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A Flexible Form for the Topp-Leone Distribution: Properties and Different Methods of Estimation

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Abstract

It is important to find new simple explicit forms for distributions instead of their old implicit forms which cause some problems in mathematical properties and generating random numbers. In this paper, a simple transformation is applied to the Topp-Leone distribution giving a new distribution called the flexible Toppe-Leone distribution having more flexibility in mathematical properties and simulation studies specially generating random numbers. Some different methods of estimation are used for the flexible Toppe-Leone distribution via classical and Bayesian approaches using a progressive Type-II censoring scheme, a simulation study is performed to compare estimators' behaviors of the estimation methods.

Keywords: Maximum likelihood estimation, maximum product spacing estimation, least square estimation, Bayesian estimation, MCMC, censored Type-II.

1. Introduction

Lifetime distributions, are used to model the life of an item to study its properties so that generalizing lifetime distributions and increasing its flexibility may provide more useful information resulting in more effective conclusions and decisions. The bounded Topp-Leone (TL) distribution, presented by Topp and Leone (1955), for empirical data with a J-shaped histogram as a powered band tool and automatically calculates machine failures. Many authors have studied the Topp-Leone distribution as Nadarajah and Kotz (2003), Ghana et al. (2005), van Dorp and Kotz (2006), Kotz and Seier (2007), Nadarajah (2009) and Genç (2012).

The cumulative distribution function (CDF) and the probability density function (PDF), for the TL distribution are given as

$$\underset{TL}{F}(x)=\left[x\left(2-x\right)\right]^{\alpha};0<x<1;\alpha>0, \tag{1}$$

and

$$\underset{TL}{f}(x)=2\alpha x^{\alpha-1}\left(2-x\right)^{\alpha-1}\left(1-x\right);0<x<1;\alpha>0, \tag{2}$$

one can see Topp and Leone (1955), Nadarajah and Kotz (2003).

(1) indicates that the TP distribution has an implicit quantile function form which gives some problems in mathematical properties and simulation studies especially generating random numbers so, in Section 2, a simple transformation will be applied to solve these problems.

There are many censoring schemes, which are used in life testing experiments such as conventional (Type-I and Type-II) censoring. Type-I censoring occurs when a study is designed to end after a fixed period T, determined before starting the experiment. The experimental time is fixed, but the number of failures is a random variable. Type-II censoring occurs when exactly k failures occur. The failure time of the k items is observed. The number of failures is fixed, but the experimental time is a random variable.

2. The Flexible Topp-Leone Distribution

A flexible form for the TL distribution can be given by using the square complete transformation as follows: Since

$$F(x)=\frac{1}{B(a,b)}\int_0^x x^{a-1}(1-x)^{b-1}\,dx;0<x<1;a,b>0, \tag{3}$$

where $B(.,.)$ is the beta function, at $a=1$ gives $F(x)=1-(1-x)^b$, then, replacing X in the last equation with (1) leads to

$$F_{FTL}(x)=1-\left(1-\left[x(2-x)\right]^{\alpha}\right)^{b};0<x<1;\alpha,b>0,$$

when $b=1$ gives the CDF of the classic TL distribution.

Using the square complete transformation and adding ± 1 inside brackets gives the CDF of flexible Topp-Leone (FTL) distribution

$$F_{FTL}(x)=1-\left(1-\left[2x-x^2\right]^{\alpha}\right)^{b};0<x<1;\ \alpha,b>0,$$

hence,

$$F_{FTL}(x)=1-\left(1-\left[1-(1-x)^2\right]^{\alpha}\right)^{b};0<x<1;\alpha,b>0, \tag{4}$$

one can see that, at $F_{FTL}(0)=0$ and $F_{FTL}(1)=1$.

Some shapes of the cumulative functions for the FTL distribution are illustrated in Figure 1. Differentiating (4) concerning x gives

$$f_{FTL}(x)=2\alpha b(1-x)\left[1-(1-x)^2\right]^{\alpha-1}\left(1-\left[1-(1-x)^2\right]^{\alpha}\right)^{b-1};0<x<1;\alpha,b>0. \tag{5}$$

Some shapes of the density functions for the FTL distribution are illustrated in Figure 2.

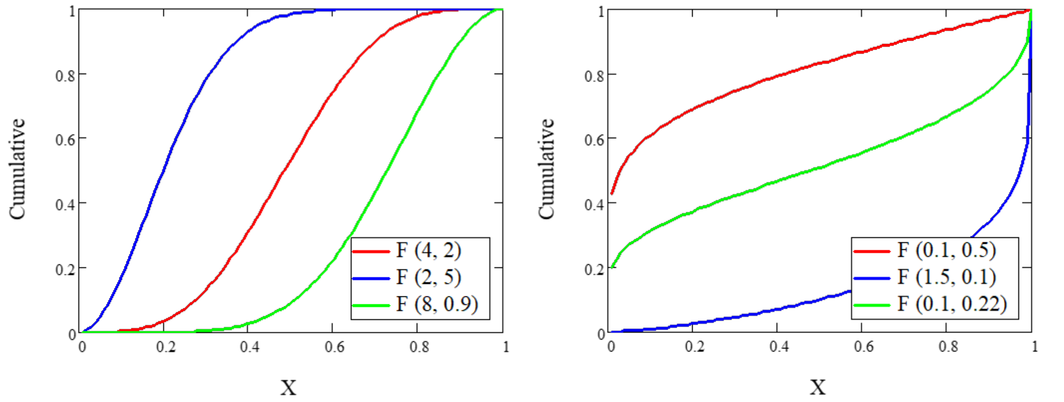

Figure 1 The FTL cumulative functions

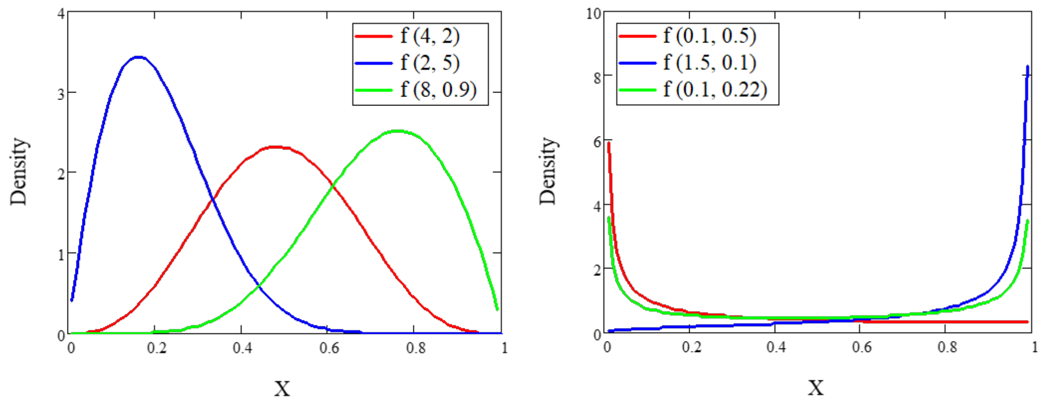

Figure 2 The FTL probability density function

One can see, in Figure 2, that the density function is unimodal and the density is suitable for lifetime, especially the bathtub lifetime curve.

2.1. Expansions for CDF and PDF

In this section, expansions for the CDF and PDF of the FTL distribution will be given.

2.1.1 An Expansion for the PDF

Using binomial expansion,

$$\left(1-\left[1-(1-x)^2\right]^{\alpha}\right)^{b-1}=\sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}\left[1-(1-x)^2\right]^{\alpha i};0<x<1;\alpha,b>0, \tag{6}$$

then, using (6) into (5) gives

$$f(x)=2\alpha b(1-x)\left[1-(1-x)^2\right]^{\alpha-1}\sum_{i=0}^{\infty}(-1)^i\binom{b-1}{i}\left[1-(1-x)^2\right]^{\alpha i};0<x<1;\alpha,b>0,$$

hence,

$$f(x) = 2\alpha b \sum_{i,j=0}^{\infty} a_{ij} (1-x)^{2j}, \tag{7}$$

where $a_{ij} = (-1)^{i+j} \binom{b-1}{i} \binom{\alpha + \alpha i - 1}{j}$.

2.1.2 An Expansion for the CDF

$$F(x) = 1 - \left(1 - \left[1 - (1-x)^2\right]^{\alpha}\right)^{b},$$

hence,

$$F(x) = 1 - \sum_{k=0}^{\infty} (-1)^k \binom{b}{k} \left[1 - (1-x)^2\right]^{\alpha k},$$

moreover,

$$F(x) = 1 - \sum_{k=0}^{\infty} (-1)^k \sum_{l=0}^{\infty} (-1)^l \binom{b}{k} \binom{\alpha k}{l} (1-x)^{2l},$$

hence,

$$F(x) = 1 - \sum_{k,l=0}^{\infty} (-1)^{k+l} \binom{b}{k} \binom{\alpha k}{l} (1-x)^{2l},$$

then,

$$F(x) = 1 - \sum_{l=0}^{\infty} D_l (1-x)^{2l}, \tag{8}$$

where $D_l = \sum_{k=0}^{\infty} (-1)^{k+l} \binom{b}{k} \binom{\alpha k}{l}$.

3. Some Statistical Properties of the FTL Distribution

In this section, some statistical properties of the FTL distribution will be illustrated as follows

3.1. Quantile and median of the FTL distribution

We derive the quantile function of the FTL distribution as follows:

The quantile function of the random variable X having the CDF of the FTL distribution is given by the nonlinear equation

$$x_u = [F(x)]^{-1} : 0 < x < 1; \alpha, b > 0,$$

$$x_u = 1 - \left\{1 - \left[1 - (1-u)^{\frac{1}{b}}\right]^{\frac{1}{\alpha}}\right\}^{\frac{1}{2}} ; 0 < u < 1; \alpha, b > 0, \tag{9}$$

in particular, the median (M) can be derived from (9) by setting $q = 0.5$ then the M is

$$M = 1-\left\{1-\left[1-\left(1-0.5\right)^{\frac{1}{b}}\right]^{\frac{1}{\alpha}}\right\}^{\frac{1}{2}}; \alpha, b > 0.$$

3.2 The r^{th} moment of the FTL distribution

Generally, the r^{th} moment of a continuous random variable X, is given by

$$E\left(X^r\right) = \int_{-\infty}^{\infty} x^r f(x)\,dx; -\infty < x < \infty, \quad (10)$$

substituting (7) into (10) yields

$$E\left(X^r\right) = 2\alpha b \sum_{i,j=0}^{\infty} a_{ij}\beta\left(r+1, 2j+1\right)dx, \quad (11)$$

where,

$$a_{ij} = \left(-1\right)^{i+j}\binom{b-1}{i}\binom{\alpha+\alpha i-1}{j}.$$

3.3 The moment generating function of the FTL distribution

The moment-generating function of the random variable X which has the PDF of the FTL distribution is given by

$$M_X\left(t\right) = E\left(e^{tX}\right) = \int_0^1 e^{tx} f\left(x\right)dx,$$

substituting (7) into the last equation yields

$$M_x\left(t\right) = 2\alpha b\int_0^1 e^{tx}\sum_{i,j=0}^{\infty} a_j\left(1-x\right)^{2j}\,dx,$$

then,

$$M_X\left(t\right) = 2\alpha b\sum_{i,j=0}^{\infty} a_{ij}\int_0^1 e^{tx}\left(1-x\right)^{2j}\,dx,$$

where $a_{ij} = \left(-1\right)^{i+j}\binom{b-1}{i}\binom{\alpha+\alpha i-1}{j}.$

3.4. The mode of the FTL distribution

The natural logarithm of the (5) is

$$\log f\left(x\right) = 2\log\alpha b + \log\left(1-x\right) + \left(\alpha-1\right)\log\left[1-\left(1-x\right)^2\right] + \left(b-1\right)\log\left(1-\left[1-\left(1-x\right)^2\right]^{\alpha}\right),$$

differentiating the last equation concerning x

$$\frac{d}{dx}\log f\left(x\right) = \frac{-1}{1-x} + \frac{\left(\alpha-1\right)\left[-2\left(1-x\right)\right]\left(-1\right)}{1-\left(1-x\right)^2} + \frac{\left(b-1\right)\left\{-\alpha\left[1-\left(1-x\right)^2\right]\right\}\left[-2\left(1-x\right)\left(-1\right)\right]}{1-\left[1-\left(1-x\right)^2\right]^{\alpha}},$$

Then,

$$\frac{d}{dx}\log f\left(x\right)=-\frac{1}{1-x}+\frac{2\left(\alpha-1\right)\left(1-x\right)}{1-\left(1-x\right)^{2}}-\frac{2\alpha\left[1-\left(1-x\right)^{2}\right]\left(1-x\right)\left(b-1\right)}{1-\left[1-\left(1-x\right)^{2}\right]^{\alpha}}, \tag{12}$$

where the second derivative is

$$\frac{d^{2}}{dx^{2}}\log f(x)=-\frac{1}{\left(1-x\right)^{2}}+\frac{2\left(\alpha-1\right)\left\{\left(3\left(1-x\right)^{2}\right)-1\right\}}{\left(1-\left(1-x\right)^{2}\right)^{2}}$$
$$-\left[\frac{4\alpha^{2}\left(1-x\right)^{2}\left(b-1\right)\left[1-\left(1-x\right)^{2}\right]\left[1-\left(1-x\right)^{2}\right]^{\alpha-1}}{\left\{1-\left[1-\left(1-x\right)^{2}\right]^{\alpha}\right\}^{2}}\right.$$
$$\left.-\frac{8\alpha^{2}\left(1-x\right)^{2}\left(b-1\right)^{2}\left[1-\left(1-x\right)^{2}\right]\left[1-\left[1-\left(1-x\right)^{2}\right]^{\alpha}\right]}{\left\{1-\left[1-\left(1-x\right)^{2}\right]^{\alpha}\right\}^{2}}\right], \tag{13}$$

Equation (12) is nonlinear and it does not have an analytic solution with respect to x, therefore it has to be solved numerically. If x_0 is a root for (12) then it must be verify that $f''\left[\log(x_0)\right]<0$.

3.5. The survival and Hazard function of the FTL distribution

Generally, the survival function of a random variable X, (Meeker and Escobar, 1998), can be given by

$$S\left(x\right)=1-F(x),$$

substituting (4) into the last equation gives

$$S\left(x\right)=\left(1-\left[1-\left(1-x\right)^{2}\right]^{\alpha}\right)^{b};0<x<1;\alpha,b>0, \tag{14}$$

simply, the hazard rate function, Meeker and Escobar (1998), can be given by

$$h\left(x\right)=\frac{f(x)}{S\left(x\right)},$$

substituting (5) and (14) into the last equation yields

$$h\left(x\right)=\frac{2\alpha b\left(1-x\right)\left[1-\left(1-x\right)^{2}\right]^{\alpha-1}}{\left(1-\left[1-\left(1-x\right)^{2}\right]^{\alpha}\right)}. \tag{15}$$

Some shapes of the hazard functions for the identified FTL distribution are illustrated in Figure 3. One can see, in Figure 3, two types of hazard functions curves of the FTL distribution are described as follows: A decreasing then stability then increasing (bathtub) hazard curve and a stability then increasing hazard curve.

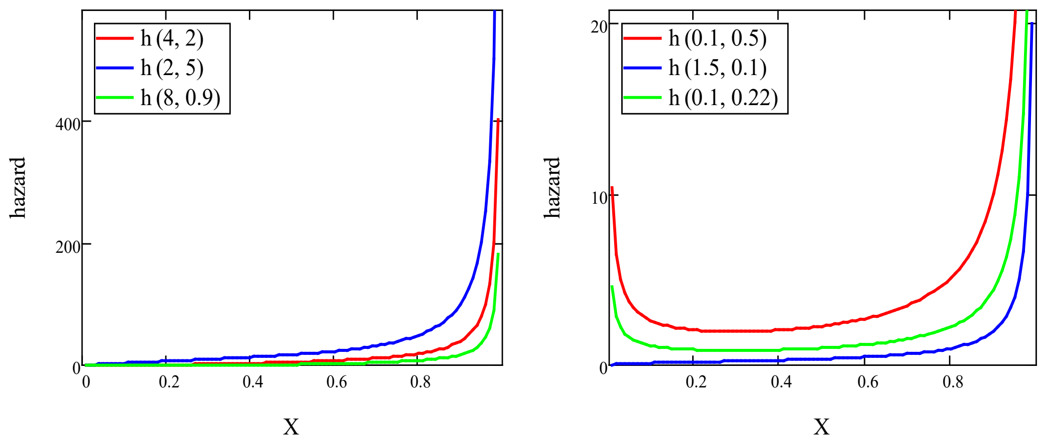

Figure 3 The FTL distribution hazard functions

3.6. The order statistic of the FTL distribution

A simple random sample $X_1, X_1, ..., X_v$ given from the FTL distribution where X's are identically independent distributed (iid) random variables, has the density $f_{u:v}(x_{u:v})$ of the u^{th} order statistic, (Arnold et al. 1992), for $u = 1, 2, ..., v$ as follows

$$f_{u:v}(x_{u:v}) = \frac{2\alpha b}{B(u, v-u+1)} \sum_{g=0}^{\infty} w\left(1-x_u\right)^{2g+1}; -\infty < x_u < \infty, \tag{16}$$

where $B(.,.)$ is the beta function.

Proof:

$$f_{u:v}(x_{u:v}) = \frac{1}{B(u, v-u+1)} f(x_u) F(x_u)^{u-1} \left\{ 1 - F(x_u) \right\}^{v-u}; -\infty < x_u < \infty, \tag{17}$$

then,

$$f_{u:v}(x_{u:v}) = \frac{1}{B(u, v-u+1)} 2\alpha b\left(1-x_u\right)\left[1-\left(1-x_u\right)^2\right]^{\alpha-1}\left(1-\left[1-\left(1-x_u\right)^2\right]^{\alpha}\right)^{b-1}$$

$$\times\left[\left(1-\left[1-\left(1-x_u\right)^2\right]^{\alpha}\right)^{b}\right]^{u-1}\left\{\left(1-\left[1-\left(1-x_u\right)^2\right]^{\alpha}\right)^{b}\right\}^{v-u}; 0 < x_u < 1,$$

hence,

$$f_{u:v}(x_{u:v}) = \frac{1}{B(u, v-u+1)} 2\alpha b\left(1-x_u\right)\left[1-\left(1-x_u\right)^2\right]^{\alpha-1}\left(1-\left[1-\left(1-x_u\right)^2\right]^{\alpha}\right)^{(b-1)+b(v-u)}$$

$$\times\left[\left(1-\left[1-\left(1-x_u\right)^2\right]^{\alpha}\right)^{b}\right]^{u-1}; \ 0 < x < 1,$$

then,

$$f_{u:v}(x_{u:v})=\frac{1}{B(u,v-u+1)}2\alpha b\left(1-x_u\right)\left[1-\left(1-x_u\right)^2\right]^{\alpha-1}\left(1-\left[1-\left(1-x_u\right)^2\right]^{\alpha}\right)^{(b-1)+b(v-u)}$$
$$\times\sum_{m=0}^{u-1}\left(-1\right)^m\binom{u-1}{m}\left(1-\left[1-\left(1-x_u\right)^2\right]^{\alpha}\right)^{bm}\ ;\ 0<x<1,$$

moreover,

$$f_{u:v}(x_{u:v})=\frac{1}{B(u,v-u+1)}2\alpha b\left(1-x_u\right)\left[1-\left(1-x_u\right)^2\right]^{\alpha-1}$$
$$\times\sum_{m=0}^{u-1}\left(-1\right)^m\binom{u-1}{m}\left(1-\left[1-\left(1-x_u\right)^2\right]^{\alpha}\right)^{(b-1)+b(v-u)+bm}\ ;\ 0<x<1,$$

furthermore,

$$f_{u:v}(x_{u:v})=\frac{1}{B(u,v-u+1)}2\alpha b\left(1-x_u\right)\left[1-\left(1-x_u\right)^2\right]^{\alpha-1}\sum_{m=0}^{u-1}\left(-1\right)^m\binom{u-1}{m}$$
$$\times\sum_{d=0}^{\infty}\left(-1\right)^d\binom{(b-1)+b(v-u)+bm}{d}\left[1-\left(1-x_u\right)^2\right]^{\alpha d}\ ;\ 0<x<1,$$

then,

$$f_{u:v}(x_{u:v})=\frac{1}{B(u,v-u+1)}2\alpha b\left(1-x_u\right)\sum_{m=0}^{u-1}\sum_{d=0}^{\infty}\left(-1\right)^{m+d}\binom{u-1}{m}$$
$$\times\binom{(b-1)+b(v-u)+bm}{d}\left[1-\left(1-x_u\right)^2\right]^{\alpha d+(\alpha-1)}\ ;\ 0<x<1,$$

since,

$$f_{u:v}(x_{u:v})=\frac{1}{B(u,v-u+1)}2\alpha b\left(1-x_u\right)\sum_{m=0}^{u-1}\sum_{d=0}^{\infty}\left(-1\right)^{m+d}\binom{u-1}{m}$$
$$\times\binom{(b-1)+b(v-u)+bm}{d}\sum_{g=0}^{\infty}\left(-1\right)^g\binom{\alpha d+(\alpha-1)}{g}\left(1-x_u\right)^{2g}\ ;\ 0<x<1,$$

then,

$$f_{u:v}(x_{u:v})=\frac{1}{B(u,v-u+1)}2\alpha b\sum_{g,d=0}^{\infty}\sum_{m=0}^{u-1}\left(-1\right)^{m+d+g}\binom{u-1}{m}$$
$$\times\binom{(b-1)+b(v-u)+bm}{d}\binom{\alpha d+(\alpha-1)}{g}\left(1-x_u\right)^{2g+1}\ ;\ 0<x<1,$$

hence,

$$f_{u:v}(x_{u:v})=\frac{2\alpha b}{B(u,v-u+1)}\sum_{g=0}^{\infty}w\left(1-x_u\right)^{2g+1}\ ;\ 0<x<1, \tag{18}$$

where $w_{mdg}=\sum_{d=0}^{\infty}\sum_{m=0}^{u-1}\left(-1\right)^{d+g+m}\binom{u-1}{m}\binom{(b-1)+b(v-u)+bm}{d}\binom{\alpha d+(\alpha-1)}{g}.$

4. Non-Bayesian Estimation

In this section, the maximum likelihood estimation, the least square and the maximum product spacing methods of the FTL distribution will be applied as follows:

4.1. Maximum likelihood estimation

Let $X_1, X_2, ..., X_n$ be iid random variables from the FTL (α, b) distribution then the likelihood function for the parameters α and b, (Garthwaite et al. 2002), is given by

$$L(\alpha, b; x) = \prod_{i=1}^{n} f(\alpha, b; x),$$

then,

$$L(\alpha, b; x) = (2\alpha b)^n \prod_{i=1}^{n}(1-x_i)\prod_{i=1}^{n}\left[1-(1-x_i)^2\right]^{\alpha-1}\prod_{i=1}^{n}\left\{1-\left[1-(1-x_i)^2\right]^{\alpha}\right\}^{b-1},$$

the log likelihood function can be written as

$$\ell(\alpha, b; x) = n\log(2\alpha b) + \sum_{i=1}^{n}\log(1-x_i) + (\alpha-1)\sum_{i=1}^{n}\log\left[1-(1-x_i)^2\right]$$
$$+(b-1)\sum_{i=1}^{n}\log\left\{1-\left[1-(1-x_i)^2\right]^{\alpha}\right\},$$

the score functions for the parameters α and b are given by

$$\frac{\partial \ell}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\log\left\{1-\left[1-(1-x_i)^2\right]^{\alpha}\right\}, \tag{19}$$

and

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(1-(1-x_i)^2\right) - (b-1)\sum_{i=1}^{n}\frac{\left(1-(1-x_i)^2\right)^{\alpha}\log\left(1-(1-x_i)^2\right)}{\left(1-\left(1-(1-x_i)^2\right)^{\alpha}\right)}. \tag{20}$$

The unknown parameters of the maximum likelihood estimators (MLEs) are obtained by solving the nonlinear (19) and (20) numerically, using a suitable iterative technique such as the Newton–Raphson algorithm to obtain the estimate.

4.2. Least square method

Let $X_1, X_2, ..., X_n$ be iid random variables from the FTL $(\alpha, b; x)$ distribution then the summation of square for the error term, (Singh et al. 2014; Dey et al. 2017), is given by

$$\sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n}\left\{F(x_i) - F_{Em}\right\}^2, \tag{21}$$

where F_{EM} is the empirical CDF of the FTL (α, b) distribution based on the mean rank, since,

$$F_{EM} = \frac{i}{n+1}, \tag{22}$$

then, substituting (4) and (22) into (21) gives

$$\sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n}\left\{\left(1-\left(1-\left[1-(1-x_i)^2\right]^{\alpha}\right)^{b}\right)-\frac{i}{n+1}\right\}^2,$$

the score functions for the parameters α and b are given by

$$\frac{\partial\left(\sum_{i=1}^{n} e_i^2\right)}{\partial\alpha} = 2b\sum_{i=1}^{n}\left\{1-\left(1-\left[1-(1-x_i)^2\right]^{\alpha}\right)^{b}-\frac{i}{n+1}\right\}\left(1-\left[1-(1-x_i)^2\right]^{\alpha}\right)^{b-1}$$
$$\times\left[1-(1-x_i)^2\right]^{\alpha}\log\left(1-(1-x_i)^2\right), \tag{23}$$

and

$$\frac{\partial\left(\sum_{i=1}^{n} e_i^2\right)}{\partial b} = -2\sum_{i=1}^{n}\left\{1-\left(1-\left[1-(1-x_i)^2\right]^{\alpha}\right)^{b}-\frac{i}{n+1}\right\}\left(1-\left[1-(1-x_i)^2\right]^{\alpha}\right)^{b}$$
$$\times\log\left(1-\left[1-(1-x_i)^2\right]^{\alpha}\right). \tag{24}$$

The unknown parameters of the least square estimators (LSEs) are obtained by solving the nonlinear equations numerically, using a suitable iterative technique.

4.3. Maximum product spacing method

Let $X_1, X_2, ..., X_n$ be iid random variables from the FTL (α, b) distribution then the geometric mean (GM) for parameters α, b, (Singh et al. 2014; Bhatti et al. 2021), is given by

$$GM = \sqrt[n+1]{\prod_{i=1}^{n+1}\left[F(x_i)-F(x_{i-1})\right]};\ i=1,2,...,n+1, \tag{25}$$

where $F(\alpha, b; x_0)=0$ and $F(\alpha, b; x_{n+1})=1$, then, taking the natural logarithm of (25) yields

$$\log(GM)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log\left\{\left[F(x_i)-F(x_{i-1})\right]\right\}, \tag{26}$$

then, the last equation can be rewritten as follows

$$\log(GM)=\frac{1}{n+1}\left\{\sum_{i=1}^{n+1}\log\left\{\left[F(x_i)-F(x_{i-1})\right]\right\}+\log\left[F(x_1)\right]+\log\left\{\left[1-F(x_n)\right]\right\}\right\},$$

substituting (4) into (26) leads to

$$\log(GM)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log\left\{\left(\left(1-\left(1-\left[1-(1-x_i)^2\right]^{\alpha}\right)^{b}\right)-\left(1-\left(1-\left[1-(1-x_{i-1})^2\right]^{\alpha}\right)^{b}\right)\right)\right\},$$

the score functions for the parameters α and b are given by

$$\frac{\partial\log(GM)}{\partial\alpha}=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{b\left(1-\left[1-(1-x_i)^2\right]^{\alpha}\right)^{b-1}\left[1-(1-x_i)^2\right]^{\alpha}\log\left(1-(1-x_i)^2\right)}{\left(\left(1-\left(1-\left[1-(1-x_i)^2\right]^{\alpha}\right)^{b}\right)-\left(1-\left(1-\left[1-(1-x_{i-1})^2\right]^{\alpha}\right)^{b}\right)\right)}$$

$$-\frac{b\left(1-\left[1-(1-x_{i-1})^2\right]^{\alpha}\right)^{b-1}\left[1-(1-x_{i-1})^2\right]^{\alpha}\log\left(1-(1-x_{i-1})^2\right)}{\left(1-\left(1-\left[1-(1-x_i)^2\right]^{\alpha}\right)^{b}\right)-\left(1-\left(1-\left[1-(1-x_{i-1})^2\right]^{\alpha}\right)^{b}\right)}, \tag{27}$$

and

$$\frac{\partial\log(GM)}{\partial b}=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{-\left(1-\left[1-(1-x_i)^2\right]^{\alpha}\right)^{b}\log\left(1-\left[1-(1-x_i)^2\right]^{\alpha}\right)}{\left(\left(1-\left(1-\left[1-(1-x_i)^2\right]^{\alpha}\right)^{b}\right)-\left(1-\left(1-\left[1-(1-x_{i-1})^2\right]^{\alpha}\right)^{b}\right)\right)}$$

$$+\frac{\left(1-\left[1-(1-x_{i-1})^2\right]^{\alpha}\right)^{b}\log\left(1-\left[1-(1-x_{i-1})^2\right]^{\alpha}\right)}{\left(\left(1-\left(1-\left[1-(1-x_i)^2\right]^{\alpha}\right)^{b}\right)-\left(1-\left(1-\left[1-(1-x_{i-1})^2\right]^{\alpha}\right)^{b}\right)\right)}. \tag{28}$$

The unknown parameters of the maximum product space estimators (MPSEs) are obtained by solving the nonlinear (27) and (28) numerically using a suitable iterative technique as will be seen in Section 7.

5. Bayesian Estimation

In this section, the Bayesian estimation for the FTL (α,b) distribution parameters is considered under the assumption that the random variables α,b prior distributions are as follows

$$\pi(\alpha)=\frac{1}{\alpha};0<\alpha<a, \tag{29}$$

and

$$\pi(b)=\frac{1}{b};0<b<c. \tag{30}$$

The joint prior density of α and b can be written as

$$\pi(\alpha,b;x)=\frac{L(\alpha,b;x)\pi(\alpha)\pi(b)}{\int_0^c\int_0^a L(\alpha,b;x)\pi(\alpha)\pi(b)\,d\alpha\,db};0<\alpha<a;0<b<c;x>0,$$

substituting (29) and (30) into the last equation gives

$$\pi(\alpha,b;x)=\frac{\left(2\alpha b\right)^{n}\prod_{i=1}^{n}(1-x)\prod_{i=1}^{n}\left[1-(1-x)^{2}\right]^{\alpha-1}\prod_{i=1}^{n}\left\{1-\left[1-(1-x)^{2}\right]^{\alpha}\right\}^{b-1}\frac{1}{\alpha}\frac{1}{b},}{\int_{0}^{c}\int_{0}^{a}\left(2\alpha b\right)^{n}\prod_{i=1}^{n}(1-x)\prod_{i=1}^{n}\left[1-(1-x)^{2}\right]^{\alpha-1}\prod_{i=1}^{n}\left\{1-\left[1-(1-x)^{2}\right]^{\alpha}\right\}^{b-1}\frac{1}{\alpha}\frac{1}{b},}d\alpha\, db.$$

The marginal posterior distribution of α and b can be given, respectively, by

$$\pi(\alpha;x)=\int_{0}^{c}\pi(\alpha,b;x)\,db;0<b<c;x>0, \tag{31}$$

and

$$\pi(b;x)=\int_{0}^{a}\pi(\alpha,b;x)\,d\alpha;0<\alpha<a;x>0. \tag{32}$$

Estimating α and b can be obtained using the squared error (SE) loss function or linear exponential (LINEX) loss function.

5.1. The SE Loss Function

In this subsection, estimation of the marginal posterior distributions will be performed using the SE loss function, or the quadratic loss function, which is a symmetric loss function for (31) and (32), (Guure et al. 2012), as follows

$$E_{SE}(\alpha;x)=\int_{0}^{a}\alpha\,\pi(\alpha;x)\,d\alpha;0<\alpha<a;x>0, \tag{33}$$

and

$$E_{SE}(b;x)=\int_{0}^{c}b\;\pi(b;x)\,db;0<b<c;x>0. \tag{34}$$

The unknown parameters of the Bayesian technique via integrations (33) and (34) are not possible to be obtained numerically so the Markov Chain Monte Carlo (MCMC) method will be used.

5.2. The LINEX loss function

In this subsection, estimation of the marginal posterior distributions will be performed using the LINEX loss function which is an asymmetric loss function for (31) and (32), (Guure et al. 2012), as follows

$$E_{LINEX}(\alpha;x)=-\frac{1}{h}\ln\left[\int_{0}^{a}e^{-h\alpha}\;\pi(\alpha;x)\,d\alpha\right];0<\alpha<a;x>0, \tag{35}$$

and

$$E_{LINEX}(b;x)=-\frac{1}{h}\ln\left[\int_{0}^{c}e^{-hb}\;\pi(b;x)\,db\right];0<b<c;x>0. \tag{36}$$

On the other hand, h is the shape parameter for the LINEX function where the sign of h reflects the direction of asymmetry and its magnitude reflects the degree of asymmetry, when h closes to zero the LINEX loss is approximately SE loss.

The unknown parameters of the Bayesian technique via integrations in (35) and (36) are not possible to be obtained numerically so the MCMC method will be used.

5.3. The MCMC method

In this subsection, the MCMC method will be discussed using the Gibbs sampling procedure. The conditional posterior densities of the parameters α and b are given respectively by:

$$p(\alpha, j) = \frac{1}{\alpha^2}\left[\prod_{i=1}^{r}\left\{\alpha\left[1-\left(1-x_{i,j}\right)^2\right]^{\alpha-1}\left\{1-\left[1-\left(1-x_{i,j}\right)^2\right]^{\alpha}\right\}^{b-1}\right\}\right]$$
$$\times\left[\left\{1-\left[1-\left(1-x_{r,j}\right)^2\right]^{\alpha}\right\}^{(n-r)}\right]^{b}, \tag{37}$$

and

$$q(b, j) = \frac{1}{b^2}\left[\left[\prod_{i=1}^{r}\left\{b.\left\{1-\left[1-\left(1-x_{i,j}\right)^2\right]^{\alpha}\right\}^{(b-1)}\right\}\right]\left[\left\{1-\left[1-\left(1-x_{r,j}\right)^2\right]^{\alpha}\right\}^{(n-r)}\right]^{b}\right]. \tag{38}$$

The Bayes estimates of the parameters α and b under squared error loss function respectively are

$$E_{SE}\left(\pi_{TI\,II(MLE)}(\alpha|b,x)\right) = \frac{1}{N}\sum_{j=1}^{N}\pi_j(\alpha|b,x), \tag{39}$$

and

$$E_{SE}\left(\pi_{TI\,II(MLE)}(b|\alpha,x)\right) = \frac{1}{N}\sum_{j=1}^{N}\pi_j(b|\alpha,x), \tag{40}$$

where N is the number of iteration in the MCMC process, the Bayes estimates of the parameter α and b under LINEX loss function respectively are

$$E_{LINEX}\left(\pi_{TI\,II(MLE)}(\alpha|b,x)\right) = -\frac{1}{h}\ln\left(\frac{1}{N}\sum_{j=1}^{N}e^{-h\pi_j(\alpha|b,x)}\right), \tag{41}$$

and

$$E_{LINEX}\left(\pi_{TI\,II(MLE)}(b|\alpha,x)\right) = -\frac{1}{h}\ln\left(\frac{1}{N}\sum_{j=1}^{N}e^{-h\pi_j(b|\alpha,x)}\right), \tag{42}$$

where N is the number of iteration in the MCMC process.

An important sub-class of MCMC methods is Gibbs sampling and more general Metropolis within Gibbs samplers. For more information about the Metropolis-Hastings algorithm see Metropolis et al. (1953), Amin (2017) and Nassar et al. (2018).

6. Estimation based on Censored Type- II Samples

In this section, estimating parameters of the FTL distribution will be used based on censored Type-II samples.

6.1. Maximum likelihood estimation

Let $X_{(1)}, X_{(2)}, \ldots, X_{(r)}$ be the ordered observed failures in a random sample from n components from the FTL(α, b) distribution after a predetermined and fixed number of failures r then the censored Type- II likelihood function for parameters α and $b,$ is given by

$$L_{CII}\left(\alpha,b;x\right)=\frac{n!}{\left(n-r\right)!}\left\{\left(2\alpha b\right)^{r}\prod_{i=1}^{r}\left(1-x_i\right)\prod_{i=1}^{r}\left[1-\left(1-x_i\right)^2\right]^{\alpha-1}\prod_{i=1}^{r}\left\{1-\left[1-\left(1-x_i\right)^2\right]^{\alpha}\right\}^{b-1}\right\}$$
$$\times\left\{1-\left[1-\left(1-x_r\right)^2\right]^{\alpha}\right\}^{b\left(n-r\right)},$$

The log likelihood function can be written as

$$\ell_{CII}\left(\alpha,b;x\right)=\log\frac{n!}{\left(n-r\right)!}+r\log\left(2\alpha b\right)+\log\sum_{i=1}^{r}\left(1-x_i\right)+\alpha\log\sum_{i=1}^{r}\left(1-\left(1-x_i\right)^2\right)+$$
$$\left(b-1\right)\log\sum_{i=1}^{r}\left(1-\left[1-\left(1-x_i\right)^2\right]^{\alpha}\right)+b\left(n-r\right)\log\left(1-\left[1-\left(1-x_r\right)^2\right]^{\alpha}\right),$$

The score functions for the parameters α and b are given by

$$\frac{\partial\ell_{CII}}{\partial\alpha}=\frac{r}{\alpha}+\log\sum_{i=1}^{r}\left(1-x_i\right)-\left(b-1\right)\sum_{i=1}^{r}\frac{\left[1-\left(1-x_i\right)^2\right]^{\alpha}\log\left[1-\left(1-x_i\right)^2\right]}{1-\left[1-\left(1-x_i\right)^2\right]^{\alpha}}$$
$$-b\left(n-r\right)\frac{\left[1-\left(1-x_i\right)^2\right]^{\alpha}\log\left[1-\left(1-x_i\right)^2\right]}{1-\left[1-\left(1-x_i\right)^2\right]^{\alpha}}, \tag{43}$$

and

$$\frac{\partial\ell_{CII}}{\partial b}=\frac{r}{b}+\log\sum_{i=1}^{r}\left(1-\left[1-\left(1-x_i\right)^2\right]^{\alpha}\right)+\left(n-r\right)\log\left(1-\left[1-\left(1-x_r\right)^2\right]^{\alpha}\right). \tag{44}$$

The MLES of the censored Type- II samples (CII-MLEs) are obtained by solving the nonlinear equations numerically, using a suitable iterative technique.

6.2. Bayesian estimation

Let $X_{(1)}, X_{(2)}, \ldots, X_{(r)}$ be the ordered observed failures in a random sample from n components from the FTL(α, b) distribution after a predetermined and fixed number of failures r then the censored Type-II likelihood function for parameters α and $b,$ is given by

$$L_{CII}\left(\alpha,b;x\right)=\frac{n!}{\left(n-r\right)!}\left\{\left(2\alpha b\right)^{r}\prod_{i=1}^{r}\left(1-x_i\right)\prod_{i=1}^{r}\left[1-\left(1-x_i\right)^2\right]^{\alpha-1}\prod_{i=1}^{r}\left\{1-\left[1-\left(1-x_i\right)^2\right]^{\alpha}\right\}^{b-1}\right\}$$
$$\times\left\{1-\left[1-\left(1-x_r\right)^2\right]^{\alpha}\right\}^{b\left(n-r\right)},$$

non-informative prior distributions for parameters α and b will be used, respectively, from (43) and (44), then, the joint posterior distribution is

$$\pi_{CII}(\alpha,b;x)=\frac{L_{CII}\left(\alpha,b;x\right)\pi(\alpha)\pi(b)}{\int_0^c\int_0^a L_{CII}\left(\alpha,b;x\right)\pi(\alpha)\pi(b)\,d\alpha\,db};0<\alpha<a;0<b<c;x>0,$$

last equation needs a numerical integration technique to be solved using a mathematical package. The marginal posterior distribution of α and b can be given respectively by,

$$\pi_{CII}(\alpha;x)=\int_0^c \pi_{CII}(\alpha,b;x)\,db;0<b<c;x>0, \tag{45}$$

and

$$\pi_{CII}(b;x)=\int_0^a \pi_{CII}(\alpha,b;x)\,d\alpha;0<\alpha<a;x>0. \tag{46}$$

Estimation of the marginal posterior distributions will be performed using the SE loss function, or the quadratic loss function, which is a symmetric loss function as follows

$$E_{CII}(\alpha;x)=\int_0^a \alpha\,\pi_{CII}(\alpha;x)\,d\alpha;0<\alpha<a;x>0, \tag{47}$$

and

$$E_{CII}(b;x)=\int_0^c b\;\pi_{CII}(b;x)\,db;0<b<c;x>0. \tag{48}$$

The unknown parameters of the Bayesian estimators are obtained by solving integrations in (47) and (48), numerically, using a suitable iterative technique.

6.3. The MCMC method

In this subsection, the MCMC method will be discussed using the Gibbs sampling procedure. The conditional posterior densities of the parameters α and b are given respectively by

$$p(\alpha,j)=\frac{1}{\alpha^2}\left[\left[\prod_{i=1}^{n}\left\{\alpha\left[1-\left(1-x_{i,j}\right)^2\right]^{\alpha-1}\left\{1-\left[1-\left(1-x_{i,j}\right)^2\right]^{\alpha}\right\}^{b-1}\right\}\right]\right], \tag{49}$$

and

$$q(b,j)=\frac{1}{b^2}\left[\left[\prod_{i=1}^{n}\left\{b.\left\{1-\left[1-\left(1-x_{i,j}\right)^2\right]^{\alpha}\right\}^{(b-1)}\right\}\right]\right]. \tag{50}$$

The Bayes estimates of the parameter α and b under squared error loss function, respectively are

$$E_{SE}\left(\pi_{FTL(MLE)}(\alpha|b,x)\right)=\frac{1}{N}\sum_{j=1}^{N}\pi_j(\alpha|b,x), \tag{51}$$

and

$$E_{SE}\left(\pi_{FTL(MLE)}(b|\alpha,x)\right)=\frac{1}{N}\sum_{j=1}^{N}\pi_j(b|\alpha,x), \tag{52}$$

where N is the number of iteration in the MCMC process, the Bayes estimates of the parameter α and b under LINEX loss function, respectively are

$$E_{LINEX}\left(\pi_{FTL(MLE)}(\alpha|b,x)\right)=\frac{-1}{h}\ln\left(\frac{1}{N}\sum_{j=1}^{N}e^{-h\pi_j(\alpha|b,x)}\right), \tag{53}$$

and

$$E_{LINEX}\left(\pi_{FTL(MLE)}(b|\alpha,x)\right)=\frac{-1}{h}\ln\left(\frac{1}{N}\sum_{j=1}^{N}e^{-h\pi_j(b|\alpha,x)}\right), \tag{54}$$

where N is the number of iteration in the MCMC process.

An important sub-class of MCMC methods are Gibbs sampling and more general Metropolis within Gibbs samplers. For more information about the Metropolis-Hastings algorithm, see Metropolis et al. (1953), Amin (2017) and Nassar et al (2018).

7. A simulation study

In this section, some simulation studies will be performed in order to investigate between estimators' behaviors of estimated methods.

7.1. Complete sample (non-Bayesian)

In this subsection, the algorithm for MLE, maximum product spacing (MPS), and least square (LS) methods under a complete sample using a non-Bayesian approach will be illustrated in the following steps:

Step (1): Generating random samples $X_1, X_2, ..., X_n$ of sizes n = (10, 20, 30, 50, 100, 300) using the FTL distribution with fixed seeds of random numbers.

Step (2): Using a set values of parameters as: $(\alpha = 3, b = 4)$.

Step (3): Solving normal equations of estimators for every method independently as follows:
In the MLE method under complete sample: Solve (19) and (20) , in the LS method with the complete sample: Solve(23) and (24), in the MPS method with the complete sample: Solve (27) and (28).

Step (4): Calculate biases, MLEs, and RMSE (the root of mean squared error) of the FTL distribution.

Step (5): Repeating Step (1) to Step (4), 1000 times.

In this study, random numbers with fixed seeds are generated via Mathcad package v15 where the conjugate gradient iteration method is performed. All results are included in the Appendix I, included in Tables 1, 2 and 3.

From study results, included in appendices; as sample size increases, biases and RMSEs decrease, moreover, when sample size increases, the distribution estimators can be more consistent.

7.2. Complete sample (Bayesian)

In this subsection, the algorithm for MLE method under a complete sample using the Bayesian approach with the MCMC method will be illustrated in the following steps:

Step (1): Generating a random sample $X_1, X_2, ..., X_n$ of sizes n = (10, 20, 30, 50, 100, 300) using the FTL distribution with fixed seeds of random numbers.

Step (2): Using a set values of parameters as: $(\alpha = 3, b = 4)$.

Step (3): Generating posterior for α and b as follows: Generate posterior for α and b from (37) and (38) where the Bayes estimate of the parameters under SE loss function is given by (39) and (40), the Bayes estimate of the parameters under LINEX loss function is given by (41) and(42).

Step (4): Calculating biases, MLEs and RMSE of the FTL distribution.

Step (5): Repeating step (1) to step (4), 1,000 times.

In this study, random numbers with fixed seeds are generated via Mathcad package v15 where the conjugate gradient iteration method is performed. All results are included in Tables 4, 5, and 6 and are indicated in the Appendix I.

From study results, included in appendices; as sample size increases, biases and RMSEs decrease, moreover, when sample size increases, the distribution estimators can be more consistent.

In Bayesian estimation methods, it is clear that the most efficient estimation method, according to biases and RMSEs, is the Bayesian estimation using LINEX loss function, on the other hand, Bayesian estimation methods give better efficiency than classical methods.

7.3. Censored Type-II sample (non-Bayesian)

In this subsection, the algorithm for MLE method under censored Type-II censoring scheme using a non-Bayesian approach will be illustrated in the following steps:

Step (1): Generating a random sample $X_1, X_2, ..., X_r$ of sizes r = (5, 10, 15, 25, 50, 150) where r represents failures for n = (10, 20, 30, 50, 100), respectively from the FTL distribution using fixed seeds.

Step (2): Using a set of values of parameters as: $(\alpha = 3, b = 4)$.

Step(3): Solving normal equations of estimators in (43) and (44).

Step (4): Calculate biases, MLEs, and RMSE of the FTL distribution.

Step (5): Repeating step (1) to step (4), 1,000 times.

From the simulation results, MLE and MPS methods under censored Type-II censoring scheme using the non-Bayesian approach, as sample size increases, biases and RMSEs decrease, moreover, when the sample size increases, the consistency of estimators increases.

One can see that the best efficient estimation method, according to biases and RMSEs, is the MPS method.

7.4. Censored sample (Bayesian)

In this subsection, the algorithm for MLE method under the Type-II censoring scheme using the Bayesian approach with the MCMC method will be illustrated in the following steps:

Step (1): Generating a random sample $X_1, X_2, ..., X_r$ of sizes r = (5, 10, 15, 25, 50, 150) where r represents failures for n = (10, 20, 30, 50, 100), respectively from the FTL distribution using fixed seeds.

Step (2): Using a set of values of parameters as: $(\alpha = 3, b = 4)$.

Step (3): Generating posterior for α and b as follows: Generate posterior for α and b from (49) and (50) where the Bayes estimate of the parameters under SE loss function is given by (51) and (52), the Bayes estimate of the parameters under LINEX loss function is given by (53) and (54).

Step (4): Calculating biases, MLEs and RMSE of the FTL distribution.

Step (5): Repeating step (1) to step (4), 1,000 times.

From the study results, as sample size increases, biases and RMSEs decrease. When sample size increases, the consistency of estimators increases. Moreover, using the Bayesian approach in estimation methods under the censored Type-II censoring scheme with the LINEX loss function gives, according to biases and RMSEs, more efficient estimators than the SE loss function estimators.

8. Conclusions

Using the complete square transformation on the FTL distribution gives big flexibility for the distribution, especially, in mathematical properties and generating random numbers which helps to use different parameter estimation methods. The MPS method is very efficient estimation method having a good performance with small biases and RMSEs. Bayesian estimation methods have a better performance with the smallest biases and RMSEs when compared with classical estimation methods in complete and censored samples. Author encourages researchers to study more about MPS and Bayesian estimation methods.

List of abbreviations

CDF	:	The cumulative distribution function
PDF	:	The probability density function
TL	:	The Topp-Leone distribution
FTL	:	Flexible Topp-Leone
LSEs	:	Least square estimators
GM	:	Geometric mean
ML	:	Maximum likelihood
MLE	:	The maximum likelihood estimation method
SE	:	Standard error
LINEX	:	Linear exponential
MCMC	:	Markov Chain Monte Carlo
MSE	:	Mean squared errors
MPS	:	Maximum product spacing
M	:	Median
MLEs	:	Maximum likelihood estimators
LS	:	Least square
RMSE	:	The root of mean squared error
MPS	:	Maximum product space
CII-MLEs	:	The MLES of the censored Type-II samples

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Appendix I

Table (1) MLE method

Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE
10	$\alpha = 3$	2.885	–0.115	1.756	0.890	7.651
	$b = 4$	5.752	1.752		7.599	
20	$\alpha = 3$	2.931	–0.069	0.526	0.625	2.296
	$b = 4$	4.522	0.522		2.209	
30	$\alpha = 3$	2.954	–0.046	0.324	0.526	1.653
	$b = 4$	4.320	0.320		1.567	
50	$\alpha = 3$	2.967	–0.033	0.180	0.409	1.195
	$b = 4$	4.177	0.177		1.123	
100	$\alpha = 3$	2.982	–0.018	0.090	0.297	0.831
	$b = 4$	4.088	0.088		0.776	
300	$\alpha = 3$	2.992	-8.436×10^{-3}	0.016	0.167	0.435
	$b = 4$	4.014	0.014		0.402	

Table (2) LS method

Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE
10	$\alpha = 3$	2.388	–0.612	2.409	1.326	46.455
	$b = 4$	6.330	2.330		46.436	
20	$\alpha = 3$	2.668	–0.332	0.335	0.844	3.191
	$b = 4$	3.953	–0.047		3.077	
30	$\alpha = 3$	2.764	–0.236	0.266	0.674	2.065
	$b = 4$	3.876	–0.124		1.952	
50	$\alpha = 3$	2.851	–0.149	0.187	0.501	1.377
	$b = 4$	3.887	–0.113		1.282	
100	$\alpha = 3$	2.923	–0.077	0.094	0.367	0.999
	$b = 4$	3.947	–0.053		0.929	
300	$\alpha = 3$	2.971	–0.029	0.047	0.204	0.531
	$b = 4$	3.963	–0.037		0.490	

Table (3) MPS method

Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE
10	$\alpha = 3$	2.145	–0.855	1.250	1.086	3.094
	$b = 4$	3.088	–0.912		2.898	
20	$\alpha = 3$	2.461	–0.539	0.935	0.757	1.722
	$b = 4$	3.236	–0.764		1.546	
30	$\alpha = 3$	2.599	–0.401	0.734	0.618	1.409
	$b = 4$	3.386	–0.614		1.267	
50	$\alpha = 3$	2.721	–0.279	0.532	0.472	1.113
	$b = 4$	3.547	–0.453		1.008	
100	$\alpha = 3$	2.835	–0.165	0.330	0.328	0.809
	$b = 4$	3.715	–0.285		0.739	
300	$\alpha = 3$	2.930	–0.070	0.159	0.178	0.445
	$b = 4$	3.858	–0.142		0.407	

Table (4) Bayesian method - SE loss function

Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE
10	$\alpha=3$	2.572	−0.428	0.432	0.595	1.536
	$b=4$	3.939	−0.061		1.416	
20	$\alpha=3$	2.779	−0.221	0.222	0.401	0.995
	$b=4$	3.975	−0.025		0.910	
30	$\alpha=3$	2.854	−0.146	0.149	0.314	0.787
	$b=4$	3.971	−0.029		0.722	
50	$\alpha=3$	2.909	−0.091	0.091	0.245	0.623
	$b=4$	3.993	-7.426×10^{-3}		0.573	
100	$\alpha=3$	2.953	−0.047	0.047	0.173	0.443
	$b=4$	3.999	-1.145×10^{-3}		0.408	
300	$\alpha=3$	2.986	−0.014	0.016	0.096	0.244
	$b=4$	3.991	-9.035×10^{-3}		0.224	

Table (5) Bayesian method - LINEX loss function $(h=1)$

Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE
10	$\alpha=3$	2.493	−0.507	0.851	0.659	1.814
	$b=4$	3.317	−0.683		1.690	
20	$\alpha=3$	2.726	−0.274	0.456	0.436	1.125
	$b=4$	3.635	−0.365		1.038	
30	$\alpha=3$	2.816	−0.184	0.313	0.335	0.865
	$b=4$	3.746	−0.254		0.797	
50	$\alpha=3$	2.884	−0.116	0.196	0.257	0.664
	$b=4$	3.842	−0.158		0.613	
100	$\alpha=3$	2.939	−0.061	0.100	0.178	0.459
	$b=4$	3.920	−0.080		0.423	
300	$\alpha=3$	2.982	−0.018	0.038	0.097	0.248
	$b=4$	3.966	−0.034		0.228	

Table (6) Bayesian method - LINEX loss function $(h=-1)$

Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE
10	$\alpha=3$	2.664	–0.336	2.398	0.540	3.724
	$b=4$	6.375	2.375		3.684	
20	$\alpha=3$	2.839	–0.161	0.569	0.376	1.262
	$b=4$	4.546	0.546		1.205	
30	$\alpha=3$	2.894	–0.106	0.301	0.300	0.887
	$b=4$	4.281	0.281		0.834	
50	$\alpha=3$	2.936	–0.064	0.183	0.238	0.668
	$b=4$	4.171	0.171		0.624	
100	$\alpha=3$	2.967	–0.033	0.093	0.171	0.459
	$b=4$	4.087	0.087		0.426	
300	$\alpha=3$	2.991	-9.208×10^{-3}	0.019	0.095	0.246
	$b=4$	4.017	0.017		0.226	

Table (7) Censored II scheme

Sample Size	Parameters	r	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE
10	$\alpha=3$	9	2.568	–0.432	0.471	0.937	5.059
	$b=4$		3.811	–0.189		4.972	
20	$\alpha=3$	18	2.484	–0.516	1.751	0.787	4.193
	$b=4$		2.327	–1.673		4.119	
30	$\alpha=3$	27	2.449	–0.551	1.673	0.733	1.958
	$b=4$		2.420	–1.580		1.815	
50	$\alpha=3$	45	2.386	–0.614	1.905	0.721	2.035
	$b=4$		2.197	–1.803		1.903	
100	$\alpha=3$	90	2.299	–0.701	2.145	0.754	2.210
	$b=4$		1.973	–2.027		2.078	

Table (8) Censored II scheme: Bayesian method - SE loss function

Sample Size	Parameters	r	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE
10	$\alpha=3$	9	2.795	−0.205	1.045	0.519	1.665
	$b=4$		2.975	−1.025		1.582	
20	$\alpha=3$	18	3.057	0.057	1.321	0.392	3.521
	$b=4$		2.680	−1.320		3.499	
30	$\alpha=3$	27	3.156	0.156	1.247	0.360	1.410
	$b=4$		2.763	−1.237		1.363	
50	$\alpha=3$	45	3.235	0.235	1.335	0.359	1.439
	$b=4$		2.686	−1.314		1.394	
100	$\alpha=3$	90	3.303	0.303	1.463	0.362	1.516
	$b=4$		2.568	−1.432		1.472	

Table (9) Censored II scheme: Bayesian method - Linex $(h=-1)$ loss function

Sample Size	Parameters	r	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE
10	$\alpha=3$	9	2.918	−0.082	0.617	0.499	2.181
	$b=4$		4.612	0.612		2.123	
20	$\alpha=3$	18	3.137	0.137	0.803	0.420	3.404
	$b=4$		3.208	−0.792		3.378	
30	$\alpha=3$	27	3.211	0.211	1.076	0.391	1.276
	$b=4$		2.945	−1.055		1.214	
50	$\alpha=3$	45	3.273	0.273	1.230	0.387	1.348
	$b=4$		2.801	−1.199		1.291	
100	$\alpha=3$	90	3.323	0.323	1.408	0.379	1.464
	$b=4$		2.630	−1.370		1.414	