Thailand Statistician
January 2026; 24(1): 80-97
http://statassoc.or.th
Contributed paper

A New Exponent-Generator Family of Statistical Distribution with Simulation and Analysis to Cancer Data

Nuzhat Ahad [a], Ahmad SP [a] and Reshi JA*[b]
[a] Department of Statistics, University of Kashmir, Srinagar, J&K, India
[b] Department of Statistics, Govt. Degree College, Pulwama, J&K, India
*Corresponding author; e-mail: reshijavaid19@gmail.com

Received: 28 May 2024
Revised: 31 May 2025
Accepted: 4 August 2025

Abstract

A novel family of distribution has been introduced, named as "exponent-Generator family of statistical distribution", designed for optimal univariate modeling. We explored the structural and characterizing properties of a newly proposed distribution, the exponent power function (EPF) distribution. We provide explicit expressions for the probability density function (PDF), cumulative distribution function (CDF), reliability function (RF), and hazard rate function (HRF). Also the r-th moment, moment generating function (MGF), and the order statistics are obtained. The manuscript also includes a detailed discussion on the shapes of PDF and HRF for selected parameter values, providing valuable insights into the behavior of distribution. Moreover, we discussed maximum likelihood estimation (MLE) and Bayesian estimation method. The adaptability of the proposed distribution is evaluated by analyzing the three real data sets related to lifetime of cancer patients as well as a simulated dataset.

Keywords: Exponent-Generator family of distribution, power function distribution, exponent power function distribution, hazard rate function, maximum likelihood estimation.

1. Introduction

In applied fields, researchers and practitioners encounter difficulties when dealing with diverse lifetime data sets in health and natural sciences. To simplify modeling, they are investigating the power function (PF) distribution, a versatile and simple lifetime distribution. Power function distribution emerges in several scientific fields and is often employed in the assessment of semiconductor devices and electrical component reliability. The PF is also called the inverse of Pareto distribution (see Dallas(Dallas, 1976)). Estimation of the PF parameters has been done through both classical and bayesian approach, by various authors, for instance; Sultan and Ahmad (Sultan et al., 2014) , Hanif et al.(Hanif et al., 2015) ,Sathar et al. (Sathar and Sathyareji, 2022), Wang and Choa (Wang, 2023) and Nuzhat et al. (Ahad et al., 2024).

Meniconi and Bery(Meniconi and Barry, 1996) proposed the probability density function (PDF) and cumulative distribution function (CDF) of two parameter PF distribution with scale parameter λ and shape parameter η respectively as,

$$f(y,\lambda,\eta)=\frac{\eta y^{\eta-1}}{\lambda^{\eta}}\ ;\quad 0<y<\lambda,\lambda>0,\eta>0$$

$$F(y,\lambda,\eta)=\left(\frac{y}{\lambda}\right)^{\eta};\quad 0<y<\lambda,\lambda>0,\eta>0.$$

Various generalizations of PF distribution has been introduced by many researchers, some recent include, exponentiated Weibull PF distribution by Hassan and Assar(Hassan and Assar, 2017) . Haq et al(Haq et al., 2018) introduced McDonald PF distribution . Hassan et al. introduced the odd generalized exponential PF distribution. Zaka and Akhter(Zaka et al., 2020) introduced Reflected PF, Haq et al(Haq et al., 2021) introduced Frechet FP. Haq et al(Ahsan-ul-Haq et al., 2023) introduced New cubic transmuted PF distribution, Alshawarbeh et al.(Alshawarbeh et al., 2024) introduced innovative model of PF distribution utilizes a newly developed logarithmic transformation method.

The random behavior of these datasets can lead to deviation from well-established probability models, such limitation demands for the larger family of probability distributions and development of new generalized probability models which are richer and more versatile. In the recent years, generated family of continuous distributions is a new evolution for generating and extending the well-known probability models by introducing one or more extra shape parameters to the baseline distribution. We present a list of some generated families as follows; the exponentiated generalized (EG) by Cordeiro et al.(Cordeiro et al., 2013), Weibull-G by Bourguignon et al.(Bourguignon et al., 2014), exponentiated Weibull-G by Hassan and Elgarhy(Hassan and Elgarhy, 2016), inverse Weibull-G by Hassan and Nassr(Hassan and Nassr, 2018) and power Lindley-G by Hassan and Nassr(Hassan and Nassr, 2019),exponentaited-G by Mutairi et al(Hassan and Nassr, 2020), Ratio transformed weibull by Murtaza et al(Lone et al., 2022) among others.

In this manuscript new family of Power Function is introduced named as Exponent-Generator family. As far as we know, the new family has not been previously discussed. and we study one of its special sub-models using the PF distributions as a baseline model, the new proposed model is named as Exponent Power Function(EPF) distribution.

The following are the key motivations for generating new Exponent-Generator family of distribution:

- A simple and efficient method to create the best models.
- To enhance the flexibility and characteristics of existing models.
- Easy to use, making models highly effective for data analysis.
- To provide superior fits than the other adapted models to complex real data sets.

2. Basic Characteristics of Exponent-Generator Family

Let y be a continuous random variable and $F(y)$ be its CDF, then CDF of new generator (Exponent-Generator family) $F(y,\zeta)$ for $0<y<\lambda$ and $\zeta\geq -1$, is defined as follows

$$F(y,\zeta)=\begin{cases}\frac{F(y)}{e^{\zeta(1-F(y))}}; & \zeta\geq -1,\zeta\neq 0\\ F(y); & \zeta=0\end{cases}\quad \text{or,}\quad F(y,\zeta)=\begin{cases}F(y)\,e^{-\zeta(F\bar{(y)})}; & \zeta\geq -1,\zeta\neq 0\\ F(y); & \zeta=0.\end{cases}$$

Clearly, $F(y,\zeta)$ is a valid CDF. If $F(y)$ is an standard CDF with the pdf $f(y)$, then $F(y,\zeta)$ is also a standard commulative distribution function with the pdf

$$f(y,\zeta)=\begin{cases}f(y)(1+\zeta F(y))e^{-\zeta F\bar{(y)}}; & \zeta\geq -1,\zeta\neq 0\\ f(y); & \zeta=0.\end{cases}$$

The RF $R(y,\zeta)$ and The HRF $h(y,\zeta)$ of new class are respectively given by

$$R(y,\zeta)=\left(e^{\zeta(F\bar{(y)})}-F(y)\right)e^{-\zeta(F\bar{(y)})};\qquad \zeta\geq -1. \tag{1}$$

$$h(y,\zeta)=\frac{f(y)(1+\zeta F(y))}{\left(e^{\zeta \bar{F(y)}}-F(y)\right)}\ ; \qquad \zeta\geq -1. \tag{2}$$

The hazard rate $h(y,\zeta)$ in terms of reliability function $\bar{F(y)}$ and HRF $h(y)$ of baseline distribution can be written as

$$h(y,\zeta)=h(y)\bar{F(y)}\frac{(1+\zeta F(y))}{\left(e^{\zeta \bar{F(y)}}-F(y)\right)}\ ; \qquad \zeta\geq -1. \tag{3}$$

From (3), clearly, it is observed that,

$$\lim_{y\to-\infty} h(y,\zeta)=\frac{1}{e^{\zeta}}\lim_{y\to-\infty} h(y) \quad \text{and} \quad \lim_{y\to\infty} h(y,\zeta)=\lim_{y\to\infty} h(y).$$

Advantages of Exponent-Generator family:

- The introduced parameter ζ in the Exponent-Generator family provides enhanced flexibility, enabling effortless adaptation to diverse dataset characteristics.
- When $\zeta = 0$, it reverts to the original CDF, ensuring consistency with base line distribution without added complexity.
- $F(y,\zeta)$ reliably maintains the core properties of a CDF, confirming its trustworthiness for statistical use.
- This Exponent-Generator family is versatile, effectively enhancing various probability distributions to improve their fit to real-life datasets.

We have developed a new distribution in Section 3 called EPF distribution by adapting the power function distribution to the exponent-Generator family, thereby enhancing its applicability and precision in modeling the time until significant events, such as failure or death. This novel distribution is particularly well-suited for lifetime data, providing exceptional flexibility in representing various hazard functions. Its adaptability enables it to accurately capture a range of failure rates, including constant, increasing, decreasing , bathtub and j patterns commonly observed in survival and reliability analysis.

3. EPF Distribution and Its Characteristics

Let $\Theta=(\zeta,\lambda,\eta)^T$. From (2), The continuous random variable y follows EPF distribution if its CDF, with scale parameter $\lambda>0$ shape parameters and $\zeta\geq -1, \eta>0$, for $0<y<\lambda$, is given by

$$F_{EPF}(y,\Theta)=\left(\frac{y}{\lambda}\right)^{\eta} e^{-\zeta\left(1-\left(\frac{y}{\lambda}\right)^{\eta}\right)}\ ;\ \zeta\geq -1 \tag{4}$$

and the corresponding PDF is

$$f_{EPF}(y,\Theta)=\frac{\eta}{\lambda^{\eta}}y^{\eta-1}\left(1+\zeta\left(\frac{y}{\lambda}\right)^{\eta}\right)e^{-\zeta\left(1-\left(\frac{y}{\lambda}\right)^{\eta}\right)}\ ;\ \zeta\geq -1. \tag{5}$$

3.1. Limiting properties of CDF and PDF

Asymptotes of the CDF and PDF at $y\to 0$ are given by

$$F(y)|y\to 0\sim 0 \text{ and } f(y)|y\to 0\sim 0.$$

Asymptotes of the CDF and PDF at $y\to\lambda$ are given by

$$F(y)|y\to\lambda\sim 1 \text{ and } f(y)|y\to\lambda\sim\frac{\eta}{\lambda}(1+\zeta).$$

The obtained expressions explores how the parameter ζ dynamically effects the asymptotic behavior of both $F(y)$ and $f(y)$.

3.2. Shapes of PDF

In this subsection, we explore diverse forms of the PDF of the Exponent power function distribution. Figure 1 presents some different curves of the PDF for different combination of EPF parameters ζ, η and for $\lambda = 2$. It is noted from Figure 1 that the density curves for EPF distribution can be decreasing, decreasing-increasing, and increasing.

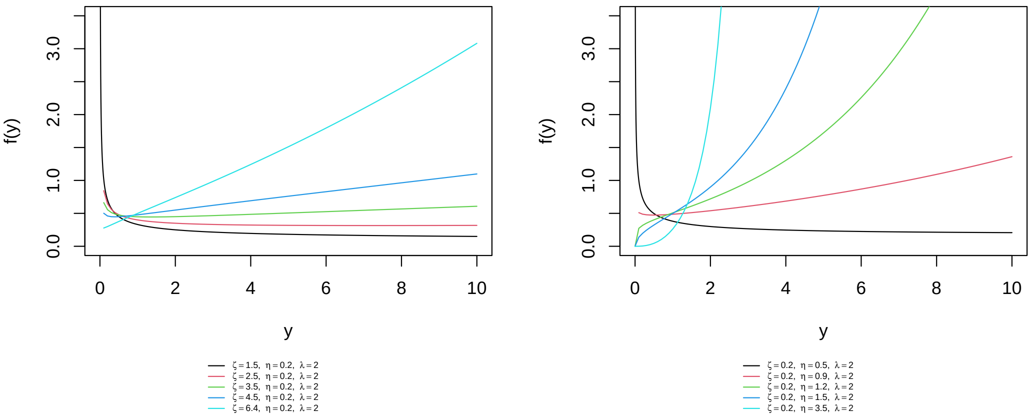

Figure 1 Density plot of EPF distribution for different combinations of ζ, η and $\lambda = 2$

3.3. Reliability and Associated measures

In this subsection, reliability and related measures of EPF distribution are obtained. The RF $R_{EPF}(y, \Theta)$ and the HRF $h_{EPF}(y, \Theta)$ for $y < \lambda$ are, respectively, given by

$$R_{EPF}(y, \Theta) = 1 - F(y, \Theta) = \frac{e^{\zeta\left(1-\left(\frac{y}{\lambda}\right)^{\eta}\right)} - \left(\frac{y}{\lambda}\right)^{\eta}}{e^{\zeta\left(1-\frac{y}{\lambda}\right)^{\eta}}}; \quad \zeta \geq -1 \tag{6}$$

$$h_{EPF}(y, \Theta) = \frac{f(y, \Theta)}{R(y, \Theta)} = \frac{\frac{\eta}{\lambda^{\eta}} y^{\eta-1}\left(1 + \zeta\left(\frac{y}{\lambda}\right)^{\eta}\right)}{e^{\zeta\left(1-\left(\frac{y}{\lambda}\right)^{\eta}\right)} - \left(\frac{y}{\lambda}\right)^{\eta}}; \quad \zeta \geq -1.$$

3.4. Shapes of HRF

In this subsection, we explore diverse forms of the HRF of the EPF distribution. Figure 2 presents some different curves of the HRF for different combination of EPF parameters ζ,η and λ. It is noted that the EPF distribution possesses increasing, decreasing, J-shaped and bathtub shape HRF.

The behavior of the HRF at maximums for different values of shape parameter η.

$$h(y) = \begin{cases} 0 & \text{for } 0 < \eta < 1, y \sim 0, \\ \frac{1}{\lambda e^{\zeta}} & \text{for } \eta = 1, y \sim 0, \\ \infty & \text{for } \eta > 1, y \sim 0, \\ \infty & \text{for } \eta > 0, y \sim \lambda. \end{cases}$$

Remark 1

- For $0 < \eta < 1$, the hazard rate starts from zero at $y = 0$ and steadily increases as $y \to \lambda$.
- For $\eta = 1$, the hazard rate is constant at $\frac{1}{\lambda e^{\zeta}}$ for $y = 0$ and increases to infinity at $y \to \lambda$.
- For $\eta > 1$, the hazard rate experiences an immediate and infinite increase at $y = 0$ and maintains an infinite level at $y = \lambda$.

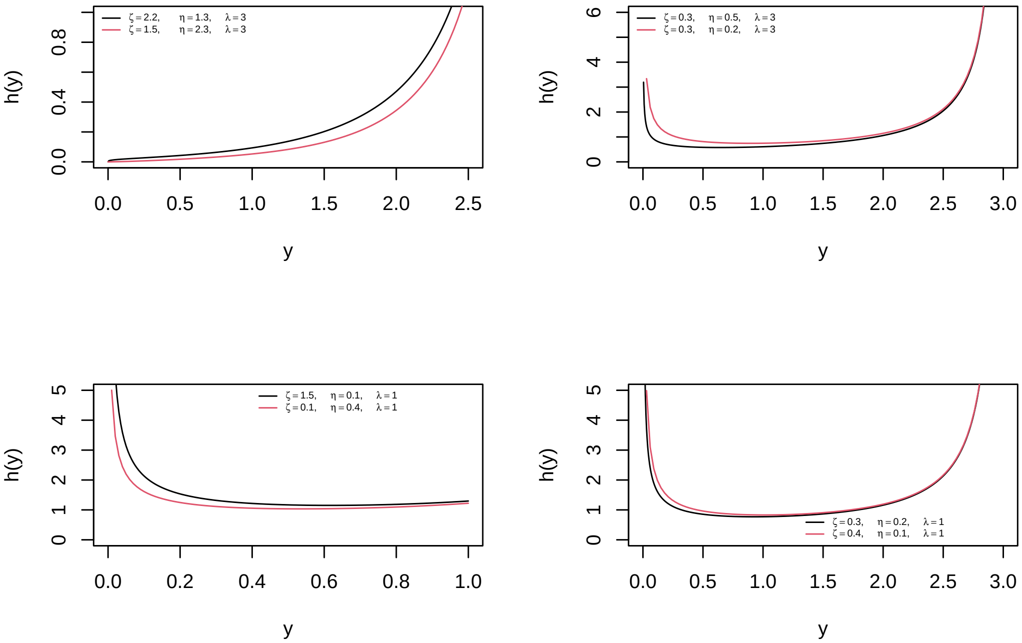

Figure 2 Hazard function plot of EPF distribution for different combinations of ζ, η andλ

The reverse HRF $h_r(y, \Theta)$ for a random varaible $y < \lambda \sim$ (EPF) distribution, is given by

$$h_r(y, \Theta) = \frac{f(y, \Theta)}{F(y, \Theta)} = \frac{\eta}{y}\left(1 + \zeta\left(\frac{y}{\lambda}\right)^{\eta}\right); \quad \zeta \geq -1.$$

The odds ratio $O(y, \Theta)$ for a random varaible $y < \lambda \sim$ (EPF) distribution, is given by

$$O(y, \Theta) = \frac{F(y, \Theta)}{f(y, \Theta)} = \frac{y}{\eta\left(1 + \zeta\left(\frac{y}{\lambda}\right)^{\eta}\right)}; \quad \zeta \geq -1.$$

The Mills ratio $M(y, \Theta)$ for a random variable $y < \lambda \sim$ (EPF) distribution, is given by

$$M(y, \Theta) = \frac{e^{\zeta\left(1 - \left(\frac{y}{\lambda}\right)^{\eta}\right)} - \left(\frac{y}{\lambda}\right)^{\eta}}{\frac{\eta}{\lambda^{\eta}} y^{\eta - 1}\left(1 + \zeta\left(\frac{y}{\lambda}\right)^{\eta}\right)}; \quad \zeta \geq -1.$$

3.5. Moment and moment generating function

In this segment, the rth moment , mean ,variance and the MGF of the EPF distribution are obtained.

Let Y follow the EPF distribution with two shape parameters $(\zeta \geq -1, \eta > 0)$ then, the rth ordinary moment (μ^r) of Y has the form

$$E[Y^r] = \lambda^r\left[1 - e^{-\zeta}\frac{r}{\eta}(-\zeta)^{-\frac{r+\eta}{\eta}}\left(\Gamma(\frac{r+\eta}{\eta}) - \Gamma(\frac{r+\eta}{\eta}, -\zeta)\right)\right]; \quad \zeta \geq -1. \tag{7}$$

where $\Gamma(b, -a) = \int\limits_{-a}^{\infty} y^{b-1} e^{-y} dx$ the first and second ordinary moments can be obtained by substituting $r = 1, 2$ in (8), respectively. The expressions of mean,second moment and variance are given, respectively

$$E[Y]=\lambda\left[1-e^{-\zeta}\frac{1}{\eta}(-\zeta)^{-\frac{1+\eta}{\eta}}\left(\Gamma(\frac{1+\eta}{\eta})-\Gamma(\frac{1+\eta}{\eta},-\zeta)\right)\right]\ ;\quad \zeta\geq -1$$
$$E[Y^2]=\lambda^2\left[1-e^{-\zeta}\frac{2}{\eta}(-\zeta)^{-\frac{2+\eta}{\eta}}\left(\Gamma(\frac{2+\eta}{\eta})-\Gamma(\frac{2+\eta}{\eta},-\zeta)\right)\right]\ ;\quad \zeta\geq -1.$$

and,

$$V[Y]=\lambda^2\Bigg\{1-e^{-\zeta}\frac{2}{\eta}(-\zeta)^{-\frac{2+\eta}{\eta}}\left(\Gamma(\frac{2+\eta}{\eta})-\Gamma(\frac{2+\eta}{\eta},-\zeta)\right)$$
$$-\left(1-e^{-\zeta}\frac{1}{\eta}(-\zeta)^{-\frac{1+\eta}{\eta}}\left(\Gamma(\frac{1+\eta}{\eta})-\Gamma(\frac{1+\eta}{\eta},-\zeta)\right)\right)^2\Bigg\};\quad \zeta\geq -1. \tag{8}$$

If $Y\sim EPF(\zeta,\eta)$, then the moment generating function (MGF) $(M_y(t))$ of Y is given by

$$M_y(t)=\sum_{i=0}^{\infty}\frac{(\lambda t)^i}{i!}\left[1-\frac{i}{\eta}e^{-\zeta}(-\zeta)^{-\frac{i+\eta}{\eta}}\left(\Gamma(\frac{i}{\eta}+1)-\Gamma(\frac{i}{\eta}+1,-\zeta)\right)\right].$$

3.6. Incomplete moments and Associated measures

The nth incomplete moment of the EPF distribution, say $I_n(t)$ is given by

$$I_n(t)=\int_0^t y^n f(y)dy \tag{9}$$

using (5) in (9),we have

$$I_n(t)=\lambda^n\left[(\frac{t}{\lambda})^{n+\eta}e^{-\zeta(1-(\frac{t}{\lambda})^\eta)}-\frac{ne^{-\zeta}}{\eta}(-\zeta)^{-\frac{n+\eta}{\eta}}\left(\Gamma(\frac{n+\eta}{\eta})-\Gamma(\frac{n+\eta}{\eta},-\zeta\left(\frac{t}{\lambda}\right)^\eta)\right)\right]. \tag{10}$$

The first incomplete moments of the EPF distribution is obtained by setting $n=1$ in (10), and is given by

$$I(t)=\int_0^t yf(y)dy$$
$$=\lambda\left[(\frac{t}{\lambda})^{1+\eta}e^{-\zeta(1-(\frac{t}{\lambda})^\eta)}-\frac{e^{-\zeta}}{\eta}(-\zeta)^{-\frac{1+\eta}{\eta}}\left(\Gamma(\frac{1+\eta}{\eta})-\Gamma(\frac{1+\eta}{\eta},-\zeta\left(\frac{t}{\lambda}\right)^\eta)\right)\right]. \tag{11}$$

The mean deviations gives us useful information about the characteristics of a population and it can be computed using the first incomplete moments. Basically, mean deviations help us see how spread out or dispersed the things are in a population. Indeed, The extent of dispersion in a population can be assessed by examining the overall deviations from both the mean and median. The mean deviations of Y about the mean μ and about the median M can be calculated from the following relations

$$\delta_1(y)=2\mu F(\mu)-2I(\mu) \tag{12}$$

and

$$\delta_2(y)=\mu-2I(M) \tag{13}$$

where $I(t) = \int_0^t yf(y)dy$ is the first incomplete moments.

By using (11), we get

$$I(\mu) = \lambda\left[(\frac{\mu}{\lambda})^{1+\eta}e^{-\zeta(1-(\frac{\mu}{\lambda})^{\eta})} - \frac{e^{-\zeta}}{\eta}(-\zeta)^{-\frac{1+\eta}{\eta}}\left(\Gamma(\frac{1+\eta}{\eta}) - \Gamma(\frac{1+\eta}{\eta}, -\zeta\left(\frac{\mu}{\lambda}\right)^{\eta})\right)\right], \quad (14)$$

$$I(M) = \lambda\left[(\frac{M}{\lambda})^{1+\eta}e^{-\alpha(1-(\frac{M}{\lambda})^{\eta})} - \frac{e^{-\zeta}}{\eta}(-\zeta)^{-\frac{1+\eta}{\eta}}\left(\Gamma(\frac{1+\eta}{\eta}) - \Gamma(\frac{1+\eta}{\eta}, -\alpha\left(\frac{M}{\lambda}\right)^{\eta})\right)\right]. \quad (15)$$

Substituting, (14), (15) and (4) in (12) and (13) , we will get the expressions for mean deviation of a continuous random variable Y from mean and median respectively. Also the first incomplete moments play a significant role in constructing Bonferroni and Lorenz curves, which are often applied in economics for understanding income distribution, also find utility in reliability studies, facilitating the analysis of failure rates and system performance. The Lorenz and Bonferroni curves are obtained, respectively, as follows

$$L(t) = \frac{I(t)}{E(t)}\frac{\left[(\frac{t}{\lambda})^{1+\eta}e^{-\zeta(1-(\frac{t}{\lambda})^{\eta})} - \frac{e^{-\zeta}}{\eta}(-\zeta)^{-\frac{1+\eta}{\eta}}\left(\Gamma(\frac{1+\eta}{\eta}) - \Gamma(\frac{1+\eta}{\eta}, -\zeta\left(\frac{t}{\lambda}\right)^{\eta})\right)\right]}{\left[1 - \frac{e^{-\zeta}}{\eta}(-\zeta)^{-\frac{1+\eta}{\eta}}\left(\Gamma(\frac{1+\eta}{\eta}) - \Gamma(\frac{1+\eta}{\eta}, -\zeta)\right)\right]},$$

$$\begin{aligned} B(t) &= \frac{L(t)}{F(t)} \\ &= \left(\frac{\lambda}{t}\right)^{\eta} e^{\zeta(1-(\frac{t}{\lambda})^{\eta})}\frac{\left[(\frac{t}{\lambda})^{1+\eta}e^{-\zeta(1-(\frac{t}{\lambda})^{\eta})} - \frac{e^{-\zeta}}{\eta}(-\zeta)^{-\frac{1+\eta}{\eta}}\left(\Gamma(\frac{1+\eta}{\eta}) - \Gamma(\frac{1+\eta}{\eta}, -\zeta\left(\frac{t}{\lambda}\right)^{\eta})\right)\right]}{\left[1 - \frac{e^{-\zeta}}{\eta}(-\zeta)^{-\frac{1+\eta}{\eta}}\left(\Gamma(\frac{1+\eta}{\eta}) - \Gamma(\frac{1+\eta}{\eta}, -\zeta)\right)\right]}. \end{aligned}$$

3.7. Mean residual life and mean waiting time

Mean residual life (MRL) is the predicted extra lifespan given that a component has survived up to certain time t. Suppose that y is a continuous random variable with RF $R(y)$,then The MRL function, say $\mu(t)$, is given by

$$\mu(t) = \frac{1}{R(t)}\left(E(t) - \int_0^t yf(y)dy\right) - t \quad (16)$$

where

$$E(t) = \lambda\left[1 - \frac{e^{-\zeta}}{\eta}(-\zeta)^{-\frac{1+\eta}{\eta}}\left(\Gamma(\frac{1+\eta}{\eta}) - \Gamma(\frac{1+\eta}{\eta}, -\zeta)\right)\right]. \quad (17)$$

Substituting (6), (17) and (11) in (16), $\mu(t)$ can be written as

$$\begin{aligned} \mu(t) = -t + &\frac{\lambda e^{\zeta(1-(\frac{t}{\lambda})^{\eta})}}{e^{\zeta(1-(\frac{t}{\lambda})^{\eta})} - (\frac{t}{\lambda})^{\eta}} \\ &\left\{1 - \left(\frac{t}{\lambda}\right)^{1+\eta} e^{-\zeta(1-(\frac{t}{\lambda})^{\eta})} + \frac{e^{-\zeta}}{\eta}(-\zeta)^{-\frac{1+\eta}{\eta}}\left[\Gamma(\frac{1+\eta}{\eta}, -\zeta) - \Gamma(\frac{1+\eta}{\eta}, -\zeta\left(\frac{t}{\lambda}\right)^{\eta})\right]\right\}. \end{aligned}$$

The mean waiting time (MWT) is the average time that has passed since the failure of an object, considering that this failure occurred within the interval [0, t]. The MWT of y, say $\bar{\mu}(t)$, is defined by

$$\bar{\mu}(t) = t - \frac{1}{F(t)}\int_0^t yf(y)dy. \quad (18)$$

Substituting (4) and (11) in (18), we get

$$\bar{\mu}(t) = t - \frac{1}{e^{\zeta(1-(\frac{t}{\lambda})^{\eta})}(\frac{t}{\lambda})^{\eta}} \left\{ \left(\frac{t}{\lambda}\right)^{1+\eta} e^{-\zeta(1-(\frac{t}{\lambda})^{\eta})} - \frac{e^{-\zeta}}{\eta}(-\zeta)^{-\frac{1+\eta}{\eta}} \left(\Gamma(\frac{1+\eta}{\eta}) - \Gamma(\frac{1+\eta}{\eta}, -\zeta\left(\frac{t}{\lambda}\right)^{\eta})\right)\right\}.$$

3.8. Renyi entropy

The measure entropy quantify the degree of variability of a random variable y. The Renyi entropy $H_R(\delta)$ of EPF distribution is defined by

$$H_R(\delta) = \frac{1}{1-\delta} \log \int_{-\infty}^{\infty} f^{\delta}(y)dy, \quad \delta > 0, \delta \neq 1$$

substituting, (5), we get, Renyi entrophy as

$$H_R(\delta) = \frac{1}{1-\delta} \log \Bigg[\eta^{\delta-1}\lambda^{1-\delta}e^{-\zeta\delta} \sum_{k=0}^{\infty} {}^{\delta}C_k \zeta^k (-\zeta\delta)^{\frac{\delta\eta+k\eta-2\eta-\delta+1}{\eta}} \left[\Gamma(\frac{\delta\eta+k\eta-\eta-\delta+1}{\eta}) - \Gamma(\frac{\delta\eta+k\eta-\eta-\delta+1}{\eta}, -\zeta\delta)\right]\Bigg], \quad \delta > 0, \delta \neq 1. \tag{19}$$

3.9. Order statistics

Let $y_1, y_2, ..., y_n$ be a random sample of size n, and let $y_{r:n}$ denote the rth order statistic, then, the CDF of $y_{r:n}$, say $F_{r:n}(y)$ is given by

$$F_{r:n}(y) = \sum_{j=r}^{n} {}^{n}C_j F^J(y) \left[1 - F(y)\right]^{n-j} \tag{20}$$

and the corresponding PDF is given by

$$f_{r:n}(y) = \frac{n!}{(r-1)!(n-r)!}[F(y)]^{r-1}f(y)[(1-F(y))]^{n-r}. \tag{21}$$

Substituting (4) and (5) in (20) and (21), we get the CDF and PDF of rth order statistics, respectively as

$$F_{r:n}(y) = \sum_{j=r}^{n} {}^{n}C_j \left[\left(\frac{y}{\lambda}\right)^{\eta} e^{-\zeta(1-(\frac{y}{\lambda})^{\eta})}\right]^{j} \left[1 - \left(\frac{y}{\lambda}\right)^{\eta} e^{-\zeta(1-(\frac{y}{\lambda})^{\eta})}\right]^{n-j}$$

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} \frac{\eta}{\lambda^{\eta r}} y^{\eta r-1} \left(1 + \zeta\left(\frac{y}{\lambda}\right)^{\eta}\right) e^{-\zeta r(1-(\frac{y}{\lambda})^{\eta})} \left[1 - \left(\frac{y}{\lambda}\right)^{\eta} e^{-\zeta(1-(\frac{y}{\lambda})^{\eta})}\right]^{n-r}.$$

4. Parameters Estimation

In this section estimation of parameters of EPF distribution has been done by both classical and Bayesian approach.

4.1. Maximum likelihood estimation

Let a random sample $(y_1, y_2, ..., y_n)$ of size n is drawn from EPF distribution, then the Log-likelihood function is

$$l = n\log\eta - n\eta\log\lambda + (\eta-1)\sum_{i=1}^{n}\log y_i + \sum_{i=1}^{n}\log\left(1+\zeta\left(\frac{y_i}{\lambda}\right)^{\eta}\right) - \zeta\sum_{i=1}^{n}\left(1-\left(\frac{y_i}{\lambda}\right)^{\eta}\right). \quad (22)$$

To obtain MLEs of ζ, λ and η, the (22) is partially differentiating with respect to the corresponding parameters and equating to zero, we have

$$\frac{\partial l}{\partial \zeta} = \sum_{i=1}^{n}\frac{(y_i)^{\eta}}{(\lambda)^{\eta}\left(1+\zeta\left(\frac{y_i}{\lambda}\right)^{\eta}\right)} - \sum_{i=1}^{n}\left(1-\left(\frac{y_i}{\lambda}\right)^{\eta}\right) \quad (23)$$

$$\frac{\partial l}{\partial \eta} = \frac{n}{\eta} - n\log\lambda + \sum_{i=1}^{n}\log y_i + \zeta\sum_{i=1}^{n}\left(\frac{y_i}{\lambda}\right)^{\eta}\log\left(\frac{y_i}{\lambda}\right)\left(\frac{2+\zeta\left(\frac{y_i}{\lambda}\right)^{\eta}}{1+\zeta\left(\frac{y_i}{\lambda}\right)^{\eta}}\right) \quad (24)$$

$$\frac{\partial l}{\partial \lambda} = -\frac{n\eta}{\lambda} - \frac{\zeta\eta\sum\limits_{i=1}^{n} {y_i}^{\eta}\left(2+\zeta\left(\frac{y_i}{\lambda}\right)^{\eta}\right)}{\lambda^{\eta+1}\left(1+\zeta\left(\frac{y_i}{\lambda}\right)^{\eta}\right)}. \quad (25)$$

The above three equations (23), (24) and (25) are not in explicit form. Thus, It's proving challenging to compute the parameter values. However the maximum likelihood estimates of the parameter, denoted by ζ, η and λ can be estimated using Newton-Raphson procedure provided by R software.

4.2. Bayesian estimation method

In this we estimated the shape parameter ζ of EPF distribution using a Bayesian approach, applying the squared error loss function (SELF) and quadratic loss function (QLF), assuming uniform prior $g(\zeta) = 1$.

Let $(y_1, y_2, ..., y_n\ )$ be a random sample of size n is drawn from EPF distribution, The Joint Probability Density Function of y and given ζ is given by:

$$L(\underline{y}|\zeta) \propto \left(1+\zeta(\frac{y}{\lambda})\right)^{n} e^{\left(-\zeta\sum_{i=1}^{n}\left(1-(\frac{x_i}{\lambda})^{\beta}\right)\right)}. \quad (26)$$

Using binomial expansion $(1+u)^n = \sum_{k=0}^{n}\binom{n}{k}u^k$, it can also be written as

$$L(\underline{y}|\zeta) \propto \sum_{k=0}^{n}\binom{n}{k}(\frac{y}{\lambda})^{k\beta}\zeta^{k}e^{\left(-\zeta\left(n-\sum_{i=1}^{n}(\frac{y_i}{\lambda})^{\beta}\right)\right)}. \quad (27)$$

The posterior probability density function of ζ for given data y is given by:

$$\pi_1(\zeta|\underline{y}) \propto \frac{c\left(n-\sum_{i=1}^{n}\left(\frac{y_i}{\lambda}\right)^{\beta}\right)^{k+1}}{\sum_{k=0}^{n}\binom{n}{k}(\frac{y}{\lambda})^{k\beta}\Gamma(k+1)}\sum_{k=0}^{n}\binom{n}{k}(\frac{y}{\lambda})^{k\beta}\zeta^{k}e^{\left(-\zeta\left(n-\sum_{i=1}^{n}(\frac{y_i}{\lambda})^{\beta}\right)\right)}.$$

The Risk Function Under SELF is given by:

$$R_{(sq)}(\hat{\zeta}) = c\hat{\zeta}^2 + \frac{\sum_{k=0}^{n}\binom{n}{k}(\frac{y}{\lambda})^{k\beta}\Gamma(k+3)}{\left(n-\sum_{i=1}^{n}\left(\frac{y_i}{\lambda}\right)^{\beta}\right)^{2}\sum_{k=0}^{n}\binom{n}{k}(\frac{y}{\lambda})^{k\beta}\Gamma(k+1)} - 2\hat{\zeta}c\frac{\sum_{k=0}^{n}\binom{n}{k}(\frac{y}{\lambda})^{k\beta}\Gamma(k+2)}{\left(n-\sum_{i=1}^{n}\left(\frac{y_i}{\lambda}\right)^{\beta}\right)\sum_{k=0}^{n}\binom{n}{k}(\frac{y}{\lambda})^{k\beta}\Gamma(k+1)}. \quad (28)$$

Differentiating w.r.t $\hat{\zeta}$ and equating to 0, we get, Baye's estimator as

$$\hat{\zeta}_{(sq)} = \frac{\sum_{k=0}^{n}\binom{n}{k}(\frac{y}{\lambda})^{k\beta}(k+1)\Gamma(k+1)}{\left(n-\sum_{i=1}^{n}\left(\frac{y_i}{\lambda}\right)^{\beta}\right)\sum_{k=0}^{n}\binom{n}{k}(\frac{y}{\lambda})^{k\beta}\Gamma(k+1)}. \quad (29)$$

The Risk Function Under QLF is given by:

$$R_{(q)}(\hat{\zeta}) = \hat{\zeta}^2 \frac{\left(n - \sum_{i=1}^{n} \left(\frac{y_i}{\lambda}\right)^{\beta}\right)^2 \sum_{k=0}^{n} \binom{n}{k} (\frac{y}{\lambda})^{k\beta} \Gamma(k-1)}{\sum_{k=0}^{n} \binom{n}{k} (\frac{y}{\lambda})^{k\beta} \Gamma(k+1)} + 1 - 2\hat{\zeta} \frac{\left(n - \sum_{i=1}^{n} \left(\frac{y_i}{\lambda}\right)^{\beta}\right) \sum_{k=0}^{n} \binom{n}{k} (\frac{y}{\lambda})^{k\beta} \Gamma(k)}{\sum_{k=0}^{n} \binom{n}{k} (\frac{y}{\lambda})^{k\beta} \Gamma(k+1)}. \tag{30}$$

Differentiating w.r.t $\hat{\zeta}$ and equating to 0, we get, Baye's estimator as

$$\hat{\zeta}_{(q)} = \frac{\sum_{k=0}^{n} \binom{n}{k} (\frac{y}{\lambda})^{k\beta} (k-1)\Gamma(k-1)}{\left(n - \sum_{i=1}^{n} \left(\frac{y_i}{\lambda}\right)^{\beta}\right) \sum_{k=0}^{n} \binom{n}{k} (\frac{y}{\lambda})^{k\beta} \Gamma(k-1)}. \tag{31}$$

Regularizing the sum over k we have,

$$\hat{\zeta}_{(q)} = \frac{\sum_{k>1}^{n} \binom{n}{k} (\frac{y}{\lambda})^{k\beta} (k-1)\Gamma(k-1)}{\left(n - \sum_{i=1}^{n} \left(\frac{y_i}{\lambda}\right)^{\beta}\right) \sum_{k>1}^{n} \binom{n}{k} (\frac{y}{\lambda})^{k\beta} \Gamma(k-1)}. \tag{32}$$

4.3. Simulation study

In this segment, we are using R Software to conduct the simulation study to evaluate the performance of the MLEs and bayes estimator of the EPF parameters with respect to sample size. This comparison is carried out by generating the two sample of sizes (n=50 and n=100), each iteration conducted 100 times with varying parameter values $\zeta = (1, 2)$, $\eta = (0.5, 1, 1.5, 2)$ and $\lambda = (1, 1.5, 2)$ from EPF distribution. For each case, we calculated the mean MLE values under MLE method and Bayes estimator of ζ in Bayesian estimation method along with their corresponding MSEs. The simulation results are showcased in Tables 1 and 2 respectively. From Tables 1 and 2, it can be seen that the estimates are consistent and pretty close to the true parameter values. And in all cases it is clearly observed with increase in sample size, the MSE decreases significantly.

Also, the random sample of 51 observations has been simulated from EPF distribution with parameters $\zeta = 0.5, \eta = 0.5, \lambda = 1000$, to demonstrate theoretical concepts and to compared fit of proposed model with several competitive models. The results presented in Table 3 and Table 4 are discussed in Section 5, with the corresponding code included in the appendix. And the data are presented below:

480.12, 667.62, 877.15, 708.32, 103.81, 547.64, 850.04, 741.9, 367.57, 338.83, 173.57, 908.03, 230.83, 557.08, 424.87, 785.56, 775.93, 535.72, 468.37, 607.26, 84.57, 605.67, 196.66, 112.76, 646.81, 496.29, 487.8, 709.57, 655.45, 403.64, 989.27, 911.52, 539.16, 604.6, 869.42, 585.49, 283.13, 283.2, 868.99, 173.23, 515.82, 918.74, 772.5, 998.12, 609.05, 660.24, 999.21, 157.19, 328.9, 686.82, 508.91.

Table 1 Mean values of likelihood estimates (MLEs) and the corresponding mean square errors (MSEs) for n=50 and 100

Parameter				MLE			MSE		
n	ζ	η	λ	$\hat{\zeta}$	$\hat{\eta}$	$\hat{\lambda}$	$\hat{\zeta}$	$\hat{\eta}$	$\hat{\lambda}$
50	1	0.5	1	1.1424	0.4933	1.0337	0.4745	0.0188	0.0415
100				1.0657	0.4953	1.0121	0.2796	0.0099	0.0176
50			1.5	1.1727	0.4861	1.5461	0.6559	0.0175	0.1302
100				1.1039	0.4950	1.5262	0.4996	0.0103	0.0646
50			2	1.1463	0.4946	2.2644	0.7148	0.0234	0.6237
100				1.1244	0.4958	2.1297	0.4386	0.0128	0.5199
50		1	1	1.1823	1.0603	0.9946	0.8783	0.1606	0.0065
100				1.1388	1.0431	0.9950	0.7568	0.1319	0.0027
50			1.5	1.2775	1.1115	1.4883	1.7713	0.2870	0.0317
100				1.2559	1.0249	1.4939	1.1349	0.1477	0.0151
50			2	1.0991	1.1887	1.9860	1.2275	0.4124	0.0597
100				1.0750	1.1652	1.9946	0.6327	0.2497	0.0333
50		1.5	1	1.0928	1.6251	0.9982	0.4633	0.3027	0.0025
100				1.0824	1.5819	0.9996	0.3193	0.2054	0.0013
50			1.5	1.1925	1.5839	1.4813	0.8462	0.3435	0.0120
100				1.1636	1.5692	1.4919	0.5434	0.3077	0.0071
50			2	1.2203	1.6189	1.9711	1.2346	0.4069	0.0279
100				1.2020	1.5831	1.9897	0.6099	0.2703	0.0121
50		2	1	1.2278	2.2882	1.0031	1.2925	1.0792	0.0006
100				1.1684	2.1558	1.0017	0.7235	0.7046	0.0006
50			1.5	1.1195	2.0309	1.4829	0.4749	0.3941	0.0062
100				1.0654	2.0109	1.4862	0.3267	0.2908	0.0032
50			2	1.2379	2.1200	1.9761	0.9888	0.7410	0.0177
100				1.0944	2.0681	1.9909	0.4591	0.3828	0.0068
50	2	0.5	1	2.2103	0.5192	0.9957	1.7901	0.0372	0.0115
100				2.1486	0.5129	0.9970	1.1389	0.0262	0.0060
50			1.5	2.2842	0.5347	1.5206	1.7704	0.0488	0.0426
100				2.2317	0.5012	1.5093	1.5890	0.0351	0.0333
50			2	1.7288	0.5860	2.1068	1.1232	0.0457	0.2039
100				1.8621	0.5633	2.0023	0.5095	0.0216	0.0674
50		1	1	2.2680	1.0983	0.9970	1.8135	0.2869	0.0028
100				2.2255	1.0788	0.9985	1.5742	0.1720	0.0014
50			1.5	2.0527	1.1215	1.5118	0.6417	0.0585	0.0051
100				2.0268	1.0654	1.5067	0.3741	0.0268	0.0030
50			2	2.1875	1.1588	2.0365	1.5011	0.2285	0.0136
100				2.0192	1.0985	2.0091	0.4140	0.0660	0.0084
50		1.5	1	2.0968	1.6639	0.9968	1.4330	0.4118	0.0007
100				2.0449	1.6569	0.9992	1.1156	0.3230	0.0003
50			1.5	2.2046	1.5891	1.4907	1.1132	0.3557	0.0022
100				2.0963	1.5671	1.4937	0.9544	0.2323	0.0015
50			2	2.2040	1.7577	1.9901	1.7665	0.8231	0.0100
100				2.1772	1.5775	1.9910	1.0845	0.3611	0.0052
50		2	1	2.0661	2.1792	0.9975	1.0503	0.6706	0.0002
100				2.0508	2.1734	0.9993	0.9248	0.5064	0.0001
50			1.5	2.3302	1.9196	1.4944	1.0503	0.2740	0.0013
100				2.3232	1.9367	1.4991	0.8002	0.2648	0.0005
50			2	2.2379	1.9271	1.9793	0.9898	0.4666	0.0051
100				2.2244	1.9980	1.9941	0.8738	0.3060	0.0015

Table 2 Bayes estimate and the corresponding mean square errors (MSE) and Bias for n=50 and 100

Sample Size	Parameter			SELF		QLF	
n	ζ	η	λ	$\hat{\zeta_{sq}}(MLE)$	$\hat{\zeta_{sq}}(MSE)$	$\hat{\zeta}_q(MLE)$	$\hat{\zeta}_q(MSE)$
50	1	0.5	1	2.7102	3.0120	2.7254	3.0730
100				2.6947	2.9176	2.7079	2.9529
50			1.5	2.6888	2.9546	2.7414	3.1077
100				2.6560	2.7916	2.7400	3.0786
50			2	2.7005	2.9578	2.6847	2.9298
100				2.6529	2.7789	2.6794	2.8590
50	1	1	1	2.6775	2.8877	2.7182	3.0368
100				2.6757	2.8506	2.7089	2.9704
50			1.5	2.6827	2.9113	2.6958	2.9211
100				2.6586	2.7870	2.7003	2.9741
50			2	2.7130	3.0248	2.6986	2.9857
100				2.6710	2.8549	2.6863	2.8806
50	1	1.5	1	2.6808	2.9103	2.7075	3.0076
100				2.6607	2.8031	2.6982	2.9125
50			1.5	2.6996	2.9998	2.7236	3.0525
100				2.6791	2.8639	2.6660	2.8024
50			2	2.7054	3.0021	2.7199	3.0524
100				2.6913	2.9008	2.6987	2.9197
50	1	2	1	2.7139	3.0341	2.7129	3.0044
100				2.6764	2.8614	2.7088	2.9616
50			1.5	2.6854	2.9348	2.7072	3.0088
100				2.6726	2.8356	2.7002	2.9308
50			2	2.6710	2.9252	2.7088	3.0041
100				2.6658	2.8141	2.6968	2.9155
50	2	0.5	1	3.4945	2.3819	3.5209	2.4672
100				3.4861	2.3115	3.4870	2.3223
50			1.5	3.5412	2.5967	3.5613	2.6357
100				3.4957	2.3262	3.5543	2.5129
50			2	3.4873	2.3485	3.5537	2.5985
100				3.4583	2.2345	3.5033	2.3320
50	2	1	1	3.5712	2.6684	3.5451	2.5052
100				3.4555	2.2139	3.4836	2.2688
50			1.5	3.4909	2.4267	3.5605	2.6766
100				3.4701	2.2641	3.5499	2.4847
50			2	3.4790	2.3436	3.5213	2.5110
100				3.4530	2.1974	3.5025	2.3601
50	2	1.5	1	3.5243	2.5279	3.5480	2.6149
100				3.4887	2.2977	3.4812	2.2706
50			1.5	3.4515	2.2791	3.5116	2.4489
100				3.4440	2.1747	3.4946	2.2969
50			2	3.4687	2.3211	3.5409	2.5026
100				3.4664	2.2328	3.5212	2.4063
50	2	2	1	3.5312	2.5312	3.5703	2.6382
100				3.5065	2.3550	3.5492	2.4899
50			1.5	3.4886	2.4216	3.5795	2.6994
100				3.4561	2.2043	3.4923	2.3047
50			2	3.4949	2.4525	3.5078	2.4697
100				3.4597	2.2196	3.4596	2.2028

Table 3 MLEs , K-S and p-value for the simulated data

	Estimates			Statistics	
Model	$\hat{\zeta}$	$\hat{\eta}$	$\hat{\lambda}$	K-S	p-value
EPF	-0.9251	2.0625	1096.2949	0.0778	0.8933
ZTP-PF	2.4673	2.0596	1095.8049	0.0867	0.8066
EPF	0.1052	1.4764	1093.2187	0.1223	0.3988
EP	3.0054	0.4172	1070.0648	0.1281	0.3435
PF	-	1.2489	1101.6386	0.14839	0.1912

Table 4 $-2l$, AIC, AICC, BIC,HQIC for the simulated data.

Model	$-2l$	AIC	BIC	AICC	HQIC
*EPF	704.0587	710.0587	715.8542	710.5694	712.2734
ZTP-PF	705.2383	711.2383	717.0337	711.7489	713.4529
EPF	709.3165	715.3165	721.1120	715.8271	717.5311
EP	708.8163	714.8163	720.6118	715.3270	717.0310
PF	712.5265	716.5265	720.3901	716.7765	718.0029

5. Applications

In this segment, Three cancer data sets were used to demonstrate the applicability and the flexibility of the proposed distribution.

The first dataset consists of the life-lengths, from the diagnosis, of 43 blood cancer patients from the ministry of Health Hospitals in Saudi Arabia, reported first by Abouammoh and Abdulghani(Abouammoh et al., 1994).

The second data set consists the survival times of first 200 patients with breast cancer obtained from the Ministry of Health in Gaza City by Okasha and Matteral(Okasha and Matter, 2015).

The third dataset consists of the life-lengths, from the diagnosis, of 43 patients suffering form granulocytic leukemia from the National Cancer Institute, reported first by Abouammoh and Abdulghani(Abouammoh et al., 1994).

We compare the fit of the proposed EPF distribution with its base-model two parameter Power function (PF) distribution and with several more related competitive models, namely Exponentiated Power Function(EPF) Al Mutairi et al. (2022), Zero Truncated Poisson Power Function(ZTP-PF)Okorie et al. (2021) and Exponentiated Power(EP)Subramanian and Rather (2019) Distribution, their corresponding density functions for $0 < y < \lambda$ are as follows

$$\text{ZTP-PF} \quad f(y) = \frac{\zeta\eta y^{\eta-1}\exp\left(-\zeta\left(\frac{y}{\lambda}\right)^{\eta}\right)}{\lambda^{\eta}\left(\exp\left(-\zeta\left(\frac{y}{\lambda}\right)^{\eta}\right)\right)-\exp(-\zeta)}$$

$$\text{EPF} \quad f(y) = \frac{\zeta\eta}{\lambda^{\eta}}\frac{1}{y^{1-\eta}}\left(1+\left(\frac{y}{\lambda}\right)^{\eta}\right)^{-(1-\zeta)}$$

$$\text{EP} \quad f(y) = \frac{\zeta\eta y^{\zeta\eta-1}}{\lambda^{\zeta\eta}}$$

The results of Tables 3-10, indicate that the suggested model demonstrates excellent performance compared to its competitors. This is apparent from the significantly lower value of information tools (AIC, AICC, BIC), and lower K-S value and highest p-value among all the other competitive models. Hence the suggested model yields the better fit than the alternative models for both real life data sets.

The results are also justified by graphs of relative frequency distribution and the estimated density functions of EPF and competitive distribution of data set first, second , third and simulated dataset, and are showcased in Figures 3 and Figures 4 respectively.

Table 5 MLEs , K-S and p-value for the first data set

	Estimates			Statistics	
Model	$\hat{\zeta}$	$\hat{\eta}$	$\hat{\lambda}$	K-S	p-value
*EPF	-0.9972	2.2787	2233.2186	0.0953	0.8293
ZTP-PF	2.5073	2.1871	2228.1315	0.10011	0.7819
EPF	0.0612	1.5022	2247.5983	0.1408	0.3613
EP	1.7811	0.7242	2233.5232	0.16101	0.2149
PF	-	1.2805	2248.4428	0.16453	0.1948

Table 6 $-2l$, AIC, AICC, BIC,HQIC for the first data set

Model	$-2l$	AIC	BIC	AICC	HQIC
*EPF	652.7874	658.7874	664.0710	659.4028	660.7359
ZTP-PF	655.3008	661.3008	666.5844	661.9162	663.2492
EPF	659.5584	665.5584	670.8420	666.1738	667.5068
EP	660.5847	666.5847	671.8683	667.2001	668.5332
PF	661.3210	665.3210	668.8434	665.6210	666.6200

Table 7 MLEs , K-S and p-value for the second data set

	Estimates			Statistics	
Model	$\hat{\zeta}$	$\hat{\eta}$	$\hat{\lambda}$	K-S	p-value
*EPF	-0.5031	1.0878	987.9469	0.0389	0.9232
ZTP-PF	1.3313	1.1322	998.0271	0.0431	0.8512
EPF	0.0271	1.0015	999.4069	0.0744	0.2181
EP	0.5186	1.6194	990.3534	0.0864	0.1008
PF	-	0.8336	994.7703	0.0883	0.0885

Table 8 $-2l$, AIC, AICC, BIC,HQIC for the second data set

Model	$-2l$	AIC	BIC	AICC	HQIC
*EPF	2744.850	2750.850	2760.745	2750.972	2754.854
ZTP-PF	2747.111	2753.111	2763.006	2753.233	2757.115
EPF	2750.094	2756.094	2765.989	2756.216	2760.098
EP	2752.653	2758.653	2768.548	2758.776	2762.657
PF	2754.144	2758.144	2764.740	2758.205	2760.813

Table 9 MLEs , K-S and p-value for the third data set.

	Estimates			Statistics	
Model	$\hat{\zeta}$	$\hat{\eta}$	$\hat{\lambda}$	K-S	p-value
*EPF	-0.9956	1.0659	3008.9946	0.0845	0.8931
ZTP-PF	3.0152	1.1206	2999.8245	0.0881	0.8629
EPF	0.0073	0.7757	3002.0089	0.1505	0.2575
EP	2.7461	0.2295	3002.4247	0.1639	0.1774
PF	-	0.6189	2998.8608	0.1574	0.2134

Table 10 $-2l$, AIC, AICC, BIC,HQIC for the third data set.

Model	$-2l$	AIC	BIC	AICC	HQIC
*EPF	667.3817	673.3817	678.6653	673.9970	675.3301
ZTP-PF	668.2388	674.2388	679.5224	674.8541	676.1872
EPF	674.5767	680.5767	685.8603	681.1921	682.5251
EP	677.3133	683.3133	688.5969	683.9287	685.2617
PF	677.2472	681.2472	684.7696	681.5472	682.5461

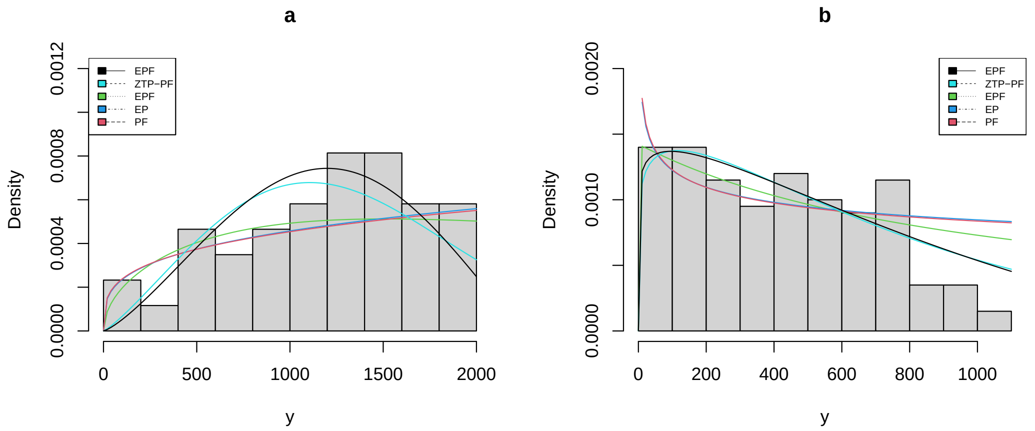

Figure 3 (a) Frequency distribution and estimated density functions for dataset first.(b) Frequency distribution and estimated density functions for dataset second

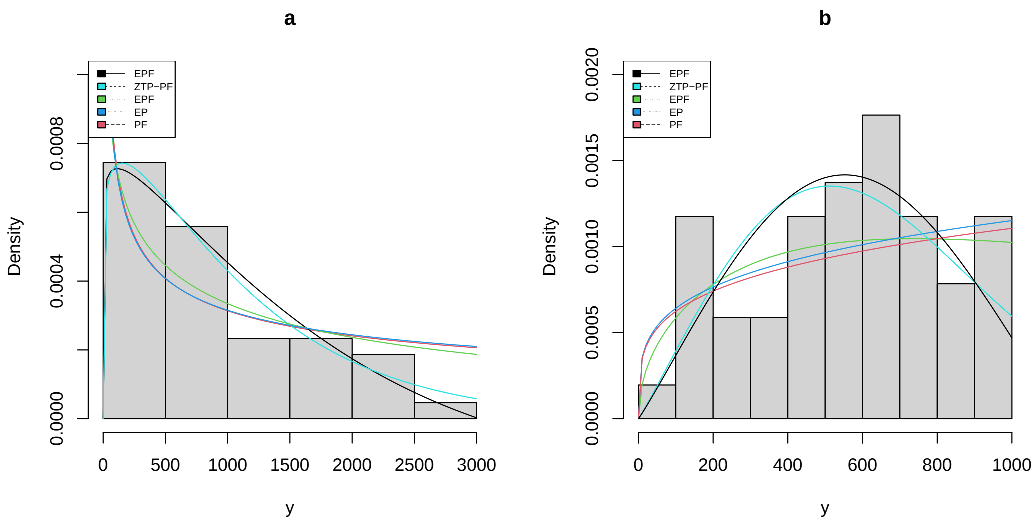

Figure 4 (a) Frequency distribution and estimated density functions for dataset third.(b) Frequency distribution and estimated density functions for simulated dataset.

6. Conclusion

A novel family of distributions has been presented called New Exponent-Generator family of distribution to introduce new uni-variate models. A particular member of this proposed family, known as the exponent power function (EPF) distribution (using two parameter Power Function model as base), is thoroughly examined. Various statistical and reliability features of the EPF distribution are explored. It has been observed that the new proposed distribution has more versatility and flexibility concerning both the HRF and the PDF. Additionally, a simulation study was conducted to evaluate the performance of the maximum likelihood estimators (MLEs) and Bayesian estimator for the distribution parameters. Furthermore, The potency of the suggested model is contrasted with base model and other competitive distributions by using goodness of fit measures. The distribution has been adapted to two different real life data sets as well as simulated data, the results show that our model fits better than all the other that we compared to it.

Our future plan is to apply the Exponent-Generator family to different types of real-world data beyond cancer data. Such as finance, engineering and medical data. Additionally, we will explore the use of various priors, including conjugate and non-informative priors, in Bayesian inference to assess their impact on posterior distributions and improve inference accuracy.

Declarations

Ethical statements Currently, the manuscript is not under consideration for publication elsewhere, in any format or language.
Conflict of interest The authors state that they do not have any competing interest.
Data availability All the references where from data is taken are mentioned within the manuscript.
Funding No funding.

Acknowledgements

We would like to thank the referees for his comments and suggestions on the manuscript.

References

Abouammoh AM, Abdulghani SA, Qamber IS. On partial orderings and testing of new better than renewal used classes. Reliab Eng Syst Safe. 1994; 43(1): 37-41.

Ahad N, Ahmad SP, Reshi JA. Bayesian analysis of extended Maxwell-Boltzmann distribution using simulated and real-life data sets. Reliab Theory Appl. 2024; 19(2): 188-201.

Ahsan-ul-Haq M, Aldahlan MA, Zafar J, Gmez HW, Afify AZ, Mahran HA. A new cubic transmuted power-function distribution: Properties, inference, and applications. PLoS One. 2023; 18(2): e0281419.

Akhter AS. Methods for estimating the parameters of the power function distribution. Pak J Stat Oper Res. 2013; 9(2): 213-224.

Al Mutairi A, Iqbal MZ, Arshad MZ, Afify AZ. A New Class of the Power Function Distribution: Theory and Inference with an Application to Engineering Data. J Math. 2022; 2022: 1-15.

Alshawarbeh E, Arshad MZ, Iqbal MZ, Ghamkhar M, Al Mutairi A, Meraou MA, Hussam E, Alrashidi A. Modeling medical and engineering data using a new power function distribution: Theory and inference. J Radiat Res Appl Sci. 2024; 17(1): 100787.

Bourguignon M, Silva RB, Cordeiro GM. The Weibull-G family of probability distributions. J Data Sci. 2014; 12(1): 53-68.

Cordeiro GM, Ortega EMM, da Cunha DCC. The exponentiated generalized class of distributions. J Data Sci. 2013; 11(1): 1-27.

Dallas AC. Characterizing the Pareto and power distributions. Ann Inst Stat Math. 1976; 28: 491-497.

Hanif S, Al-Ghamdi SD, Khan K, Shahbaz MQ. Bayesian estimation for parameters of power function distribution under various priors. Math Theory Model. 2015; 5(13): 1-15.

Hassan AS, Nassr SG. A new generalization of power function distribution: Properties and estimation based on censored samples. Thail Stat. 2020; 18(2): 215-234.

Hassan AS, Nassr SG. Power Lindley-G family of distributions. Ann Data Sci. 2019; 6(2): 189-210.

Hassan AS, Nassr SG. The inverse Weibull generator of distributions: Properties and applications. J Data Sci. 2018; 16(4): 723-742.

Hassan AS, Assar SM. The exponentiated Weibull power function distribution. J Data Sci. 2017; 16(2): 589-614.

Hassan AS, Elgarhy M. A new family of exponentiated Weibull-generated distributions. Int J Math Appl. 2016; 4(1-D): 135-148.

Lee ET, Wang J. Statistical methods for survival data analysis. New Jersey: John Wiley & Sons. 2003.

Lone MA, Dar IH, Jan TR. A new method for generating distributions with an application to Weibull distribution. Reliab Theory Appl. 2022; 17(1): 223-239.

Meniconi M, Barry DM. The power function distribution: A useful and simple distribution to assess electrical component reliability. Microelectron Reliab. 1996; 36(9): 1207-1212.

Naveed-Shahzad M, Asghar Z, Shehzad F, Shahzadi M. Parameter estimation of power function distribution with TL-moments. Rev Colomb Estad. 2015; 38(2): 321-334.

Okorie IE, Akpanta AC, Ohakwe J, Chikezie DC, Onyemachi CU, Rastogi MK. Zero-truncated Poisson-power function distribution. Ann Data Sci. 2021; 8(1): 107-129.

Okasha MK, Matter MY. On the three-parameter Burr type XII distribution and its application to heavy tailed lifetime data. J Adv Math. 2015; 10(4): 3429-3442.

Saran J, Pandey A. Estimation of parameters of a power function distribution and its characterization by k-th record values. Statistica. 2004; 64(3): 523-536.

Sathar EIA, Sathyareji GS. Estimation of the parameters of power function distribution based on progressively Type-II right censoring with binomial removal. Statistica. 2022; 82(3): 201-227.

Subramanian C, Rather A. Exponentiated power distribution: properties and estimation. Sci Technol Dev. 2019; VIII(IX): 213-223.

Sultan R, Sultan H, Ahmad SP. Bayesian analysis of power function distribution under double priors. J Stat Appl Probab. 2014; 3(2): 239-249.

Tahir MH, Alizadeh M, Mansoor M, Cordeiro GM, Zubair M. The Weibull-power function distribution with applications. Hacet J Math Stat. 2016; 45(1): 245-265.

Haq MA, Usman RM, Bursa N, zel G. McDonald power function distribution with theory and applications. Int J Stat Econ. 2018; 19(1): 89-107.

Haq MA, Albassam M, Aslam M, Hashmi S. Statistical inferences on odd Frechet power function distribution. J Reliab Stat Stud. 2021; 14(1): 141-172.

Wang C. Parameter estimation for power function-lognormal composite distribution. Commun Stat Theory. 2023; 52(9): 2966-2982.

Zaka A, Akhter AS, Jabeen R. The new reflected power function distribution: Theory, simulation & application. AIMS Math. 2020; 5(5): 5031-5054.

Zarrin S, Saxena S, Kamal M, Islam AU. Reliability computation and Bayesian Analysis of system reliability of Power function distribution. Int J Adv Eng Sci Technol. 2013; 2(4): 76-86.

Appendix

```
Data = function(n, r, zeta, eta,lambda) {
U = runif(n, 0, 1)
library(zipfR)
cdf = function(x,zeta, eta,lambda) {
((x / lambda)^(eta)) * exp(-zeta * (1 - (x / lambda)^(eta)))
}
data = c() Create an empty vector
```

```
for (i in 1:length(U)) {
fn = function(x) { cdf(x, zeta, eta,lambda) - U[i] }
uni = uniroot(fn, c(0, 100000))
data = c(data, uni$root)
}
return(data)
}
Simulateddata ¡- Data(51, 1,0.5, 0.5, 1000)
x=round(Simulateddata,4)
cat(x)
```