Thailand Statistician
January 2026; 24(1): 98-115
http://statassoc.or.th
Contributed paper

A New Class of Models with Bathtub Shaped Hazard Rate Functions

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Received: 21 May 2024
Revised: 25 October 2024
Accepted: 4 August 2025

Abstract

In the present research, we compound the alpha power transformation distribution with the power series distribution to create a novel distribution known as the alpha power transformed-G Power Series (APT-GPS) distribution. The APT-GPS distribution's statistical characteristics are discovered. The density and hazard rate life plots are shown in a range of forms using a Weibull baseline distribution. Additionally, a simulation study is carried out to evaluate the effectiveness of the maximum likelihood estimates. Lastly, some samples of bladder cancer data and United Kingdom (UK) Covid-19 data are evaluated for illustrative purposes. The results show that, in comparison to other non-nested distributions taken into consideration in this article, the proposed distribution offers a better fit as it exhibits the lowest goodness of fit statistics values and the highest accuracy rate on both data sets.

Keywords: Alpha power transformation, residual life function, reliability model, hazard rate function, quantile function, moment generating function.

1. Introduction

Over the years, lifetime distributions have drawn a lot of attention. These distributions are derived using many different techniques such as the exponentiated-G which includes the Lehmann alternative of type 1 and Lehmann alternative of type 2 Gupta et al. (1998), exponentiated generalised-G (Cordeiro et al., 2013), transmuted-G (Rahman et al., 2018), T-X technique (Alzaatreh et al., 2013), cubic rank transmuted-G (Aslam et al., 2018), Marchall-Olkin-G (Marshall and Olkin, 1997), alpha log power transformed-G (Musekwa et al., 2024), generalized flexible-G (Makubate and Musekwa, 2024) and the new flexible generalized-G (Tahir et al., 2022). These methos have been widely used in literature, including research by Oluyede et al. (2024), Oluyede et al. (2024), Chuncharoenkit et al. (2024), Ashraf et al. (2024), Zidana et al. (2024), Sengweni et al. (2023), Chamunorwa et al. (2024) and Chamunorwa et al. (2022).

Recently, Mahdavi and Kundu (2017) introduced a new technique of creating families of distributions called the alpha power transformation (APT). This family was primarily designed to take account of the non-symmetrical behavior of the parent distribution. Following the notation of Mahdavi and Kundu (2017), for any arbitrary baseline cdf distribution $G(z)$, the cdf and pdf of the APT are given by

$$F_{APT}(z) = \frac{\alpha^{G(z)} - 1}{\alpha - 1}, \tag{1}$$

and

$$f_{APT}(z) = \log(\alpha) g(z) \frac{\alpha^{G(z)}}{\alpha - 1}, \tag{2}$$

respectively, where $z \in \mathbb{R}$, $\alpha > 0$, and $\alpha \neq 1$.

More research on the APT has been done including the alpha-power exponentiated Inverse Rayleigh distribution (Ali et al., 2021), the Gull alpha-power Weibull distribution (Ijaz et al., 2020), the alpha-power inverse Weibull distribution (Basheer, 2019), the alpha-power transformed extended exponential distribution (Hassan et al., 2018), the alpha-power Weibull Fretchet distribution (Eghwerido, 2020), the alpha power inverted exponential (Ceren et al., 2018) and the alpha power transmuted (Eghwerido et al., 2021) distributions to mention a few.

Traditional models used to analyze real world data sets often fall short in accurately capturing the dynamic and complex data behavior. These models may inadequately account for cascading failures and the intricate interdependencies. For instance, the exponential, Poisson, and uniform models are characterized by constant hazard rates over time, the Weibull model accommodates constant, increasing, or decreasing hazard rates, and the Pareto model exhibits a decreasing hazard rate. However, these models may not fully reflect the complexity of monotonic and non-monotonic hazard rate. Consequently, there is an increasing demand for more sophisticated and comprehensive models that can better these dynamic data sets.

The key objective of this study is to provide a new and wider class of distributions with bathtub hazard rate functions (hrfs). The new class called the alpha power transformed-G power series (APT-GPS) is formed by compounding the power series distribution and the alpha power transformation distribution.

Let $Z_1, Z_2, ..., Z_N$ be a random sample from a distribution with pdf defined in (2), where N is a discrete random variable with a zero truncated power series distribution (Makubate et al., 2022). Suppose $Z = min\{Z_1, Z_2, ..., Z_N\}$. Then, by the compounding method (Rannona et al., 2022; Musekwa et al., 2024), the marginal cdf and pdf of Z are given by

$$F_{APT-GPS}(z; \theta, \alpha, \Delta) = 1 - \frac{C\left(\theta\left(1 - \frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)\right)}{C(\theta)}, \tag{3}$$

and

$$f_{APT-GPS}(z; \theta, \alpha, \Delta) = \frac{\theta \log(\alpha) g(z; \Delta) \alpha^{G(z;\Delta)} C'\left(\theta\left(1 - \frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)\right)}{(\alpha - 1) C(\theta)}, \tag{4}$$

respectively where, α and θ are shape parameters, $C(\theta) = \sum_{k=1}^{n} a_k \theta^k$, $C'(\theta) = dC(\theta)/d\theta$, $G(z; \Delta)$ is the baseline distribution, and Δ is the parameter vector. Equations (3) and (4) are respectively, the cdf and pdf of the alpha power transformation-G power series (APT-GPS). Consequently, the quantile function of the APT-GPS is given by

$$Q(u) = G^{-1}\left\{\frac{\log\left\{(\alpha - 1)\left(1 - \frac{C^{-1}[C(\theta)(1-u)]}{\theta}\right) + 1\right\}}{\log(\alpha)}\right\}, u \in (0, 1). \tag{5}$$

Note that Equation (4) can be expressed as

$$f_{APT-GPS}(z; \theta, \alpha, \Delta) = \sum_{d=0}^{\infty} p^*_{d+1} g^*_{d+1}(z; \Delta), \tag{6}$$

where

$$g^*_{d+1}(z; \Delta) = (d + 1) g(z; \Delta) G^d(z; \Delta), \tag{7}$$

is the pdf of an exponentiated-G (exp-G) distribution with power parameter $(d+1)$ and

$$p^*_{d+1} = \sum_{n,a,b=0}^{\infty} \frac{(n+1)a_{n+1}\theta^{n+1}(b+1)^d(\log(\alpha))^{d+1}(-1)^{a+b+1}}{d!(d+1)(\alpha-1)^{a+1}C(\theta)} \binom{n}{a}\binom{a}{b}, \tag{8}$$

is a constant (see Appendix A for proof). Note also that, the alpha power transformation defined in Equation (1) is a limiting special case of the APT-GPS when $\theta \to 0^+$ (see proof in Appendix B).

Table 1 shows cdfs of some classes of the APT-GPS namely, the alpha power transformed-G poisson (APT-GP), the alpha power transformed-G logarithmic (APT-GL), the alpha power transformed-G geometric (APT-GG) and the alpha power transformed-G binomial (APT-GBin) together with $C(\theta), C'(\theta), a_n$ and the parameter space (Ω).

Table 1 Some Submodels of the APT-GPS

Model	$C(\theta)$	$C'(\theta)$	a_n	Ω	*cdf*
APT-GP	$e^\theta - 1$	e^θ	$(n!)^{-1}$	$(0,\infty)$	$1-\frac{\exp\left(\theta\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)\right)-1}{e^\theta-1}$
APT-GL	$-\log(1-\theta)$	$(1-\theta)^{-1}$	$(n)^{-1}$	$(0,1)$	$1-\frac{\log\left(1-\theta\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)\right)}{\log(1-\theta)}$
APT-GG	$\theta(1-\theta)^{-1}$	$(1-\theta)^{-2}$	1	$(0,1)$	$1-\theta\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)\frac{\left(1-\theta\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)\right)^{-1}}{\theta(1-\theta)^{-1}}$
APT-GBin	$(1+\theta)^m - 1$	$m(1+\theta)^{m-1}$	$\binom{m}{n}$	$(0,1)$	$1-\frac{\left(1+\theta\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)\right)-1}{(1+\theta)^m-1}$

Here is a summary of the remaining portion of the article: In Section 2, we derived some mathematical properties of the APT-GPS. Section 3 covers a few special cases of the APT-GPS, with the Weibull distribution serving as the baseline distribution. In Section 4, a Monte Carlo simulation analysis is presented. In Section 5, we illustrate the usefulness of the APT-GPS distribution using two different real-life data sets, followed by concluding remarks.

2. Mathematical Properties

2.1. Order statistics

The pdf of the k^{th} order statistic of the random sample of size $n \in \mathbb{Z}^+$ from the APT-GPS is given by

$$\begin{aligned} f_{k:m}(z) &= \frac{m!}{(k-1)(m-k)!}\sum_{l=0}^{\infty}(-1)^l\binom{m-k}{l} f_{APT-GPS}(z)F^{k+l-1}_{APT-GPS}(z) \\ &= \sum_{d=0}^{\infty} q^*_{d+1} g^*_{d+1}(z;\Delta), \end{aligned} \tag{9}$$

for $k = 1,2,3,...,n$, where

$$\begin{aligned} q^*_{d+1} &= \frac{m!}{(k-1)(m-k)!}\sum_{n,l,i,r,a,b=0}^{\infty}(-1)^{l+i+a+b+1}\binom{m-k}{l}\binom{k+l-1}{i} \\ &\times \binom{n+i}{a}\binom{a}{b}\frac{\log(\alpha)(b+1)^d(\log(\alpha))^d}{d!(\alpha-1)^{a+1}(d+1)}\frac{(n+1)a_{n+1}\theta^{n+i+1}d_{r,i}}{[C(\theta)]^{i+1}}, \end{aligned} \tag{10}$$

is a constant and, $f_{APT-GPS}(z)$ and $F_{APT-GPS}(z)$ are respectively, the pdf and cdf of the APT-GPS (see Appendix C for more details).

2.2. Moments

2.2.1 Moments and moment generating function

The j^{th} moment, the j^{th} incomplete moment and the moment generating function (mgf) of Z, follow from Equation (6) as

$$\mu'_j = E[Z^j] = \sum_{d=0}^{\infty} p^*_{d+1} \int_0^{\infty} z^j g^*_{d+1}(z;\Delta)dz, \tag{11}$$

$$I_j(t) = E[Z|Z>t] = \sum_{d=0}^{\infty} p^*_{d+1} \int_0^{t} z^j g^*_{d+1}(z;\Delta)dz, \tag{12}$$

and

$$M_Z(t) = E[e^{tz}] = \sum_{d=0}^{\infty} p^*_{d+1} M_{d+1}(t), \tag{13}$$

respectively, where $M_{d+1}(t)$ is the mgf of Z_{d+1}. The mean and variance can be easily derived from Equation (11).

2.2.2 Residual and reverse residual life function

Let n be an integer value greater than 1 ($n > 1$) and $z > t$, then the j^{th} moment of residual life is given by

$$\kappa^*_j(t) = \frac{1}{\bar{F}(z;\Delta)} \sum_{d=0}^{\infty} p^*_{d+1} \sum_{s=0}^{\infty} \binom{n}{s} (-t)^s \int_t^{\infty} z^{n-s} g^*_{d+1}(z;\Delta)dz, \tag{14}$$

where $\bar{F}(z;\Delta) = 1 - F(z;\Delta)$ is the survival function of the APT-GPS.

Now, if n is an integer value greater than 1 ($n > 1$) and $z < t$, then the j^{th} moment of reverse residual life is given by;

$$\iota^*_j(t) = \frac{1}{F(z;\Delta)} \sum_{d=0}^{\infty} p^*_{d+1} \sum_{s=0}^{\infty} \binom{n}{s} (t)^{n-s} (-1)^s \int_0^{t} z^s g^*_{d+1}(z;\Delta)dz. \tag{15}$$

2.2.3 Probability weighted moment

The probability weighted moment (PWM) is the expectation of a function of a random variable given that the mean of the variable exists (Musekwa and Makubate, 2023). The $(l,j)^{th}$ PWM of a random variable Z, say $(\chi_{l,j})$ is given by

$$\begin{aligned} \chi_{l,j} &= \int_0^{\infty} z^l f_{APT-GPS}(z;\theta,\alpha,\Delta) F^j_{APT-GPS}(z;\theta,\alpha,\Delta)dz \\ &= \sum_{d=0}^{\infty} e^*_{d+1} \int_0^{\infty} z^l g^*_{d+1}(z;\Delta)dz = \sum_{d=0}^{\infty} e^*_{d+1} E(Z^l_{d+1}), \end{aligned} \tag{16}$$

where

$$\begin{aligned} e^*_{d+1} &= \sum_{n,i,r,a,b=0}^{\infty} (-1)^{i+a+b+1} \binom{j}{i} \binom{n+i}{a} \binom{a}{b} \\ &\times \frac{\log(\alpha)(b+1)^d (\log(\alpha))^d}{d!(\alpha-1)^{a+1}(d+1)} \frac{(n+1)a_{n+1}\theta^{n+i+1} d_{r,i}}{[C(\theta)]^{i+1}}, \end{aligned} \tag{17}$$

is a constant (see Appendix D for more details).

2.3. Reliability model

Let Y be the strength of a system that is subjected to stress Z. The stress-strength reliability model is then defined as the probability of not failing ($P(Y > Z)$). Assume Y and Z are independent APT-GPS random variables with pdfs $f(y;\theta_1,\alpha_1,\Delta_1)$ and $f(z;\theta_2,\alpha_2,\Delta_2)$, respectively. Then expression for the reliability model of the APT-GPS is given by

$$\begin{aligned} R = P(Z < Y) &= \int_0^\infty \int_0^y f(y;r_1)f(z;r_2)dzdy = \int_0^\infty f(y;r_1)F(y;r_2)dy \\ &= \int_0^\infty \left(\frac{\theta_1 \log(\alpha_1) g(y;\Delta_1)\alpha_1^{G(y;\Delta_1)} C'\left(\theta_1\left(1-\frac{\alpha_1^{G(y;\Delta_1)}-1}{\alpha_1-1}\right)\right)}{(\alpha_1-1)C(\theta_1)} \right. \\ &\times \left. \left(1-\frac{C\left(\theta_2\left(1-\frac{\alpha_2^{G(y;\Delta_2)}-1}{\alpha_2-1}\right)\right)}{C(\theta_2)}\right)\right)dy, \end{aligned} \tag{18}$$

where $r_1 = (\theta_1,\alpha_1,\delta_1)$ and $r_2 = (\theta_2,\alpha_2,\delta_2)$.

2.4. Rényi entropy

The Rényi entropy of the APT-GPS distribution is given by

$$\begin{aligned} I_R(\delta) &= (1-\delta)^{-1}\log\left(\int_0^\infty \left(\frac{\theta\log(\alpha)}{\alpha-1}\right)^\delta g^\delta(z;\Delta)\alpha^{\delta G(z;\Delta)} \right. \\ &\times \left. \left(\sum_{n=0}^\infty \frac{(n+1)a_{n+1}\theta^{n+1}}{C(\theta)}\left[1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right]^n\right)^\delta dz\right), \end{aligned} \tag{19}$$

for $\delta \neq 1, \delta > 0$. Now, we then apply results of a power series raised to a positive integer with $a_n = (n+1)a_{n+1}\theta^n$ (Gradshteyn and Ryzhik, 2014) and using the following expansions

$$\begin{aligned} \left[1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right]^{ns} &= \sum_{a=0}^\infty (-1)^a \binom{ns}{a}(\alpha-1)^{-a}(\alpha^{G(z;\Delta)}-1)^a \\ (\alpha^{G(z;\Delta)}-1)^a &= \sum_{b=0}^\infty (-1)^{b+1}\binom{a}{b}\alpha^{bG(z;\Delta)} \\ \alpha^{(b+\delta)G(z;\Delta)} &= \sum_{d=0}^\infty \frac{(\log(\alpha))^d(b+\delta)^d}{d!}G^d(z;\Delta), \end{aligned}$$

we have the Rényi entropy of the APT-GPS given by

$$\begin{aligned} I_R(\delta) &= (1-\delta)^{-1}\log\left(\sum_{n,s,k,a,b,d=0}^\infty \left(\frac{\theta\log(\alpha)}{(\alpha-1)C(\theta)}\right)^\delta \frac{b_{s,k}(-1)^{a+b+1}(b+\delta)^d(\log(\alpha))^d}{d!(\alpha-1)^a} \right. \\ &\times \left. \binom{ns}{a}\binom{a}{b}\int_0^\infty g^\delta(z;\Delta)\alpha^{\delta G(z;\Delta)}dz\right), \end{aligned} \tag{20}$$

where $b_{s,k}$ is as defined in Chipepa et al. (2021) and Nyamajiwa et al. (2024).

2.5. Estimation

Assume $z_1, z_2, z_3, ..., z_n$ is a random sample of size n from the APT-GPS with pdf given in Equation (4). Let $r = (\theta, \alpha, \Delta)$ be the vector of model parameters. The log-likelihood function

$\ell_n(r) = \ell_n$ based on a random sample of size n from the APT-GPS distribution is given by

$$\begin{aligned} \ell_n &= n\log\left(\frac{\theta\log(\alpha)}{(\alpha-1)}\right) + \sum_{i=1}^{n} g(z;\Delta) + \sum_{i=1}^{n} G(z;\Delta)\log(\alpha) - n\log(C(\theta)) \\ &+ \sum_{i=1}^{n}\log\left(C'\left(\theta\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)\right)\right). \end{aligned} \tag{21}$$

To obtain the maximum likelihood estimates (MLEs), we therefore have to take the partial derivatives of (21) with respect to $r = (\theta, \alpha, \Delta)$, set the derivatives equal to zero and solve for $r = (\theta, \alpha, \Delta)$. The partial derivatives corresponding to Equation (21) are given in Appendix E. These partial derivatives are not linear in the parameters. Hence numerical methods such as the Newton-Raphson method are required to solve them (Musekwa et al., 2024).

3. Special Case

We consider a Weibull baseline distribution with the cdf and pdf given by $G(z;\phi,\psi) = 1 - e^{-\phi z^{\psi}}$ and $g(z;\phi,\psi) = \phi\psi z^{\psi-1}e^{-\phi z^{\psi}}$, respectively, for $\phi, \psi > 0$. Table 2 shows the cdf and pdf of some special models of the alpha power transformed Weibull power series namely the alpha power transformed-Weibull poisson (APT-WP), alpha power transformed-Weibull logarithmic (APT-WL), alpha power transformed-Weibull geometric (APT-WG) and alpha power transformed-Weibull binomial (APT-WBin).

Table 2 Some Special Cases of the APT-GPS

Model	*cdf*	*hrf*
APT-WP	$1-\frac{\exp\left(\theta\left(1-\frac{\alpha^{1-e^{-\phi z^{\psi}}}-1}{\alpha-1}\right)\right)-1}{e^{\theta}-1}$	$\frac{\theta\log(\alpha)\phi\psi z^{\psi-1}e^{-\phi z^{\psi}}\alpha^{1-e^{-\phi z^{\psi}}}\exp\left(\theta\left(1-\frac{\alpha^{1-e^{-\phi z^{\psi}}}-1}{\alpha-1}\right)\right)}{(\alpha-1)(e^{\theta}-1)}$
APT-WL	$1-\frac{\log\left(1-\theta\left(1-\frac{\alpha^{1-e^{-\phi z^{\psi}}}-1}{\alpha-1}\right)\right)}{\log(1-\theta)}$	$\frac{\theta\log(\alpha)\phi\psi z^{\psi-1}e^{-\phi z^{\psi}}\alpha^{1-e^{-\phi z^{\psi}}}\left(1-\theta\left(1-\frac{\alpha^{1-e^{-\phi z^{\psi}}}-1}{\alpha-1}\right)\right)^{-1}}{(\alpha-1)(\log(1-\theta))}$
APT-WG	$1-\theta\left(1-\frac{\alpha^{1-e^{-\phi z^{\psi}}}-1}{\alpha-1}\right)\frac{\left(1-\theta\left(1-\frac{\alpha^{1-e^{-\phi z^{\psi}}}-1}{\alpha-1}\right)\right)^{-1}}{\theta(1-\theta)^{-1}}$	$\frac{\theta\log(\alpha)\phi\psi z^{\psi-1}e^{-\phi z^{\psi}}\alpha^{1-e^{-\phi z^{\psi}}}\left(1-\theta\left(1-\frac{\alpha^{1-e^{-\phi z^{\psi}}}-1}{\alpha-1}\right)\right)^{-2}}{(\alpha-1)(\theta(1-\theta)^{-1})}$
APT-WBin	$1-\frac{\left(1+\theta\left(1-\frac{\alpha^{1-e^{-\phi z^{\psi}}}-1}{\alpha-1}\right)\right)^{m}-1}{(1+\theta)^{m}-1}$	$\frac{\theta\log(\alpha)\phi\psi z^{\psi-1}e^{-\phi z^{\psi}}\alpha^{1-e^{-\phi z^{\psi}}}m\left(1+\theta\left(1-\frac{\alpha^{1-e^{-\phi z^{\psi}}}-1}{\alpha-1}\right)\right)^{m-1}}{(\alpha-1)((1+\theta)^{m}-1)}$

Figures 1 and 2 clearly shows that for some set parameter values, the APT-GPS family can model data sets which take different shapes including but not limited to symmetric data sets, almost symmetric, left and right skewed, and heavily tailed data sets.

Figures 3 and 4 indicates that, the APT-GPS can model data sets which exhibits different hazard rate shapes including the upside down bathtub, bathtub followed by upside bathtub, upside down buthtub followed by bathtub, J-shaped, increasing and decreasing shapes.

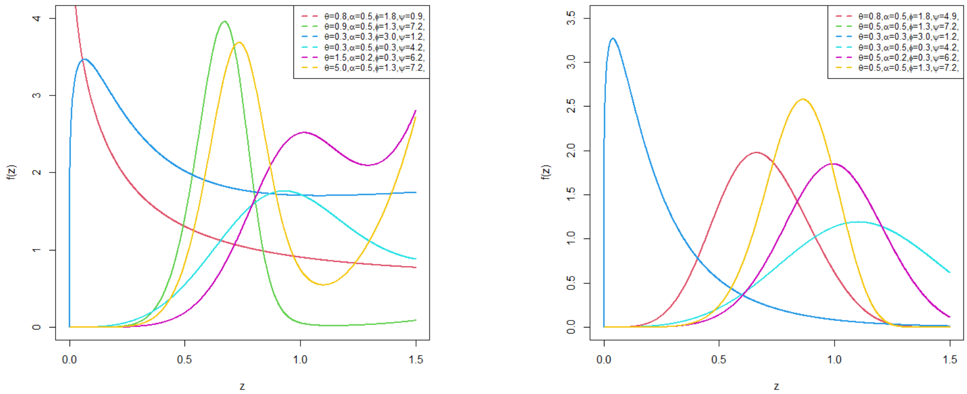

Figure 1 Plots of the pdf for the APT-WP and APT-WL distribution respectively

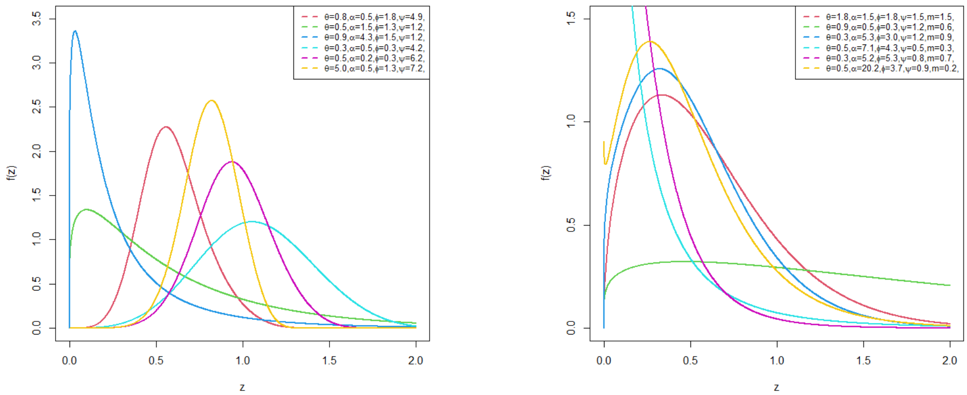

Figure 2 Plots of the pdf for the APT-WG and APT-WBin distribution respectively

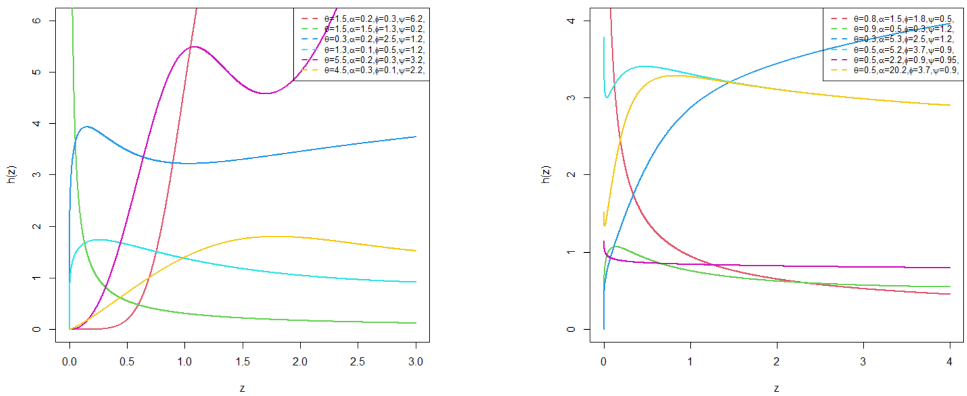

Figure 3 Plots of the hrf for the APT-WP and APT-WL distribution respectively

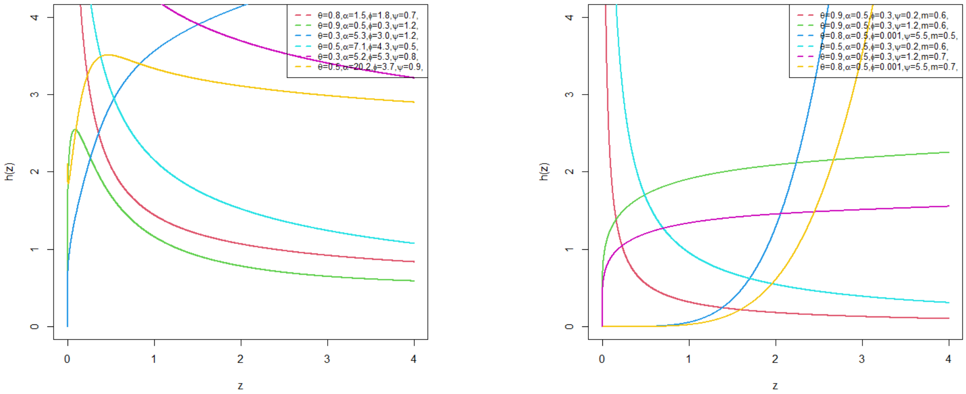

Figure 4 Plots of the hrf for the APT-WG and APT-WBin distribution respectively

4. Simulation Study

In this section, we offer a simulation exercise to assess how well the APT-GPS distribution's MLEs perform. The following algorithm was used:

1. a sample of $z_1, z_2, ..., z_k$ of sizes $n = (25, 50, 75, 100, 200, 400)$ are randomly generated from the APT-WP distribution;
2. initial values for each parameter are selected;
3. parameter estimates are obtained from each sample size;
4. the above steps are executed 3000 times for every sample, and MLE's of the parameters, their average biases (ABiase) and root mean square errors (RMSE) are recorded.

Table 3 Parameter estimation from the APT-WP Results

			$(0.5, 2.5, 0.5, 1.0)$			$(0.5, 0.5, 0.5, 1.0)$			$(0.5, 0.5, 2.5, 1.0)$	
Sample Size	Parameter	Mean	RMSE	ABias	Mean	RMSE	ABias	Mean	RMSE	ABias
50	θ	1.2886	1.4080	0.7886	0.7598	0.5656	0.2598	0.8969	0.7214	0.3969
	α	7.9669	25.0894	5.4669	1.7281	23.5734	1.2281	1.2906	11.1584	0.7906
	ϕ	0.3515	0.2470	-0.1484	0.4968	0.1671	-0.0031	2.4631	0.7180	-0.0368
	ψ	1.1351	0.2458	0.1351	1.0287	0.1275	0.0287	1.0498	0.1445	0.0498
75	θ	1.5649	1.7793	1.0649	0.6937	0.5008	0.1937	0.8346	0.6390	0.3346
	α	8.3359	28.642	5.8359	0.7243	1.5345	0.2243	0.6685	1.4531	0.1685
	ϕ	0.3474	0.2571	-0.1525	0.5019	0.1572	0.0019	2.4490	0.6101	-0.0509
	ψ	1.1288	0.2432	0.1288	1.0224	0.1133	0.0224	1.0499	0.1328	0.0499
100	θ	1.4857	1.8393	0.9857	0.7093	0.6584	0.2093	0.8004	0.8362	0.3004
	α	4.1176	9.1288	1.6176	0.8395	1.9750	0.3395	1.1299	4.2622	0.6299
	ϕ	0.3370	0.2321	-0.1629	0.5149	0.1693	0.0149	2.5199	0.6040	0.0199
	ψ	1.1240	0.1923	0.1240	1.0132	0.1010	0.0132	1.0222	0.1171	0.0222
200	θ	1.3544	1.6823	0.8544	0.7184	0.6377	0.2184	0.7462	0.5835	0.2462
	α	3.4665	5.9325	0.9665	0.8128	1.2708	0.3128	0.8180	2.2976	0.3180
	ϕ	0.3568	0.2029	-0.1431	0.5103	0.1438	0.0103	2.4543	0.5189	-0.0456
	ψ	1.0909	0.1504	0.0909	0.9987	0.0796	-0.0012	1.0111	0.0879	0.0111
400	θ	1.2992	1.6887	0.7992	0.6837	0.6011	0.1837	0.7188	0.5635	0.2188
	α	3.5843	5.0507	1.0843	0.8948	1.8117	0.3948	0.8430	1.6605	0.3430
	ϕ	0.3963	0.1797	-0.1036	0.5206	0.1355	0.0206	2.4861	0.4055	-0.0138
	ψ	1.0584	0.1182	0.0584	0.9918	0.0711	-0.0081	0.9993	0.0711	-0.0006

Table 3 presents the simulation results based on a complete sample, and it is evident that;

1. the RMSE tends to decrease when increasing sample size thus demonstrating the consistency of these estimators and
2. ABiase tend to decrease as sample size increases, demonstrating the MLEs' accuracy.

5. Data Analysis

In this section, we used data sets from Lee and Wang (2003) and Al-Dayel et al. (2022), respectively, to illustrate the utility of the APT-GPS class of distributions. We used the APT-WP distribution to analyse both data sets. Model performance was assessed using the following goodness-of-fit (GoF) statistics, namely, $-2\,loglikelihood\ (-2\,log\ L)$, Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramér-Von Mises (W^{*}), Anderson-Darling (A^{*}) and the Kolmogorov-Smirnov (K-S) statistic. We used R software to estimate the model parameters via the **nlm** function. Model parameter estimates (standard errors in parenthesis) are given in Tables 4 and 5. The GoF of the APT-WP and non-nested comparative models are presented in Tables 6 and 7. The model with the smallest values of these statistics and a higher p-value for the K-S statistic, is identified as the best-fitting model. Figures 5, 6, 7 and 8 present plots of the fitted density, the histogram of the data, probability plots, Kaplain Meier (KM), empirical cumulative distribution function (ECDF) and total-time-on-test (TTT) plots (Cleveland and McGill, 1984) to show how well our model fits the observed data sets.

We compared the MO-WL distribution with other competing non-nested models, namely the Weibull-Lomax (WL) distribution (Tahir et al., 2015), Exponentiated Power Lindley Poisson (EPLP) distribution (Pararai et al., 2017), Topp-Leone- Gompertz-Weibull (TLGW) (Oluyede et al., 2022), Exponentiated Generalized Logarithmic (EGEL) distribution (Oluyede et al., 2020), Alpha power inverse Weibull (APIW) distribution (Basheer, 2019), Generalized Gompertz-Poisson (GGP) (Tahmasebi et al., 2015), and the Odd Weibull-Topp-Leone-Log Logistic Logarithmic (OW-TL-LLoGL) distribution (Oluyede et al., 2021).

Data set 1 is a cancer data set which consists of the remission times of 128 bladder cancer patients. The data is as follows:

0.08, 4.98, 25.74, 3.70, 10.06, 2.69, 7.62, 1.26, 7.87, 4.40, 2.02, 21.73, 2.09, 6.97, 0.50, 5.17, 14.77, 4.18, 10.75, 2.83, 11.64, 5.85, 3.31, 2.07, 3.48, 9.02, 2.46, 7.28, 32.15, 5.34, 16.62, 4.33, 17.36, 8.26, 4.51, 3.36, 4.87, 13.29, 3.64, 9.74, 2.64, 7.59, 43.01, 5.49, 1.40, 11.98, 6.54, 6.93, 6.94, 0.40, 5.09, 14.76, 3.88, 10.66, 1.19, 7.66, 3.02, 19.13, 8.53, 8.65, 8.66, 2.26, 7.26 , 26.31, 5.32, 15.96, 2.75, 11.25, 4.34, 1.76, 12.03, 12.63, 13.11, 3.57, 9.47, 0.81, 7.39, 36.66, 4.26, 17.14, 5.71, 3.25, 20.28, 22.69, 23.63, 5.06, 14.24, 2.62, 10.34, 1.05, 5.41, 79.05, 7.93, 4.5, 2.02, 0.2 ,7.09, 25.82, 3.82, 14.83, 2.69, 7.63, 1.35, 11.79, 6.25, 3.36, 2.23, 9.22, 0.51, 5.32, 34.26 ,4.23, 17.12, 2.87, 18.1, 8.37, 6.76, 3.52, 13.8, 2.54, 7.32, 0.9, 5.41, 46.12 ,5.62, 1.46, 12.02 ,12.07

Data set 11 represents the Covid-19 data from the United Kingdom (UK) within 62 days, from 21 July to 21 August 2020. These data describe the drought mortality rate as follows:

0.2992, 0.1303, 0.0587, 0.3926, 0.3622, 0.4110, 0.3188, 0.1652, 0.1277, 0.0863, 0.2173, 0.3969, 0.1673, 0.1995, 0.1300, 0.0771, 0.0445, 0.2180, 0.2296, 0.1246, 0.1362, 0.0680, 0.0359, 0.0399, 0.1749, 0.1031, 0.0949, 0.1025, 0.0354, 0.0432, 0.0392, 0.0977, 0.0662, 0.0350, 0.1240, 0.0580, 0.0309, 0.0116, 0.0809, 0.1229, 0.0077, 0.0763, 0.0495, 0.0190, 0.0038, 0.0679, 0.0526, 0.0674, 0.0448, 0.0112, 0.0185, 0.0666, 0.0479, 0.0734, 0.0658, 0.0400, 0.00109, 0.0180, 0.0180, 0.0430, 0.0572, 0.0214

Tables 4 and 5 show some estimates of parameters with standard errors in parenthesis for various models fitted for cancer and UK Covid-19 data sets respectively.

Table 4 Parameter estimates for various models fitted for the cancer data set

Estimates and standard errors in parenthesis				
Model	θ	α	ϕ	ψ
APT-WP	2.9931	0.0766	0.0070	1.3802
	(2.5882)	(0.2737)	(0.0065)	(0.1072)
	a	b	α	β
WL	3.0841×10^{2}	1.5118	0.0311	8.3425
	(0.0109)	(0.2908)	(7.4258×10^{-3})	(7.5939)
	θ	α	β	ω
EPLP	3.2118×10^{-7}	0.5807	0.7745	2.5835
	(0.0386)	(0.0995)	(0.2873)	(1.1380)
	θ	γ	b	λ
TLGW	0.0235	4.5145	5.7081	0.0913
	(0.0166)	(1.1908)	(1.9001)	(0.0296)
	p	α	β	λ
EGEL	1.2642×10^{-5}	6.6635×10^{-2}	1.2180	1.8184
	(0.0126)	(7.4553×10^{-3})	(0.1488)	(2.7292×10^{-4})
	α	λ	β	
APIW	6.5282×10^{2}	0.6359	1.0023	
	(1.0545×10^{-5})	(7.4490×10^{-2})	(5.7444×10^{-2})	
	θ	α	β	γ
GGP	1.1012	0.5063	0.2062	1.8068×10^{-10}
	(0.5128)	(0.0570)	(0.0363)	(0.0035)
	α	λ	γ	θ
OW-TL-LLoGL	0.2075	0.8335	0. 9156	5.9250×10^{-9}
	(0.0388)	(0.1288)	(0.2535)	(0.0074)

Table 5 Parameter estimates for various models fitted for the UK Covid-19 data set

Estimates and standard errors in parenthesis				
Model	θ	α	ϕ	ψ
APT-WP	1.4240	0.3398	8.5506	1.3293
	(2.3645)	(0.8325)	(4.2121)	(0.1737)
	a	b	α	β
WL	93.0970	1.3248	0.0748	0.1909
	(1.7223×10^{-4})	(0.3614)	(0.0448)	(0.3359)
	θ	α	β	ω
EPLP	8.3048×10^{-8}	0.8091	3.2896	0.6524
	(0.0277)	(0.1739)	(0.6256)	(0.2336)
	θ	γ	b	λ
TLGW	0.1072	5.5867	4.3071	0.1357
	(1.2231)	(13.5666)	(7.0040)	(0.3269)
	p	α	β	λ
EGEL	3.6811×10^{-8}	2.0277	0.8414	2.0361
	(0.0264)	(0.4233)	(0.2113)	(0.4251)
	α	λ	β	
APIW	5.2188×10^{2}	8.2093×10^{-3}	0.9656	
	(1.3723×10^{-6})	(2.5893×10^{-3})	(7.8403×10^{-2})	
	θ	α	β	γ
GGP	7.8682×10^{-8}	0.5121	2.3917	0.9089
	(0.0265)	(0.0831)	(0.5813)	(1.8510)
	α	λ	γ	θ
OW-TL-LLoGL	17.4360	0.0510	2.1777	0.8730
	(3.6169×10^{-4})	(1.3609×10^{-2})	(2.7294×10^{-2})	(0.2579)

Tables 6 and 7 show the GoF statistics for various models fitted for the cancer and UK Covid-19 data sets respectively. It is observed that the APT-WP fit these data sets better as compared to other models presented in the tables as it has the lowest values of GoF statistics and the highest p-value statistic for both data sets.

Table 6 GoF statistics for various models fitted for the cancer data set

	Goodness of Fit Statistics							
Model	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
APT-WP	819.2246	827.2246	827.5498	838.6327	0.0222	0.1433	0.0373	0.9942
WL	820.0106	828.0106	828.3358	839.4187	0.0284	0.1883	0.0405	0.9845
EPLP	820.8894	828.8893	829.2146	840.2975	0.0405	0.2621	0.0434	0.9689
TLGW	821.1583	829.1583	829.4836	840.5665	0.0417	0.2747	0.0438	0.9662
EGEL	826.1552	834.1552	834.4804	845.5634	0.1122	0.6741	0.0725	0.5112
APIW	854.2629	860.2629	860.4565	868.819	0.4020	2.5320	0.0957	0.1910
GGP	934.7457	942.7488	943.0740	954.1569	0.2243	1.3329	0.3406	2.518×10^{-13}
OW-TL-LLoGL	1135.282	1143.2820	1143.6080	1154.6900	4.1521	21.1961	0.9196	2.2×10^{-16}

Table 7 GoF statistics for various models fitted for UK Covid-19 data set

	Goodness of Fit Statistics							
Model	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
APT-WP	-157.3653	-149.3653	-148.6635	-140.8567	0.04635	0.2989	0.0710	0.9131
WL	-156.5847	-148.5847	-147.8829	-140.0761	0.0477	0.3083	0.0710	0.9130
TLGW	-156.7566	-148.7566	-148.0549	-140.2481	0.0532	0.3370	0.0728	0.8975
APIW	-137.2541	-131.2541	-130.8403	-124.8727	0.2525	1.5726	0.1284	0.2578
EPLP	-117.8569	-109.8569	-109.1551	-101.3483	0.0586	0.3690	0.2390	0.0016
GGP	-120.2179	-112.2183	-111.5165	-103.7097	0.0817	0.4988	0.2570	0.0005
EGEL	-130.3298	-122.3300	-121.6283	-113.8215	0.0653	0.4053	0.2986	3.146×10^{-5}
OW-TL-LLoGL	-156.7886	-148.7886	-148.0868	-140.2800	0.0875	0.5876	0.7214	2.2×10^{-16}

Based on Figures 5-8 the histograms demonstrate that the APT-WP can accommodate highly tailed data sets, the data points are located along the probability plots which resembles a good fit. Additionally, we see that the fitted ECDFs' and Kaplan-Meiers curves are in close proximity to one another, indicating improved model performance. The fitted hrfs' also display an inverted bathtub shapes, thus displaying a strong fit between the hazard rate plots of our model and the TTT plots, which provides a hazard rate based on the data sets.

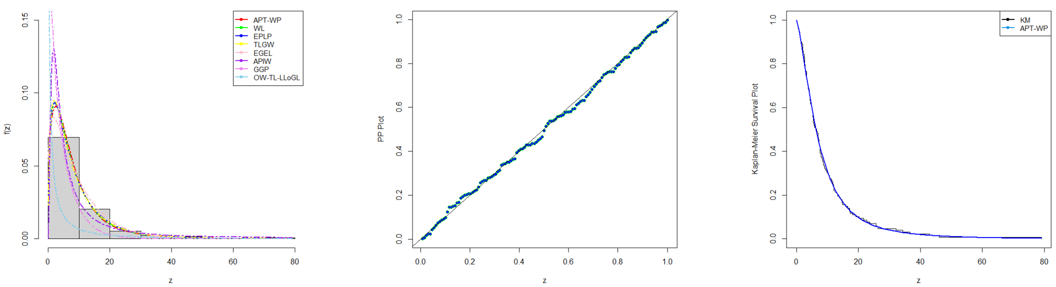

Figure 5 Fitted density, probability plot and Kaplan-Meier survival plot of the APT-WP distribution for the cancer data set

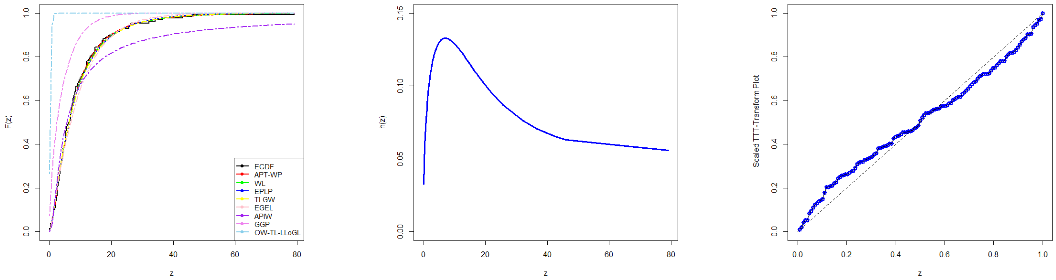

Figure 6 Estimated cdf, estimated hrf plot and Total-Time-on-Test of the APT-WP distribution for cancer data set

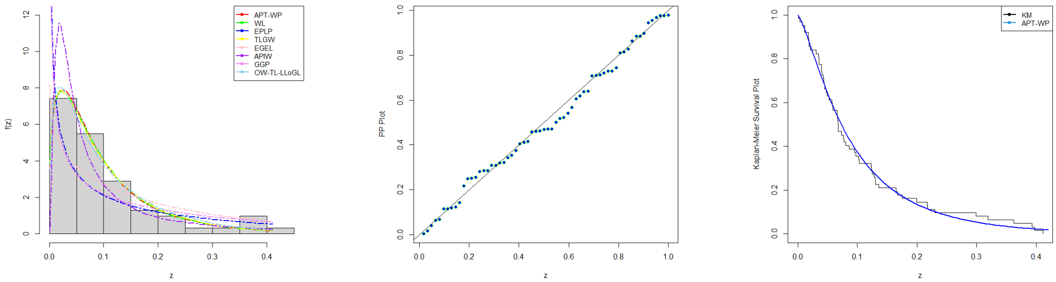

Figure 7 Fitted density, probability plot and Kaplan-Meier survival plot of the APT-WP distribution for the UK Covid-19 data set

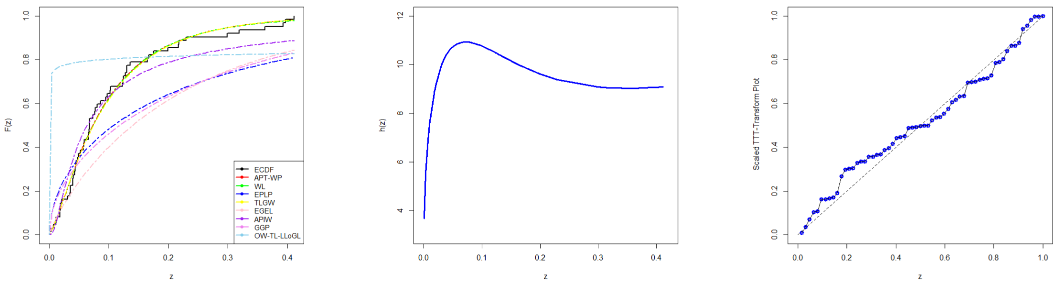

Figure 8 Estimated cdf, estimated hrf plot and Total-Time-on-Test of the APT-WP distribution for the UK Covid-19 data set

6. Concluding Remarks

This article proposes a new class of distributions called the alpha power transformed-G power series (APT-GPS). This novel class of distributions can be expressed as an infinite linear combination of the exponentiated-G family of distributions. As a result, it is easy to infer several important statistical properties of the new class from those of the exponentiated-G distribution such as the incomplete moments, moment generating function, residual and reverse residual life function. The new class of distributions can be applied to data with unimodal, decreasing, increasing, upside-down bathtub followed by bathtub, and inverted bathtub-like failure rates. A specific example of the novel distribution, the alpha power transformed-Weibull poisson (APT-WP) distribution, was applied to cancer data set and UK Covid-19 data set. The fits of the APT-WP were compared to the fits of some selected competitive models. The results clearly show that the APT-WP performs better than the non nested models presented in this article. We conclude that the our new class of distributions can be a candidate for modeling survival data sets.

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A. Density Expansion

$$
\begin{aligned}
f(z;\theta,\alpha,\Delta) &= \frac{\theta\log(\alpha)}{\alpha-1}\sum_{n=1}^{\infty}\frac{na_n\theta^n}{C(\theta)}g(z;\Delta)\alpha^{G(z;\Delta)}\left[1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right]^{n-1} \\
&= \frac{\theta\log(\alpha)}{\alpha-1}\sum_{n=0}^{\infty}\frac{(n+1)a_{n+1}\theta^{n+1}}{C(\theta)}g(z;\Delta)\alpha^{G(z;\Delta)}\left[1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right]^{n}.
\end{aligned}
$$

Now, using the following expansions

$$
\begin{aligned}
\left[1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right]^{n} &= \sum_{a=0}^{\infty}(-1)^a\binom{n}{a}(\alpha-1)^{-a}(\alpha^{G(z;\Delta)}-1)^a \\
(\alpha^{G(z;\Delta)}-1)^a &= \sum_{b=0}^{\infty}(-1)^{b+1}\binom{a}{b}\alpha^{bG(z;\Delta)} \\
\alpha^{(b+1)G(z;\Delta)} &= \sum_{d=0}^{\infty}\frac{(\log(\alpha))^d(b+1)^d}{d!}G^d(z;\Delta),
\end{aligned}
$$

we have

$$
f_{APT-GPS}(z;\theta,\alpha,\Delta)=\sum_{d=0}^{\infty}p^*_{d+1}g^*_{d+1}(z;\Delta),
$$

where

$$
p^*_{d+1} = \sum_{n,a,b=0}^{\infty}\frac{(n+1)a_{n+1}\theta^{n+1}(b+1)^d(\log(\alpha))^{d+1}(-1)^{a+b+1}}{d!(d+1)(\alpha-1)^{a+1}C(\theta)}\binom{n}{a}\binom{a}{b}.
$$

B. Limiting Distribution

Note $C_n(\theta) = \sum_{k=1}^{n} a_k\theta^k$, then

$$\begin{aligned}
\lim_{\theta\to 0^+} F(z;\theta,\alpha,\Delta) &= \lim_{\theta\to 0^+}\left(1-\frac{C\left(\theta\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)\right)}{C(\theta)}\right)\\
&= \lim_{\theta\to 0^+}\left(1-\frac{\sum_{n=1}^{\infty} a_n\theta^n\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)^n}{\sum_{n=1}^{\infty} a_n\theta^n}\right)\\
&= 1-\lim_{\theta\to 0^+}\left(\frac{a_1\theta\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)+\sum_{n=2}^{\infty} a_n\theta^n\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)^n}{a_1\theta+\sum_{n=2}^{\infty} a_n\theta^n}\right)\\
&= 1-\lim_{\theta\to 0^+}\left(\frac{\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)+\sum_{n=2}^{\infty} a_n^{-1}\theta^{n-1}\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)^n}{1+\sum_{n=2}^{\infty} a_n^{-1}\theta^{n-1}}\right)\\
&= 1-\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)\\
&= \frac{\alpha^{G(z;\Delta)}-1}{\alpha-1} = F_{APT}(z),
\end{aligned}$$

since

$$\lim_{\theta\to 0^+}\sum_{n=2}^{\infty} a_n^{-1}\theta^{n-1}\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)^n = 0$$

and

$$\lim_{\theta\to 0^+}\sum_{n=2}^{\infty} a_n^{-1}\theta^{n-1} = 0.$$

C. Order Statistics

$$\begin{aligned}
f_{k:m}(z) &= \frac{m!}{(k-1)(m-k)!}\sum_{l=0}^{\infty}(-1)^l\binom{m-k}{l}\frac{\theta\log(\alpha)}{\alpha-1}\sum_{n=0}^{\infty}\frac{(n+1)a_{n+1}\theta^{n+1}}{C(\theta)}\\
&\times\; g(z;\Delta)\alpha^{G(z;\Delta)}\left[1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right]^n\left[1-\frac{C\left(\theta\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)\right)}{C(\theta)}\right]^{k+l-1}.
\end{aligned}$$

Using the following binomial series expansion

$$\left[1-\frac{C(\theta A)}{C(\theta)}\right]^{k+l-1} = \sum_{i=0}^{\infty}(-1)^i\binom{k+l-1}{i}\left[\frac{C(\theta A)}{C(\theta)}\right]^i,$$

where $A = 1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}$ and power series raised to a positive integer result (see Violet et al) we have

$$\left[\frac{C(\theta A)}{C(\theta)}\right]^i = \sum_{r=0}^{\infty}\frac{d_{r,i}\theta^i}{C^i(\theta)}A^i,$$

where $d_{r,i}$ and $d_{0,i}$ are given by Violet et al and Makubate et al, we get

$$\begin{aligned} f_{k:m}(z) &= \frac{m!}{(k-1)(m-k)!}\sum_{l=0}^{\infty}(-1)^{l+i}\binom{m-k}{l}\binom{k+l-1}{i}\frac{\log(\alpha)}{\alpha-1}g(z;\Delta) \\ &\times \frac{(n+1)a_{n+1}\theta^{n+i+1}d_{r,i}}{[C(\theta)]^{i+1}}\alpha^{G(z;\Delta)}\left[1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right]^{n+i}. \end{aligned}$$

Now, using the following expansions

$$\begin{aligned} \left[1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right]^{n+i} &= \sum_{a=0}^{\infty}(-1)^{a}\binom{n+i}{a}(\alpha-1)^{-a}(\alpha^{G(z;\Delta)}-1)^{a} \\ (\alpha^{G(z;\Delta)}-1)^{a} &= \sum_{b=0}^{\infty}(-1)^{b+1}\binom{a}{b}\alpha^{bG(z;\Delta)} \\ \alpha^{(b+1)G(z;\Delta)} &= \sum_{d=0}^{\infty}\frac{(\log(\alpha))^{d}(b+1)^{d}}{d!}G^{d}(z;\Delta), \end{aligned}$$

the order statistics of the APT-GPS is given by

$$f_{k:m}(z) = \sum_{d=0}^{\infty}q^{*}_{d+1}g^{*}_{d+1}(z;\Delta),$$

where

$$\begin{aligned} q^{*}_{d+1} &= \frac{m!}{(k-1)(m-k)!}\sum_{n,l,i,r,a,b=0}^{\infty}(-1)^{l+i+a+b+1}\binom{m-k}{l}\binom{k+l-1}{i} \\ &\times \binom{n+i}{a}\binom{a}{b}\frac{\log(\alpha)(b+1)^{d}(\log(\alpha))^{d}}{d!(\alpha-1)^{a+1}(d+1)}\frac{(n+1)a_{n+1}\theta^{n+i+1}d_{r,i}}{[C(\theta)]^{i+1}}. \end{aligned}$$

D. Probability Weighted Moment

$$\begin{aligned} f_{k:m}(z) &= \int_{0}^{\infty}\frac{\theta\log(\alpha)}{\alpha-1}\sum_{n=0}^{\infty}\frac{(n+1)a_{n+1}\theta^{n+1}}{C(\theta)}g(z;\Delta)\alpha^{G(z;\Delta)} \\ &\times \left[1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right]^{n}\left[1-\frac{C\left(\theta\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)\right)}{C(\theta)}\right]^{j}. \end{aligned}$$

Using the following binomial series expansion

$$\left[1-\frac{C(\theta A)}{C(\theta)}\right]^{k+l-1} = \sum_{i=0}^{\infty}(-1)^{i}\binom{j}{i}\left[\frac{C(\theta A)}{C(\theta)}\right]^{i},$$

where $A = 1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}$ and power series raised to a positive integer result (Nyamajiwa et al., 2024) we have

$$\left[\frac{C(\theta A)}{C(\theta)}\right]^{i} = \sum_{r=0}^{\infty}\frac{d_{r,i}\theta^{i}}{C^{i}(\theta)}A^{i},$$

where $d_{r,i}$ and $d_{0,i}$ are given by Chipepa et al. (2021) and Nyamajiwa et al. (2024), we get

$$\begin{aligned} f_{k:m}(z) &= \frac{m!}{(k-1)(m-k)!}\sum_{l=0}^{\infty}(-1)^{l+i}\binom{m-k}{l}\binom{k+l-1}{i}\frac{\log(\alpha)}{\alpha-1}g(z;\Delta) \\ &\times \frac{(n+1)a_{n+1}\theta^{n+i+1}d_{r,i}}{[C(\theta)]^{i+1}}\alpha^{G(z;\Delta)}\left[1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right]^{n+i}. \end{aligned}$$

Now, using the following expansions

$$\begin{aligned} \left[1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right]^{n+i} &= \sum_{a=0}^{\infty}(-1)^{a}\binom{n+i}{a}(\alpha-1)^{-a}(\alpha^{G(z;\Delta)}-1)^{a} \\ (\alpha^{G(z;\Delta)}-1)^{a} &= \sum_{b=0}^{\infty}(-1)^{b+1}\binom{a}{b}\alpha^{bG(z;\Delta)} \\ \alpha^{(b+1)G(z;\Delta)} &= \sum_{d=0}^{\infty}\frac{(\log(\alpha))^{d}(b+1)^{d}}{d!}G^{d}(z;\Delta), \end{aligned}$$

the order statistics of the APT-GPS is given by

$$f_{k:m}(z) = \sum_{d=0}^{\infty}e^{*}_{d+1}g^{*}_{d+1}(z;\Delta),$$

where

$$\begin{aligned} e^{*}_{d+1} &= \sum_{n,i,r,a,b=0}^{\infty}(-1)^{i+a+b+1}\binom{j}{i}\binom{n+i}{a}\binom{a}{b} \\ &\times \frac{\log(\alpha)(b+1)^{d}(\log(\alpha))^{d}}{d!(\alpha-1)^{a+1}(d+1)}\frac{(n+1)a_{n+1}\theta^{n+i+1}d_{r,i}}{[C(\theta)]^{i+1}}. \end{aligned}$$

E. Estimation

Score vectors corresponding to Equation (21) are as follows;

$$\begin{aligned} U_{\theta}(r) &= \frac{n}{\theta}-\frac{nC'(\theta)}{C(\theta)}+\sum_{i=1}^{n}\frac{C''\left(\theta\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)\right)\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)}{C'\left(\theta\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)\right)}, \\ U_{\alpha}(r) &= \frac{n[(1-1/\alpha)-\log(\alpha)]}{\log(\alpha)}+\sum_{i=1}^{n}\frac{G(z;\Delta)}{\alpha}-\sum_{i=1}^{n}\frac{C''\left(\theta\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)\right)}{C'\left(\theta\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)\right)} \\ &\times \theta\frac{(\alpha-1)G(z;\Delta)\alpha^{G(z;\Delta)-1}-(\alpha^{G(z;\Delta)}+1)}{(\alpha-1)^{2}}, \end{aligned}$$

and

$$\begin{aligned} U_{z_k}(r) &= \sum_{i=1}^{n}\partial\left(g(z;\Delta)\right)\partial z_k+\sum_{i=1}^{n}\partial\left(G(z;\Delta)\log(\alpha)\right)\partial z_k \\ &+ \sum_{i=1}^{n}\partial\left(\log\left(C'\left(\theta\left(1-\frac{\alpha^{G(z;\Delta)}-1}{\alpha-1}\right)\right)\right)\right)\partial z_k. \end{aligned}$$