λ=3	λ=4	λ=5	λ=6
s	r	n	$\mu_{r,s:n}$	$\mu_{r,s:n}$	$\mu_{r,s:n}$	$\mu_{r,s:n}$
9	1	9	0.00136	0.00459	0.01049	0.01911
		10	0.00060	0.00248	0.00640	0.01264
	2	9	0.00591	0.01608	0.03148	0.05098
		10	0.00261	0.00869	0.01920	0.03370
	3	9	0.01579	0.03621	0.06298	0.09349
		10	0.00695	0.01957	0.03841	0.06180
	4	9	0.03347	0.06653	0.10512	0.14558
		10	0.01472	0.03592	0.06407	0.09619
	5	9	0.06193	0.10871	0.15828	0.20681
		10	0.02713	0.05855	0.09631	0.13647
	6	9	0.10546	0.16523	0.22366	0.27771
		10	0.04575	0.08846	0.1355	0.18264
	7	9	0.17250	0.24167	0.30512	0.36119
		10	0.07305	0.12737	0.18276	0.23543
	8	9	0.29078	0.35679	0.41685	0.46874
		10	0.11437	0.17945	0.24137	0.29760
10	1	10	0.00106	0.00378	0.00895	0.01673
	2	10	0.00458	0.01322	0.02687	0.04463
	3	10	0.01223	0.02977	0.05375	0.08183
	4	10	0.02587	0.05464	0.08964	0.12737
	5	10	0.04766	0.08904	0.13473	0.18069
	6	10	0.08031	0.13447	0.18951	0.24178
	7	10	0.12798	0.19341	0.25543	0.31153
	8	10	0.19943	0.27179	0.3368	0.39334
	9	10	0.32425	0.38958	0.44879	0.49956

5. Applicaton of Moments

In this section, we have calculated the *BLUEs* for the location and scale parameters μ and σ of IL-LL distribution using relations established in the this section based on Type II censored sample. Assume that $X_1, X_2, \ldots, X_n$ be a random sample of size n from IL-LL distribution with *pdf*

$$f(x) = \alpha\beta\lambda\left(\frac{x-\mu}{\sigma}\right)^{-\lambda-1}\left(1+\beta\left(\frac{x-\mu}{\sigma}\right)^{-\lambda}\right)^{-\alpha-1} \quad ; \; x>\mu, \;\; \sigma, \; \mu, \; \alpha, \; \beta, \; \lambda>0$$

and therefore its *cdf* is

$$F(x) = \left(1+\beta\left(\frac{x-\mu}{\sigma}\right)^{-\lambda}\right)^{-\alpha} \quad ; \; x>\mu, \;\; \sigma, \; \alpha, \; \beta, \; \lambda>0.$$

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n-c:n}, \; c = 0, 1, \ldots, n-1$ denote Type-II censored sample for the location-scale parameter from IL-LL distribution in (3). Let us denote $Z_{r:n} = \frac{(X_{r:n}-\mu)}{\sigma}, \; E\left(Z_{r:n}\right) = \mu^{(1)}_{r:n}, \; 1 \leq r \leq (n-c)$, and $\mathrm{Cov}\left(Z_{r:n}, \; Z_{s:n}\right) = \sigma_{r,s:n} = \mu^{(1,1)}_{r,s:n}$ $1 \leq r < s \leq (n-c)$. Then we can write the best linear unbiased estimators (BLUEs) of μ^* and σ^* (Arnold et al., 2008) as

$$\mu^* = a_1X_1 + a_2X_2 + \cdots + a_nX_n \tag{10}$$
$$\sigma^* = b_1X_1 + b_2X_2 + \cdots + b_nX_n \tag{11}$$

Here a_i's and b_i's are the entries of the matrix $C = \left(A'V^{-1}A\right)^{-1} A'V^{-1}$, with

$$\mathbf{A} = \begin{bmatrix} 1 & \mu \end{bmatrix}, \quad \mathbf{1}' = (1, \ldots, 1)_{1\times(n-c)}, \quad \mu' = (\mu_{1:n}, \ldots, \mu_{n-c:n})_{1\times(n-c)}$$

where μ is the mean of the first $n-c$ order statistics and V^{-1} is the inverse of the covariance matrix $V = (\sigma_{r,s:n})\,, 1 \le r,\ s \le n-c$.

Variances and covariance of these estimators are given by

$$\text{Var}(\mu^*) = d_{11}\sigma^2, \quad \text{Var}(\sigma^*) = d_{22}\sigma^2 \quad \text{and} \quad \text{Cov}(\mu^*, \sigma^*) = d_{12}\sigma^2,$$

where

$$D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \sigma^2 = (A'V^{-1}A)^{-1}.$$

Table 3 Variances and covariance of the *BLUEs*.

α= 0.1, β=0.2					
λ	n	c	Var(μ^*)	Var(σ^*)	Cov(μ^*,σ^*)
3	7	0	0.00011	0.39091	-0.00044
		1	0.00011	0.43995	-0.00051
		2	0.00011	0.63292	-0.00078
	10	0	0.00002	0.26354	-0.00007
		1	0.00002	0.27912	-0.00008
		2	0.00002	0.34748	-0.00009
		3	0.00002	0.46816	-0.00014
		4	0.00002	0.65594	-0.00016
4	7	0	0.00045	0.21733	-0.00123
		1	0.00046	0.25843	-0.00145
		2	0.00046	0.37980	-0.00207
	10	0	0.00012	0.14700	-0.00030
		1	0.00012	0.16032	-0.00033
		2	0.00012	0.20356	-0.00040
		3	0.00012	0.27581	-0.00051
		4	0.00012	0.38643	-0.00075
5	7	0	0.00123	0.17006	-0.00344
		1	0.00121	0.17586	-0.00306
		2	0.00123	0.26270	-0.00418
	10	0	0.00040	0.09540	-0.00079
		1	0.00040	0.10609	-0.00088
		2	0.00040	0.13653	-0.00106
		3	0.00040	0.18636	-0.00135
		4	0.00040	0.26116	-0.00178

Tables 4 and 5 display the necessary coefficients of the *BLUEs* for Type-II censored samples of order statistics for the sample sizes $n = 7,\ 10,\ \alpha = 0.1,\ \beta = 0.2,\ \lambda = 3,\ 4,\ 5$ and different censoring cases $c = 0(1)([n/2]-1)$. Similarly, we can obtain BLUE for any other choice of the parameters $\alpha,\ \beta,\ \lambda$. The coefficients of the *BLUEs* in Tables 4 and 5 are checked by using the conditions

$$\sum_{r=1}^{n-c} a_r = 1 \ \text{ and } \ \sum_{r=1}^{n-c} b_r = 0.$$

Table 4 Coefficients for the *BLUEs* of μ.

α= 0.1, β=0.2			a_i, $i = 1, 2, \ldots, (n-c)$				
λ	n	c					
3	7	0	1.09959	-0.07254	-0.01302	-0.00892	-0.00192
			-0.00262	-0.00057			
		1	1.10045	-0.07262	-0.01307	-0.00900	-0.00206
			-0.00370				
		2	1.10411	-0.07285	-0.01334	-0.00939	-0.00853
	10	0	1.08673	-0.03841	-0.05212	0.00972	-0.00017
			-0.00593	0.00138	-0.00099	-0.00010	-0.00010
		1	1.08683	-0.03840	-0.05214	0.00972	-0.00018
			-0.00594	0.00137	-0.00101	-0.00026	
		2	1.08704	-0.03842	-0.05215	0.00972	-0.00018
			-0.00595	0.00134	-0.00140		
		3	1.08794	-0.03850	-0.05213	0.00967	-0.00023
			-0.00602	-0.00074			
		4	1.08854	-0.03863	-0.05217	0.00969	-0.00025
			-0.00718				
4	7	0	1.12203	-0.07645	-0.02481	-0.00621	-0.00571
			-0.00558	-0.00328			
		1	1.12519	-0.07673	-0.02495	-0.00645	-0.00621
			-0.01085				
		2	1.13302	-0.07742	-0.02543	-0.00711	-0.02305
	10	0	1.13720	-0.10104	-0.02035	-0.00397	-0.00827
			0.00241	-0.00366	-0.00200	-0.00134	-0.00078
		1	1.13781	-0.10108	-0.02037	-0.00399	-0.00829
			0.00238	-0.00372	-0.00030	-0.00245	
		2	0.00061	1.17489	-0.10262	-0.04471	-0.01690
			-0.00632	-0.00338	-0.00156		
		3	1.14088	-0.10139	-0.02051	-0.00400	-0.00842
			0.00220	-0.00876			
		4	1.14529	-0.10179	-0.02044	-0.00421	-0.00857
			-0.01028				
5	7	0	1.16148	-0.24529	0.23205	-0.22549	0.14935
			-0.05337	-0.01873			
		1	1.14331	-0.07652	-0.02490	-0.00927	-0.00823
			-0.02438				
		2	1.15696	-0.07744	-0.02569	-0.01038	-0.04345
	10	0	1.12172	-0.06505	-0.02709	-0.01127	-0.00547
			-0.00275	-0.00112	-0.00215	-0.00384	-0.00298
		1	1.12358	-0.06635	-0.02538	-0.01201	-0.00550
			-0.00284	-0.00128	-0.00245	-0.00776	
		2	1.12656	-0.06643	-0.02550	-0.01207	-0.00563
			-0.00302	-0.00164	-0.01227		
		3	1.13121	-0.06693	-0.02546	-0.01222	-0.00574
			-0.00336	-0.01750			
		4	1.13811	-0.06705	-0.02587	-0.01229	-0.00606
			-0.02684				

Table 5 Coefficients for the *BLUEs* of σ.

α= 0.1, β=0.2			b_i, $i = 1, 2, \ldots, (n-c)$				
λ	n	c					
3	7	0	-4.56480	0.42277	0.31942	0.56447	1.04025
			1.76375	0.45413			
		1	-5.25013	0.48361	0.35907	0.63195	1.15481
			2.62070				
		2	-7.83538	0.64749	0.55010	0.90826	5.72953
	10	0	-3.97646	0.05683	0.34123	0.05003	0.18425
			0.36736	0.57957	0.94780	1.20980	0.23960
		1	-4.24000	0.01074	0.38815	0.04818	0.19333
			0.38775	0.61029	0.99469	1.60687	
		2	-5.50717	0.16559	0.45056	0.06898	0.23634
			0.47764	0.75407	3.35399		
		3	-7.66863	0.34602	0.39310	0.18401	0.34244
			0.64783	5.75524			
		4	-12.34638	1.41511	0.70069	0.06198	0.50356
			9.66504				
4	7	0	-3.02314	0.25216	0.16544	0.25571	0.5717
			1.18399	0.59413			
		1	-3.59501	0.30348	0.19152	0.30001	0.66274
			2.13725				
		2	-5.13648	0.43962	0.28650	0.42949	3.98086
	10	0	-2.83845	0.26859	0.06820	0.04409	0.10629
			0.15906	0.33319	0.59823	0.92414	0.33667
		1	-3.10326	0.28562	0.07512	0.04947	0.11372
			0.17236	0.35915	0.64164	1.40617	
		2	1.00066	-1.12853	0.07293	0.03612	0.00783
			0.00802	0.00035	0.00263		
		3	-5.10258	0.49898	0.16416	0.05576	0.20081
			0.29087	3.89200			
		4	-7.06217	0.67493	0.13493	0.14956	0.26876
			5.83400				
5	7	0	-3.17777	5.08991	-7.32495	6.73626	-4.55094
			2.29002	0.93747			
		1	-2.88752	0.22091	0.12407	0.19589	0.46263
			1.88403				
		2	-3.94266	0.29199	0.18584	0.28109	3.18374
	10	0	-2.23366	-0.02272	0.29136	-0.04497	0.05559
			0.11166	0.22275	0.45410	0.77747	0.38839
		1	-2.47597	0.14608	0.06797	0.05185	0.06059
			0.12291	0.24396	0.49352	1.28909	
		2	-2.97156	0.15999	0.08837	0.06098	0.08114
			0.15182	0.30433	2.12393		
		3	-3.77758	0.24668	0.08195	0.10045	0.21117
			3.05035				
		4	-4.98038	0.26753	0.15384	0.09955	0.15573
			4.30373				

5.1. Numerical illustration

In this section, we have presented the usefulness of coefficients of *BLUEs* of location and scale parameters as given in Tables 4 and 5, respectively. For this purpose, we have simulated here three data sets from IL-LL distribution by using the inverse transform method for given values as follows;

Data Set 1: For n=7, μ=0, σ=1, α=0.3, β=0.4, λ=0.5, we have

0.00190, 0.00020, 2.23419, 0.01129, 0.01475, 1.19372, 0.00005.

For n=10, μ=0, σ=1, α=0.3, β=0.4, λ=0.5, we have

0.18190, 0.00198, 0.05419, 0.00143, 0.00006, 0.00461, 0.18395, 0.00286, 0.09526, 0.01999.

Data Set 2: For n=7, μ=0, σ=1, α=1, β=0.4, λ=0.3, we have

0.00022, 0.06287, 0.00261, 0.00019, 0.00010, 2.06713, 3.32024.

For n=10, μ=0, σ=1, α=1, β=0.4, λ=0.3, we have

0.00035, 0.00682, 0.12522, 0.00003, 0.27655, 0.21628, 0.03953, 0.00004, 9.34846, 2.24118.

Data Set 3: For n=7, μ=0, σ=1, α=1.5, β=0.4, λ=0.3, we have

0.00100, 0.00807, 0.12479, 5.69157, 0.01808, 0.00314, 0.00626.

For n=10, μ=0, σ=1, α=1.5, β=0.4, λ=0.3, we have

1.85629, 0.12294, 8.40780, 3.47068, 0.05849, 0.08453, 2.67296, 3.17099, 0.81415, 0.03492.

Next, by using these samples, the *BLUEs* of μ and σ are computed based on complete as well as Type-II censored samples using (10) and (11). By using the *BLUEs* coefficients, we have

$$\begin{aligned}\mu^* &= \sum_{r=1}^{n-c} a_r X_{r:n}\\ &= (0.00190 \times 1.09959) + (0.00020 \times -0.07254) + (2.23419 \times -0.01302) +\\ &\quad (0.01129 \times -0.00892) + (0.01475 \times -0.00192) + (1.19372 \times -0.00262)\\ &\quad + (0.00005 \times -0.00057)\\ &= -0.03027\end{aligned}$$

and

$$\begin{aligned}\sigma^* &= \sum_{r=1}^{n-c} b_r X_{r:n}\\ &= (0.00190 \times -4.56480) + (0.00020 \times 0.42277) + (2.23419 \times 0.31942)\\ &\quad + (0.01129 \times 0.56447) + (0.01475 \times 1.04025) + (1.19372 \times 1.76375)\\ &\quad + (0.00005.45413)\\ &= 2.83221\end{aligned}$$

Remark: For $\alpha = 1, \beta = 1$, we get all the above results for Log-Logistic distribution.

6. Conclusion

In this study, we consider a novel model within the inverse Lomax-G family, referred to as the IL-LL distribution. We explored several statistical properties of the IL-LL distribution, including explicit expressions for the single and product moments of order statistics. Using these expressions, we derived the means, variances, and covariances for the order statistics and used these results to compute the *BLUEs* of the location and scale parameters. The probability density function plots presented in Figure 1 demonstrate that the distribution exhibits a right-skewed shape. Additionally, the hazard function plots reveal a decreasing pattern, further characterizing the distribution's behavior. These findings highlight the flexibility and applicability of the IL-LL distribution in modeling diverse datasets. The best linear invariant (BLI) estimators of the scale and location-scale parameters of the IL-LL distribution, as well as best linear unbiased (BLU) and best linear invariant (BLI) predictors of future unobserved order statistics, might be developed using these findings; see, for example, Balakrishnan and Cohen (2014).

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