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Estimation of $P(X < Y)$ for Morgenstern Type Bivariate Exponential Distribution Based on Ranked Set Sample

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Abstract

Estimation of $R = P(X < Y)$ has been intensively investigated in the literature using parametric and nonparametric approaches under different sampling schemes when X and Y are independent random variables. In this paper, we consider the problem of estimation of R when X and Y are dependent random variables based on ranked set sample (RSS). The maximum likelihood estimates (MLEs) and Bayes estimates (BEs) of R are obtained based on RSS when (X, Y) follows Morgenstern type bivariate exponential distribution. BEs are obtained based on both symmetric and asymmetric loss functions. The percentile bootstrap and HPD confidence intervals for R are also obtained. Simulation studies are carried out to find the accuracy of the proposed estimators. A real data is also used to illustrate the inferential procedures developed in this paper.

Keywords: Ranked set sampling, Morgenstern type bivariate exponential distribution, maximum likelihood estimator, Bayesian estimation.

1. Introduction

If X and Y are two random variables, then from the available literature we observe that making inference on $R = P(X < Y)$ attracted wide interest in several areas of studies such as quality control, engineering statistics, reliability, medicine, psychology, biostatistics, stochastic precedence and probabilistic mechanical design (see, Kotz et al. (2003), for a review). In the context of reliability the stress-strength model describes the life of a component which has a random strength Y and is subjected to a random stress X. The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever $X < Y$. Thus $R = P(X < Y)$ is a measure of component reliability. It has many applications especially in engineering concepts such as structures, deterioration of rocket motors, fatigue failure of aircraft structures and the aging of concrete pressure vessels (for details see, Nadarajah and Kotz (2006)). The applications of $R = P(X < Y)$ is not limited to reliability and engineering, it has lot of applications in medicine, psychology, environmental studies etc. For example, in medical studies, if X and Y represent the outcome of control and experimental treatments, then R can be interpreted as the effectiveness of the treatment. In the study of water quality in freshwater, if Y represent the concentration of dissolved trace metals such as zinc, copper or lead in water and X represents the corresponding worldwide water quality standards of that metal in water, then $P(Y < X) = 1 - R$ can be considered as the probability that the metal concentration in freshwater is lower than the corresponding worldwide standard.

The concept of ranked set sampling was first introduced by McIntyre (1952) as a process of improving the precision of the sample mean as an estimator of the population mean. Ranked set sampling as described in McIntyre (1952) is applicable whenever ranking of a set of sampling units can be done easily by a judgement method or any other inexpensive methods (for a detailed discussion on the theory and applications of RSS, see Chen et al. (2004)). In this paper, we consider a bivariate ranked set sample for estimating $P(X < Y)$ when X and Y are dependent. The procedure of bivariate ranked set sampling of size n is as follows: Choose n^2 independent units, arrange them randomly into n sets each with n units and observe the value of the variable X on each of these units. In the first set, that unit for which the measurement on the variable X is the smallest is chosen. In the second set, that unit for which the measurement on the variable X is the second smallest is chosen. The procedure is repeated until in the last set, that unit for which the measurement on the variable X is the largest is chosen. Then make measurements on the Y variate of the selected units. The resulting new set of n bivariate observations corresponding to the units chosen by one from each set as described above is called the bivariate ranked set sample (BRSS). If $X_{(r)}$ is the observation measured on the variable X from the unit chosen from the rth set then, we write $Y_{[r]}$ to denote the corresponding measurement made on the variable Y on this unit. Clearly, $Y_{[r]}$ is the concomitant of the rth order statistic of a sample of size n arising from the rth sample. The joint probability density function (pdf) of $(X_{(r)}, Y_{[r]})$ is given by

$$f_{X_{(r)},Y_{[r]}}(x_{(r)}, y_{[r]}) = f(y_{[r]}|x_{(r)})f_{(r)}(x_r),$$

where $f(y|x)$ is the conditional pdf of Y given X and $f_{(r)}(x)$ is the pdf of rth order statistic of a random sample of size n and is given by

$$f_{(r)}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x).$$

Since $(X_{(1)}, Y_{[1]}), (X_{(2)}, Y_{[2]}), \ldots, (X_{(n)}, Y_{[n]})$ are independent, the joint probability density function (pdf) of $(X_{(n)}, Y_{[n]}) = (X_{(1)}, Y_{[1]}), (X_{(2)}, Y_{[2]}), \ldots, (X_{(n)}, Y_{[n]})$ is given by

$$f_{X_{(n)},Y_{[n]}}(x_{(n)}, y_{[n]}) = \prod_{r=1}^{n} f(y_{[r]}|x_{(r)})f_{(r)}(x_r). \tag{1}$$

The estimation of R based on RSS has been extensively investigated in the literature when X and Y are independent random variables. Sengupta and Mukhuti (2008) considered the problem of estimation of R based on RSS and compared with simple random sampling (SRS) in terms of the variance of the unbiased estimator of $P(X < Y)$. They proved that the unbiased estimator based on RSS data has a smaller variance compared with the unbiased estimator based on SRS, even when the rankings of RSS are imperfect. Hussian (2014) discussed the problem of estimating R for generalized inverted exponential distribution based on RSS and SRS. Muttlak et al. (2010) considered the problem of estimation of $P(X < Y)$, where X and Y are independently distributed exponential random variables using ranked set sampling data. Dong et al. (2013) considered the problem of estimation of $P(X < Y)$ for a system, when strength X and stress Y follow independent exponential distributions. Akgul and Senoglu (2017) considered the problem of estimation of R based on RSS when X and Y are independent Weibull distributions. Chacko and Mathew (2019) considered the problem of estimation of $P(X < Y)$ for bivariate normal distribution based on RSS.

In this paper, we focus on estimation of $R = P(X < Y)$ for Morgenstern type bivariate exponential distribution (MTBED) based on BRSS. A random variable (X, Y) follows MTBED if its pdf is given by (see Kotz et al. (2000), p.353)

$$f(x,y) = \begin{cases} \theta_1\theta_2 exp(-\theta_1 x - \theta_2 y)[1 + \alpha(1 - 2exp(-\theta_1 x))(1 - 2exp(-\theta_2 y))], \\ \qquad x > 0, y > 0, -1 \leq \alpha \leq 1, \theta_1 > 0, \text{ and } \theta_2 > 0 \\ 0, \qquad \text{otherwise.} \end{cases} \tag{2}$$

It may be noted that if (X, Y) has a MTBED as defined in (2). Then the marginal distributions of both X and Y have exponential distributions. The correlation between X and Y is $\alpha/4$ (see, Lai and Balakrishnan (2009), p. 69). As α lies between -1 and 1, MTBED accommodates correlation in the range of (-1/4,1/4). Chacko and Thomas (2008) considered the problem of estimation of parameters of MTBED based on RSS. Chacko (2017) obtained the Bayes estimates of parameters of MTBED based on RSS when ranking is not perfect. Chacko and Mathew (2022) consider the problem of estimation of $P(X < Y)$ for MTBED based on record values.

The organization of this paper is as follows. In Section 2, we obtain the expression for $P(X < Y)$, when (X, Y) follows MTBED and consider the maximum likelihood estimation of R. The bootstrap confidence interval (CI) of R is also obtained in this section. In Section 3, we obtain the Bayes estimates of R under symmetric as well as asymmetric loss functions using importance sampling method. Section 4 is devoted to some simulation studies. In Section 5, a real data is used to illustrate the inferential procedures developed in this paper and finally in Section 6, we give some concluding remarks.

2. Maximum Likelihood Estimation

Let (X,Y) follows MTBED with pdf defined in (2). Then, R is given by

$$\begin{aligned}
R &= P(X < Y) = \int_0^\infty \int_0^y f(x,y)dxdy \\
&= \int_0^\infty \int_0^y \theta_1\theta_2 e^{-\theta_1 x - \theta_2 y}\left[1 + \alpha(1 - 2e^{-\theta_1 x})(1 - 2e^{-\theta_2 y})\right] dxdy \\
&= \int_0^\infty \theta_2 e^{-\theta_2 y}\left[(1 - 2e^{-\theta_1 y}) + \alpha[(1 - e^{-\theta_1 y}) \right. \\
&\quad \left. -(1 - e^{-2\theta_1 y})](1 - 2e^{-\theta_2 y})\right] dy \\
&= \frac{\theta_1}{\theta_1 + \theta_2}\left[1 + \alpha\frac{\theta_2(\theta_1 - \theta_2)}{(2\theta_1 + \theta_2)(2\theta_2 + \theta_1)}\right].
\end{aligned}$$

Since R is a function of α, θ_1 and θ_2, we can write

$$R = R(\theta_1, \theta_2, \alpha) = \frac{\theta_1}{\theta_1 + \theta_2}\left[1 + \alpha\frac{\theta_2(\theta_1 - \theta_2)}{(2\theta_1 + \theta_2)(2\theta_2 + \theta_1)}\right].$$

In this section, we obtain the MLE of R for MTBED using BRSS. Let $(X_{(r)}, Y_{[r]}), r = 1, 2, \ldots, n$ be the vector of BRSS arising from MTBED. Then from (1) the likelihood function is given by

$$\begin{aligned}
L(\theta_1, \theta_2, \alpha) &= (\theta_1\theta_2)^n \prod_{r=1}^n exp(-\theta_1 x_{(r)} - \theta_2 y_{[r]}) \\
&\quad [1 + \alpha(1 - 2exp(-\theta_1 x_{(r)}))(1 - 2exp(-\theta_2 y_{[r]}))] \\
&\quad \frac{n!}{(r-1)!(n-r)!}(1 - exp(-\theta_1 x_{(r)}))^{r-1}(exp(-\theta_1 x_{(r)}))^{n-r}.
\end{aligned}$$

Then the log-likelihood function is given by

$$\begin{aligned}
\log L(\theta_1, \theta_2, \alpha) &= \log k + n\log\theta_1 + n\log\theta_2 - \theta_1\sum_{r=1}^n x_{(r)}(n - r + 1) - \theta_2\sum_{r=1}^n y_{[r]} \\
&\quad + \sum_{r=1}^n \log[1 + \alpha(1 - 2exp(-\theta_1 x_{(r)}))(1 - 2exp(-\theta_2 y_{[r]}))] \\
&\quad + \sum_{r=1}^n (r-1)\log(1 - exp(-\theta_1 x_{(r)})),
\end{aligned}$$

where

$$k = \prod_{r=1}^{n} \frac{n!}{(r-1)!(n-r)!}.$$

Thus we have

$$\begin{aligned}\frac{\partial \log L}{\partial \theta_1} &= \frac{n}{\theta_1} - \sum_{r=1}^{n} x_{(r)}(n-r+1) + \sum_{r=1}^{n}(r-1)\frac{x_{(r)}exp(-\theta_1 x_{(r)})}{1-exp(-\theta_1 x_{(r)})} \\ &\quad + \sum_{r=1}^{n} \frac{2\alpha x_{(r)}(1-2exp(-\theta_2 y_{[r]}))exp(-\theta_1 x_{(r)})}{[1+\alpha(1-2exp(-\theta_1 x_{(r)}))(1-2exp(-\theta_2 y_{[r]}))]},\end{aligned}$$

$$\frac{\partial \log L}{\partial \theta_2} = \frac{n}{\theta_2} - \sum_{r=1}^{n} y_{[r]} + \sum_{r=1}^{n} \frac{2\alpha y_{[r]}(1-2exp(-\theta_1 x_{(r)}))exp(-\theta_2 y_{[r]})}{[1+\alpha(1-2exp(-\theta_1 x_{(r)}))(1-2exp(-\theta_2 y_{[r]}))]}$$

and

$$\frac{\partial \log L}{\partial \alpha} = \sum_{r=1}^{n} \frac{(1-2exp(-\theta_1 x_{(r)}))(1-2exp(-\theta_2 y_{[r]}))}{[1+\alpha(1-2exp(-\theta_1 x_{(r)}))(1-2exp(-\theta_2 y_{[r]}))]}.$$

The MLEs of θ_1, θ_2 and α can be obtained by solving the following non-linear equations using the Newton-Raphson method or any other numerical methods.

$$\begin{aligned}&\frac{n}{\theta_1} - \sum_{r=1}^{n} x_{(r)}(n-r+1) + \sum_{r=1}^{n}(r-1)\frac{x_{(r)}exp(-\theta_1 x_{(r)})}{1-exp(-\theta_1 x_{(r)})} \\ &+ \sum_{r=1}^{n} \frac{2\alpha x_{(r)}(1-2exp(-\theta_2 y_{[r]}))exp(-\theta_1 x_{(r)})}{[1+\alpha(1-2exp(-\theta_1 x_{(r)}))(1-2exp(-\theta_2 y_{[r]}))]} = 0,\end{aligned}$$

$$\frac{n}{\theta_2} - \sum_{r=1}^{n} y_{[r]} + \sum_{r=1}^{n} \frac{2\alpha y_{[r]}(1-2exp(-\theta_1 x_{(r)}))exp(-\theta_2 y_{[r]})}{[1+\alpha(1-2exp(-\theta_1 x_{(r)}))(1-2exp(-\theta_2 y_{[r]}))]} = 0$$

and

$$\sum_{r=1}^{n} \frac{(1-2exp(-\theta_1 x_{(r)}))(1-2exp(-\theta_2 y_{[r]}))}{[1+\alpha(1-2exp(-\theta_1 x_{(r)}))(1-2exp(-\theta_2 y_{[r]}))]} = 0.$$

Let $\hat{\theta_1}$, $\hat{\theta_2}$ and $\hat{\alpha}$ be the MLEs of θ_1, θ_2 and α. By the invariant property of ML estimators, the MLE of R is given by

$$\hat{R} = \frac{\hat{\theta_1}}{\hat{\theta_1}+\hat{\theta_2}}\left[1+\hat{\alpha}\frac{\hat{\theta_2}(\hat{\theta_1}-\hat{\theta_2})}{(2\hat{\theta_1}+\hat{\theta_2})(2\hat{\theta_2}+\hat{\theta_1})}\right]. \tag{3}$$

2.1. Bootstrap confidence interval

In this section, we consider percentile bootstrap CI for R based on MLEs. For that, we do the following.

1. Compute the MLEs $\hat{\theta_1}^{(0)}$, $\hat{\theta_2}^{(0)}$ and $\hat{\alpha}^{(0)}$ of θ_1, θ_2 and α using original observations and set k=1.
2. Generate a bootstrap RSS from MTBED using $\hat{\theta_1}^{(0)}, \hat{\theta_2}^{(0)}$ and $\hat{\alpha}^{(0)}$ and obtain the MLEs $\hat{\theta_1}^{(k)}$, $\hat{\theta_2}^{(k)}$ and $\hat{\alpha}^{(k)}$ using the bootstrap sample.
3. Obtain the MLE of $\hat{R}_k = R(\hat{\theta_1}^{(k)}, \hat{\theta_2}^{(k)}, \hat{\alpha}^{(k)})$ and put k=k+1.
4. Repeat steps 2 and 3 B times to have $\hat{R}_k$ for $k = 1, 2, \ldots, B$ (B sufficiently large).
5. Arrange $\hat{R}_k$ for $k = 1, 2, \ldots, B$ in ascending order as $\hat{R}_{(1)} \leq \hat{R}_{(2)}, \ldots, \leq \hat{R}_{(B)}$. Then, for $0 < \nu < 1$, the $100(1-\nu)$ percentile bootstrap CI for R is given by $\left(\hat{R}_{([B(\nu/2)])}, \hat{R}_{([B(1-\nu/2)])}\right)$.

3. Bayesian Estimation

In this section, we consider Bayesian estimation of R for MTBED under symmetric as well as asymmetric loss functions. For symmetric loss function we consider squared error loss (SEL) function and for asymmetric loss function we consider both LINEX loss (LL) and the general entropy loss (EL) function. The Bayes estimate of any parameter μ under the LINEX loss (LL) function is given by

$$\hat{d}_{LB} = \frac{-1}{h}\log\{E_\mu(e^{-h\mu})\}, h \neq 0. \tag{4}$$

The Bayes estimate of any parameter μ under general entropy loss (EL) function is given by

$$\hat{d}_{EB} = (E_\mu(\mu^{-q}))^{\frac{-1}{q}}, q \neq 0. \tag{5}$$

Let $(X_{(r)}, Y_{[r]}), r = 1, 2, \ldots, n$ be the vector of bivariate RSS drawn from MTBED $(\theta_1, \theta_2, \alpha)$. Then from (2) the likelihood function is given by

$$\begin{aligned} L(\theta_1, \theta_2, \alpha) \quad = \quad & (\theta_1\theta_2)^n \prod_{r=1}^{n} exp(-\theta_1 x_{(r)} - \theta_2 y_{[r]}) \\ & [1 + \alpha(1 - 2exp(-\theta_1 x_{(r)}))(1 - 2exp(-\theta_2 y_{[r]}))] \\ & \frac{n!}{(r-1)!(n-r)!}(1 - exp(-\theta_1 x_{(r)}))^{r-1}(exp(-\theta_1 x_{(r)}))^{n-r}. \end{aligned}$$

Assume that the prior distributions of $\theta_1 \sim$ Gamma(a, b), $\theta_2 \sim$ Gamma(c, d) and $\alpha \sim U[-1, 1]$. Therefore the prior density functions of θ_1, θ_2 and α are respectively given by

$$\pi_1(\theta_1|a, b) = \frac{b^a}{\Gamma(a)}\theta_1^{a-1}e^{-b\theta_1}; a > 0, b > 0, \tag{6}$$

$$\pi_2(\theta_2|c, d) = \frac{d^c}{\Gamma(c)}\theta_2^{c-1}e^{-d\theta_2}; c > 0, d > 0 \tag{7}$$

and

$$\pi_3(\alpha) = \frac{1}{2}, -1 \leq \alpha \leq 1. \tag{8}$$

Then, the joint prior density of $\theta = (\theta_1, \theta_2, \alpha)$ is given by

$$\pi(\theta) = \frac{1}{2}\frac{b^a}{\Gamma(a)}\frac{d^c}{\Gamma(c)}\theta_1^{a-1}\theta_2^{c-1}e^{-b\theta_1}e^{-d\theta_2}. \tag{9}$$

Then, the joint posterior density of θ given the data is

$$\pi^*(\theta|data) = \frac{L(\theta)\pi(\theta)}{\int L(\theta)\pi(\theta)d\theta}. \tag{10}$$

Therefore, the Bayes estimate of any function $R(\theta)$ of θ under SEL, LL and EL functions are respectively given by

$$\hat{R}_S(\theta) = \frac{\int_\theta R(\theta)L(\theta)\pi(\theta)d\theta}{\int_\theta L(\theta)\pi(\theta)d\theta}, \tag{11}$$

$$\hat{R}_L(\theta) = \frac{-1}{h}\log\frac{\int_\theta e^{-hR(\theta)}L(\theta)\pi(\theta)d\theta}{\int_\theta L(\theta)\pi(\theta)d\theta}, \tag{12}$$

and

$$\hat{R}_E(\theta) = \left[\frac{\int_\theta R(\theta)^{-q} L(\theta)\pi(\theta)d\theta}{\int_\theta L(\theta)\pi(\theta)d\theta} \right]^{\frac{-1}{q}}. \quad (13)$$

It is not possible to compute (11)-(13) explicitly. Thus, we propose importance sampling method to find the Bayes estimates for R.

3.1. Importance sampling method

In this subsection, we consider the importance sampling method to generate samples from the posterior distributions and then find the Bayes estimates of R under different loss functions (see Tokdar and Kass (2010)). The numerator in the posterior distribution given in (10) can be written as

$$L(\theta)\pi(\theta) = Q(\theta) f_1(\theta_1) f_2(\theta_2) f_3(\alpha),$$

where

$$Q(\theta) = \prod_{r=1}^{n} (1 - exp(-\theta_1 x_{(r)}))^{r-1}[1 + \alpha(1 - 2exp(-\theta_1 x_{(r)}))(1 - 2exp(-\theta_2 y_{[r]}))], \quad (14)$$

$$f_1(\theta_1) \propto \theta_1^{n+a-1} exp[-\theta_1(\sum_{r=1}^{n} x_{(r)}(n - r + 1) + b)], \quad (15)$$

$$f_2(\theta_2) \propto \theta_2^{n+c-1} exp[-\theta_2(\sum_{r=1}^{n} y_{[r]} + d)], \quad (16)$$

$$\text{and } f_3(\alpha) = \frac{1}{2}. \quad (17)$$

Thus, from (15), we can see that θ_1 follows gamma distribution with parameters $(n + a)$ and $\left(\sum_{r=1}^{n} x_{(r)}(n - r + 1) + b\right)$. Again, from (16), one can see that θ_2 follows gamma distribution with parameters $(n + c)$ and $\left(\sum_{r=1}^{n} y_{[r]} + d\right)$. From (17), we can see that α follows $U(-1, 1)$. Let $\theta_1^{(t)}$, $\theta_2^{(t)}$ and $\alpha^{(t)}$, $t = 1, 2, \ldots, N$ be the observations generated from (15), (16) and (17), respectively. By importance sampling method, the Bayes estimators under SEL, LL and EL given in (11), (12) and (13) can be respectively obtained as

$$\hat{R}_S = \frac{\sum_{t=1}^{N} R(\theta^{(t)})Q(\theta^{(t)})}{\sum_{t=1}^{N} Q(\theta^{(t)})}, \quad (18)$$

$$\hat{R}_L = \frac{-1}{h} \log \left[\frac{\sum_{t=1}^{N} exp(-hR(\theta^{(t)})Q(\theta^{(t)})}{\sum_{t=1}^{N} Q(\theta^{(t)})} \right], \quad (19)$$

and

$$\hat{R}_E = \left[\frac{\sum_{t=1}^{N} (R(\theta^{(t)}))^{-q} Q(\theta^{(t)})}{\sum_{t=1}^{N} Q(\theta^{(t)})} \right]^{-1/q}. \quad (20)$$

3.2. HPD Interval

In this subsection, we construct HPD intervals for R as described in Chen and Shao (1999). Define $R_t = R(\theta^{(t)})$, where $\theta^{(t)}$ for $t = 1, 2, \ldots, N$ are posterior samples generated respectively from (15), (16) and (17) for θ_1, θ_2 and α. Define

$$w_i = \frac{Q(\theta^{(i)})}{\sum_{t=1}^{N} Q(\theta^{(t)})}.$$

Let $R_{(t)}$ be the ordered values of R_t for $t = 1, 2, \ldots, N$. Then the pth quantile of R can be estimated as

$$\hat{R}^{(p)} = \begin{cases} R_{(1)} & \text{if} \quad p = 0 \\ R_{(i)} & \text{if} \quad \sum_{j=1}^{i-1} w_{(j)} < p < \sum_{j=1}^{i} w_{(j)}, \end{cases}$$

where $w_{(j)}$ is the weight associated with jth ordered value $R_{(j)}$. Then, the $100(1-\nu)\%$, $0 < \nu < 1$, confidence interval for R is given by $(\hat{R}^{(j/N)}, \hat{R}^{(j+[(1-\nu)N])/N)})$, $\quad j = 1, 2, \ldots, N - [(1-\nu)N]$, where [.] is the greatest integer function. Then, the required HPD interval for R is the interval with smallest width.

4. Simulation Study

In this section, we carry out a simulation study for illustrating the estimation procedures developed in previous sections. First, we obtain the MLE of R using (3). We have obtained the bias and MSE of MLEs for different combinations of θ_1, θ_2 and α and are provided in Table 1 and Table 2. The bootstrap CI for R are also obtained. The average interval length (AIL) and coverage probability (CP) are also obtained and are included in Table 1 and Table 2. For Bayes estimation, we take the hyperparameters $a = 2, b = 2, c = 2$, and $d = 2$ for the prior distributions of θ_1 and θ_2. We have obtained the Bayes estimators for R using RSS under SEL, LL and EL functions using importance sampling method.

For that, we use the following algorithm:

1. Generate a bivariate RSS of size n from MTBED distribution with parameters θ_1, θ_2 and α.
2. Calculate the Bayes estimators of R using importance sampling method as described below:
   (a) Set t=1
   (b) Generate $\theta_1^{(t)}$ from Gamma distribution with parameters $n+a$ and $\sum_{r=1}^{n} x_{(r)}(n-r+1)+b$.
   (c) Generate $\theta_2^{(t)}$ from Gamma distribution with parameters $n+c$ and $\sum_{r=1}^{n} y_{[r]} + d$.
   (d) Generate $\alpha^{(t)}$ from uniform $(-1, 1)$.
   (e) Calculate $\hat{R}^{(t)}$ using (3) and put t=t+1.
   (f) Repeat steps (b) to (e) $N = 50000$ times.
   (g) Calculate the Bayes estimators for R using (18)-(20).
3. Repeat steps 1 and 2 for 500 times.
4. Calculate the average bias and MSE of all the estimators.

Table 1 The bias and MSE of MLEs for R and AIL and CP for CIs when α is negative

α	n	θ_1	θ_2	R	MLE		Bootstrap		HPD	
					Bias	MSE	AIL	CP	AIL	CP
-0.25	10	1.5	0.5	0.73929	0.17350	0.01416	0.29141	0.81	0.14741	0.91
		1.5	1	0.59464	0.17875	0.08695	0.28821	0.82	0.15560	0.92
		0.5	1	0.34167	-0.17460	0.06241	0.28375	0.80	0.14378	0.91
		0.5	1.5	0.26071	-0.15409	0.05155	0.27857	0.81	0.14216	0.90
	20	1.5	0.5	0.73929	0.13245	0.01358	0.31888	0.83	0.13233	0.94
		1.5	1	0.59464	0.16490	0.07046	0.31606	0.84	0.14893	0.93
		0.5	1	0.34167	-0.14530	0.06034	0.31238	0.83	0.14761	0.93
		0.5	1.5	0.26071	-0.13605	0.04170	0.30463	0.83	0.14126	0.94
	30	1.5	0.5	0.73929	0.11863	0.01268	0.33855	0.85	0.13747	0.95
		1.5	1	0.59464	-0.12412	0.06425	0.35097	0.87	0.14120	0.96
		0.5	1	0.34167	-0.13510	0.05174	0.34934	0.86	0.13889	0.95
		0.5	1.5	0.26071	-0.12392	0.03842	0.33967	0.85	0.13301	0.95
-0.5	10	1.5	0.5	0.72857	0.17280	0.09752	0.28190	0.79	0.14898	0.91
		1.5	1	0.58929	0.17024	0.06844	0.28066	0.81	0.14215	0.90
		0.5	1	0.35000	-0.16343	0.06534	0.27887	0.82	0.13199	0.91
		0.5	1.5	0.27143	-0.15338	0.03724	0.27619	0.81	0.12185	0.90
	20	1.5	0.5	0.72857	0.14747	0.09247	0.28999	0.85	0.14309	0.93
		1.5	1	0.58929	0.15100	0.06107	0.29007	0.85	0.14291	0.94
		0.5	1	0.35000	-0.13030	0.05529	0.28865	0.84	0.14221	0.92
		0.5	1.5	0.27143	-0.13903	0.03191	0.28655	0.83	0.14021	0.93
	30	1.5	0.5	0.72857	0.11608	0.01164	0.28604	0.85	0.14357	0.95
		1.5	1	0.58929	-0.10142	0.01055	0.29066	0.86	0.14006	0.94
		0.5	1	0.35000	-0.12772	0.05111	0.29318	0.87	0.13866	0.94
		0.5	1.5	0.27143	-0.13141	0.02956	0.29236	0.85	0.13556	0.95
-0.75	10	1.5	0.5	0.71786	0.16445	0.08189	0.28562	0.80	0.13180	0.89
		1.5	1	0.58393	-0.17122	0.06250	0.28316	0.81	0.15175	0.91
		0.5	1	0.35833	-0.18248	0.05008	0.27836	0.81	0.14161	0.90
		0.5	1.5	0.28214	-0.16579	0.05519	0.27272	0.80	0.15800	0.90
	20	1.5	0.5	0.71786	0.14765	0.07256	0.31541	0.84	0.15153	0.93
		1.5	1	0.58393	0.15571	0.05923	0.31276	0.84	0.14664	0.92
		0.5	1	0.35833	-0.14327	0.04573	0.30561	0.83	0.14402	0.93
		0.5	1.5	0.28214	-0.12114	0.04764	0.29781	0.84	0.14123	0.92
	30	1.5	0.5	0.71786	0.08486	0.06507	0.34168	0.85	0.14625	0.94
		1.5	1	0.58393	-0.14040	0.01054	0.35192	0.86	0.14184	0.94
		0.5	1	0.35833	-0.12462	0.03585	0.34653	0.87	0.14564	0.95
		0.5	1.5	0.28214	-0.11834	0.04407	0.33531	0.86	0.14614	0.94

Table 2 The bias and MSE of MLEs for R and AIL and CP for CIs when α is positive

α	n	θ_1	θ_2	R	MLE		Bootstrap		HPD	
					Bias	MSE	AIL	CP	AIL	CP
0.25	10	1.5	0.5	0.76071	-0.18361	0.08460	0.27433	0.84	0.02169	0.90
		1.5	1	0.60536	-0.16413	0.06891	0.27342	0.81	0.02188	0.89
		0.5	1	0.32500	0.16493	0.04663	0.27029	0.80	0.02012	0.91
		0.5	1.5	0.23929	0.14292	0.04860	0.26728	0.81	0.01843	0.90
	20	1.5	0.5	0.76071	-0.13138	0.07741	0.28367	0.85	0.05146	0.93
		1.5	1	0.60536	-0.13160	0.05801	0.28361	0.84	0.05104	0.92
		0.5	1	0.32500	0.13082	0.03753	0.28122	0.85	0.04684	0.94
		0.5	1.5	0.23929	0.12800	0.03772	0.27848	0.83	0.04476	0.94
	30	1.5	0.5	0.76071	-0.10364	0.07298	0.28043	0.85	0.14316	0.95
		1.5	1	0.60536	-0.11349	0.05469	0.28544	0.86	0.14179	0.95
		0.5	1	0.32500	0.10272	0.02823	0.28614	0.87	0.13897	0.96
		0.5	1.5	0.23929	0.06498	0.03127	0.28545	0.86	0.13563	0.96
0.5	10	1.5	0.5	0.77143	-0.15169	0.09177	0.26875	0.81	0.02575	0.89
		1.5	1	0.61071	-0.15924	0.04851	0.26817	0.82	0.02345	0.90
		0.5	1	0.31667	0.12684	0.04534	0.26632	0.81	0.02229	0.91
		0.5	1.5	0.22857	0.12793	0.04239	0.26416	0.82	0.01956	0.90
	20	1.5	0.5	0.77143	-0.14605	0.08131	0.27436	0.83	0.04521	0.94
		1.5	1	0.61071	-0.15232	0.03783	0.27461	0.84	0.04570	0.93
		0.5	1	0.31667	0.11812	0.04193	0.27314	0.83	0.04427	0.94
		0.5	1.5	0.22857	0.10912	0.03732	0.27108	0.84	0.04098	0.93
	30	1.5	0.5	0.77143	-0.12895	0.06648	0.27214	0.86	0.13545	0.95
		1.5	1	0.61071	-0.10390	0.02101	0.27500	0.85	0.13534	0.96
		0.5	1	0.31667	0.10660	0.03567	0.27571	0.86	0.13279	0.95
		0.5	1.5	0.22857	0.06698	0.03555	0.27506	0.87	0.12968	0.95
0.75	10	1.5	0.5	0.78214	-0.17319	0.08961	0.26516	0.81	0.02355	0.88
		1.5	1	0.61607	-0.14566	0.07391	0.26464	0.81	0.02227	0.90
		0.5	1	0.30833	0.16220	0.06635	0.26305	0.82	0.02040	0.89
		0.5	1.5	0.21786	0.13259	0.04267	0.26130	0.82	0.01914	0.91
	20	1.5	0.5	0.78214	-0.14939	0.07892	0.26902	0.84	0.05173	0.91
		1.5	1	0.61607	-0.13712	0.06463	0.26936	0.83	0.04898	0.92
		0.5	1	0.30833	0.13771	0.05534	0.26839	0.83	0.04589	0.93
		0.5	1.5	0.21786	0.12987	0.04084	0.26691	0.84	0.04343	0.92
	30	1.5	0.5	0.78214	-0.10638	0.01756	0.26745	0.85	0.13681	0.95
		1.5	1	0.61607	-0.12901	0.05032	0.26935	0.85	0.13801	0.94
		0.5	1	0.30833	0.10347	0.05150	0.26956	0.87	0.13505	0.94
		0.5	1.5	0.21786	0.10319	0.03789	0.26919	0.88	0.13192	0.95

We repeat the simulation study for different values of θ_1, θ_2, α and $n = 10, 20, 30$. The bias and MSE of Bayes estimators under different loss functions are given in Table 3 and Table 4. The AILs and CP for HPD interval are also obtained and are included in Table 1 and Table 2. From the tables, we can see that bias and MSE of all estimators decrease when sample size n increases. The bias and MSE of Bayes estimators under squared error loss function are smaller than bias and MSE of all other estimators. Also, bias and MSE of MLEs are smaller than bias and MSE of Bayes estimators under LL and EL functions. AILs of HPD intervals are smaller and the associated CPs are higher than that of bootstrap confidence intervals.

Table 3 The bias and MSE for Bayes estimator for R when α is negative

α	n	θ_1	θ_2	SEL		LL		EL	
				Bias	MSE	Bias	MSE	Bias	MSE
-0.25	10	1.5	0.5	-0.15122	0.03785	-0.18144	0.03791	-0.18186	0.03805
		1.5	1	-0.16239	0.05493	-0.22263	0.05504	-0.22342	0.05540
		0.5	1	-0.15142	0.03840	-0.19159	0.03845	-0.19282	0.03889
		0.5	1.5	-0.14149	0.02125	-0.14158	0.02128	-0.14249	0.02153
	20	1.5	0.5	-0.10223	0.01627	-0.10280	0.01636	-0.10369	0.01653
		1.5	1	-0.14339	0.02622	-0.14380	0.02636	-0.14485	0.02673
		0.5	1	-0.12159	0.01883	-0.12197	0.01891	-0.12405	0.01937
		0.5	1.5	-0.09086	0.01044	-0.09106	0.01047	-0.09240	0.01064
	30	1.5	0.5	-0.01562	0.00523	-0.01718	0.00525	-0.01915	0.00533
		1.5	1	-0.04583	0.00735	-0.04769	0.00749	-0.05113	0.00787
		0.5	1	-0.04570	0.00684	-0.04713	0.00690	-0.05243	0.00736
		0.5	1.5	-0.00143	0.00311	-0.00248	0.00318	-0.00758	0.00321
-0.5	10	1.5	0.5	-0.13205	0.04565	-0.20560	0.04573	-0.20601	0.04590
		1.5	1	-0.14218	0.05187	-0.21851	0.05194	-0.21899	0.05216
		0.5	1	-0.17683	0.03400	-0.17694	0.03403	-0.17778	0.03434
		0.5	1.5	-0.13309	0.01885	-0.13317	0.01887	-0.13400	0.01908
	20	1.5	0.5	-0.12624	0.02233	-0.12672	0.02246	-0.12752	0.02271
		1.5	1	-0.12946	0.02321	-0.12992	0.02333	-0.13100	0.02364
		0.5	1	-0.11843	0.01753	-0.11883	0.01760	-0.12096	0.01809
		0.5	1.5	-0.09208	0.01045	-0.09234	0.01049	-0.09435	0.01085
	30	1.5	0.5	-0.02697	0.00689	-0.02883	0.00693	-0.03107	0.00708
		1.5	1	-0.04445	0.00831	-0.04605	0.00846	-0.04906	0.00887
		0.5	1	-0.02994	0.00591	-0.03149	0.00595	-0.03736	0.00634
		0.5	1.5	-0.00649	0.00188	-0.00776	0.00198	-0.01393	0.00209
-0.75	10	1.5	0.5	-0.15785	0.05244	-0.21803	0.05253	-0.21843	0.05273
		1.5	1	-0.16226	0.05462	-0.22278	0.05469	-0.22333	0.05494
		0.5	1	-0.17176	0.03170	-0.17185	0.03172	-0.17251	0.03193
		0.5	1.5	-0.12250	0.01674	-0.12259	0.01676	-0.12344	0.01697
	20	1.5	0.5	-0.11996	0.02231	-0.12064	0.02244	-0.12162	0.02266
		1.5	1	-0.12763	0.02218	-0.12801	0.02227	-0.12893	0.02252
		0.5	1	-0.11133	0.01629	-0.11171	0.01634	-0.11369	0.01666
		0.5	1.5	-0.07183	0.00793	-0.07208	0.00795	-0.07370	0.00810
	30	1.5	0.5	-0.06974	0.00944	-0.07164	0.00965	-0.07400	0.01000
		1.5	1	-0.04503	0.00682	-0.04657	0.00695	-0.04938	0.00727
		0.5	1	-0.02137	0.00487	-0.02293	0.00489	-0.02894	0.00519
		0.5	1.5	0.01588	0.00438	0.02458	0.00445	0.01888	0.00491

Table 4 The bias and MSE for Bayes estimator for R when α is positive

α	n	θ_1	θ_2	SEL		LL		EL	
				Bias	MSE	Bias	MSE	Bias	MSE
0.25	10	1.5	0.5	-0.12402	0.06317	-0.24052	0.06328	-0.24102	0.06349
		1.5	1	-0.12638	0.05557	-0.22661	0.05569	-0.22735	0.05606
		0.5	1	-0.15479	0.02618	-0.15491	0.02621	-0.15577	0.02648
		0.5	1.5	-0.11157	0.01370	-0.11164	0.01371	-0.11237	0.01386
	20	1.5	0.5	-0.11745	0.03911	-0.17513	0.03928	-0.17607	0.03958
		1.5	1	-0.11381	0.02387	-0.13866	0.02402	-0.14001	0.02442
		0.5	1	-0.09250	0.01136	-0.09284	0.01142	-0.09475	0.01175
		0.5	1.5	-0.05568	0.00522	-0.05596	0.00525	-0.05805	0.00550
	30	1.5	0.5	-0.07928	0.01106	-0.08114	0.01129	-0.08327	0.01161
		1.5	1	-0.05381	0.00736	-0.05554	0.00754	-0.05873	0.00798
		0.5	1	-0.00946	0.00314	-0.01086	0.00344	-0.01636	0.00328
		0.5	1.5	0.01934	0.00497	0.02800	0.00484	0.02180	0.00446
0.5	10	1.5	0.5	-0.13277	0.08094	-0.27720	0.08108	-0.27767	0.08134
		1.5	1	-0.14227	0.05641	-0.22680	0.05648	-0.22730	0.05671
		0.5	1	-0.14022	0.02193	-0.14042	0.02198	-0.14187	0.02240
		0.5	1.5	-0.10414	0.01198	-0.10418	0.01199	-0.10466	0.01210
	20	1.5	0.5	-0.12821	0.03930	-0.18263	0.03947	-0.18345	0.03974
		1.5	1	-0.11811	0.02448	-0.13866	0.02461	-0.13982	0.02491
		0.5	1	-0.08408	0.01070	-0.08451	0.01075	-0.08677	0.01108
		0.5	1.5	-0.05543	0.00493	-0.05562	0.00495	-0.05698	0.00509
	30	1.5	0.5	-0.10178	0.02144	-0.12347	0.02181	-0.12555	0.02235
		1.5	1	-0.05899	0.00847	-0.06056	0.00865	-0.06339	0.00907
		0.5	1	0.00706	0.00490	0.00856	0.00484	-0.00812	0.00480
		0.5	1.5	0.02417	0.00450	0.03302	0.00440	0.04774	0.00406
0.75	10	1.5	0.5	-0.16981	0.09357	-0.29840	0.09376	-0.29904	0.09415
		1.5	1	-0.13334	0.05870	-0.23357	0.05880	-0.23427	0.05912
		0.5	1	-0.13742	0.02096	-0.13752	0.02099	-0.13836	0.02122
		0.5	1.5	-0.09631	0.01066	-0.09639	0.01068	-0.09723	0.01083
	20	1.5	0.5	-0.12209	0.05524	-0.22146	0.05548	-0.22230	0.05587
		1.5	1	-0.11076	0.02734	-0.15123	0.02750	-0.15239	0.02793
		0.5	1	-0.06489	0.00740	-0.06525	0.00743	-0.06697	0.00761
		0.5	1.5	-0.04065	0.00399	-0.04091	0.00401	-0.04277	0.00416
	30	1.5	0.5	-0.10729	0.02366	-0.12916	0.02408	-0.13131	0.02465
		1.5	1	-0.05219	0.00946	-0.05395	0.00961	-0.05696	0.00997
		0.5	1	0.02668	0.00566	0.02821	0.00551	0.02959	0.00521
		0.5	1.5	0.03002	0.00405	0.03883	0.00392	0.03333	0.00347

5. Illustration using Real Data

To illustrate the estimation procedures based on bivariate RSS given in Section 2 and Section 3, we consider a data set given in Hanagal (2011). The data set was reported by Huster et al. (1989) related to Diabetic Retinopathy Study. The 197 Patients with diabetic retinopathy in both eyes and visual acuity of 20/100 or better in both eyes were eligible for the study. From the data set, we took variates X and Y in which first component X is the time (in months) to blindness (measured from a suitable experimental starting time point) in the untreated eye and the second component Y is the time (in months) to blindness in the treated eye. We want to estimate the probability that time to blindness of untreated eye is less than treated eye, that is, $P(X < Y)$. Hanagal (2011) fit an exponential distribution to the data on both X and Y. Out of the 197 patients, 159 patients experienced some form of censoring. Thus, the remaining 38 patients experienced failures in both eyes have been used for the present study. We have performed the Kolmogrov-Smirnov (KS) goodness of fit test to both

X and Y observations. The KS statistic value and the corresponding p-value for X observations are 0.154 and 0.34 respectively. Again, the KS statistic value and the corresponding p-value for Y observations are 0.092 and 0.90 respectively. Since the p-values for both X and Y are greater than 0.05 we fail to reject the hypothesis that the data follows exponential distribution for both X and Y. Thus, we assume that (X, Y) follows MTBED. We selected 6 random samples each of size 6 from the 38 patients and rank the sampling units of each sample according to the X values. To obtain the data for BRSS, from the rth sample, we choose the unit corresponding to the rth order statistic of the X variate. The obtained BRSS observations are given in Table 5.

Table 5 BRSS from Diabetic Retinopathy data

r	1	2	3	4	5	6
$X_{(r)}$	5.9	5.67	13.33	33.9	14.27	42.43
$Y_{[r]}$	35.53	13.83	9.6	14.8	7.6	46.63

Table 6 MLE and Bayes estimates for $P(X < Y)$ based on BRSS from Diabetic Retinopathy data

MLE (Bootstrap CI)	Bayes estimates		
	SEL (HPD Interval)	LL	EL
0.4011 (0.2347,0.6452)	0.4140 (0.2425,0.5475)	0.4120	0.3452

We have obtained the MLE and Bayes estimates for $P(X < Y)$ based on BRSS observations and are given in Table 6. The bootstrap CI and HPD interval have also been obtained and are included in Table 6. For Bayesian estimation, we took non informative priors (by putting $a = 0$, $b = 0$, $c = 0$ and $d = 0$) to both θ_1 and θ_2. From the table one can see that all the estimates of $PX < Y)$ are less than 0.5. Since, the estimate for the probability that time to blindness of untreated eye is less than treated eye is not too high, we cannot claim that time to blindness of untreated eye is less than treated eye.

6. Conclusion

In this work, we have considered the problem of estimation of $R = P(X < Y)$ for Morgenstern type bivariate exponential distribution using ranked set sample. The maximum likelihood and Bayesian estimators have been obtained for R. For obtaining the Bayes estimates, importance sampling method has been applied. Based on the simulation study, we have concluded that among the estimators, Bayes estimators under squared error loss function perform better in terms of bias and MSE. Also, MLEs perform better than the corresponding Bayes estimators under LL and EL functions in terms of bias and MSE. The AILs of HPD intervals are smaller and the associated CPs are higher than that of bootstrap confidence intervals.

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