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The Power Lomax-Generated Family of Distributions

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Abstract

In this paper, a family of distributions named the power Lomax-generated family is presented. It is derived from the T-X family by using the power Lomax as its generator. Some distributions are derived from the proposed family, such as the power Lomax-normal, power Lomax-Weibull, and power Lomax-beta distributions. Various properties of the proposed family, including the quantile function, skewness, kurtosis, order statistics, and moments, are provided. The maximum likelihood (ML) method is used to estimate the parameters of the proposed distribution. Moreover, the simulation study demonstrates how well the ML method performs when estimating the parameters of some submodels in the proposed family, such as the power Lomax-Weibull distribution. Finally, real data sets are utilized to showcase the practicality of this newly established family.

Keywords: T-X family, power Lomax generated family, power Lomax-normal, power Lomax-Weibull, power Lomax-beta, maximum likelihood estimation.

1. Introduction

Numerous classical distributions find extensive application across various domains, prompting the necessity for extended variants particularly in fields such as lifetime analysis, finance, and insurance. A newly developed distribution is being introduced to offer enhanced flexibility in data modeling. Continuous distributions are extensively used in various applications, leading researchers to develop new distributions. Eugene et al. (2002) introduced the beta-generated family of distributions and emphasized its excellent flexibility in modeling data.

In the past, numerous novel distributions have been developed by augmenting one or more parameters of existing distributions using the method of the T-X family proposed by Alzaatreh et al. (2013), which incorporates various generators. Some examples are presented in Table 1, which presents various distribution methods by the T-X family, each employing different distribution methods. The Weibull-X family of distributions by Alzaatreh et al. (2015) utilizes the Weibull distribution as a generator, and the baseline is a logistic distribution. The Gompertz-G family of distributions by Alizadeh et al. (2017) utilizes the Gompertz as a generator and baseline, comprising several distributions such as normal, gamma, beta, log-logistic, and exponential Weibull distributions. Similarly, the Zubair-G family of distributions by Zubair (2020) uses the Zubair distribution as a generator,

and the baseline is a Weibull distribution. The Power Garima-G family of distributions by Aryuyuen et al. (2021) utilizes the Power Garima distribution as a generator, and the baseline is the Frchet, Weibull, and Lindley distributions. Additionally, the Beta Topp-Leone-G family of distributions by Watthanawisut et al. (2022) and the Topp-Leone Weibull-G family of distributions by Elbatal et al. (2022) utilize the Beta Topp-Leone distribution and the Topp-Leone Weibull distribution as generators, each with a baseline is a Weibull distribution, and in the case of Topp-Leone Weibull-G, it has a Lindley distribution as well. These family of distribution examples represent the T-X family methodology, widely recognized within the statistical and computational research spheres.

Table 1 Examples of a developed distribution utilizing the T-X family method

T-X Family of Distribtutions	Author (s)	Generator	Baseline distribution
the Weibull-X	Alzaatreh et al. (2015)	Weibull distribution	Logistic distribution
the Gompertz-G	Alizadeh et al. (2017)	Gompertz distribution	Normal distribution Gamma distribution Beta distribution Log-Logistic distribution Exponentiated Weibull distribution
the Zubair-G	Zubair (2020)	Zubair distribution	Weibull distribution
Power Garima-G	Aryuyuen et al. (2021)	Power Garima distribution	Fréchet distribution Weibull distribution Lindley distribution
the Beta Topp-Leone-G	Watthanawisut et al. (2022)	Beta Topp-Leone distribution	Weibull distribution
Topp-Leone Weibull-G	Elbatal et al. (2022)	Topp-Leone Weibull distribution	Weibull distribution Lindley distribution

Let $r(t)$ be the probability density function (pdf) of a random variable $T \in$ [a, b] for $-\infty < a < b < \infty$ and $W[G(x;\boldsymbol{\xi})]$ be a function of the cumulative distribution function (cdf) of a random variable X satisfying the following conditions:

$$\begin{cases} (i) \quad W[G(x;\boldsymbol{\xi})] \in [a,b] \\ (ii) \quad W[G(x;\boldsymbol{\xi})] \text{ is differentiable and monotonically non - decreasing, and} \\ (iii) \quad W[G(x;\boldsymbol{\xi})] \to a \text{ as } x \to -\infty, \text{ and } W[G(x;\boldsymbol{\xi})] \to b \text{ as } x \to \infty. \end{cases} \tag{1}$$

Alzaatreh et al. (2013) defined the cdf of the T-X family of distributions by

$$F(x) = \int_a^{W\left[G(x;\boldsymbol{\xi})\right]} r(t)dt, \tag{2}$$

where $W[G(x;\boldsymbol{\xi})]$ satisfies the conditions in Equation (1). The pdf that corresponds to Equation (2) is given by

$$f(x) = \frac{d}{dx} W\left[G(x;\boldsymbol{\xi})\right] r\left\{W[G(x;\boldsymbol{\xi})]\right\}, \tag{3}$$

where $G(x;\boldsymbol{\xi})$ is the cdf of the baseline distribution with a vector parameter $\boldsymbol{\xi}$.

The power Lomax-generated family added parameters to improve distributional flexibility, enabling better fitting across diverse data types. This enhancement allows for greater adaptability in modeling complex datasets, particularly in reliability analysis. The objective of this paper is to introduce a new family of continuous distributions, utilizing the power Lomax (PL) distribution as the generator. The PL distribution was originally introduced by Rady et al. (2016). The structure of this paper is as follows: In Section 2, we establish the power Lomax-generated (PL-G) family of distributions. Section 3 shows the investigation of the statistical properties of this newly proposed family.

Section 4 offers three sub-models of this family. In Section 5, the estimation of model parameters using the maximum likelihood method is provided. Section 6 is dedicated to a simulation study, while Section 7 presents three real-data applications that showcase the potential utility of this innovative family. Finally, a discussion and conclusion are provided.

2. The Power Lomax-Generated Family of Distributions

In this section, we introduce a new family of continuous distributions from the T-X family. This family is defined by using the PL random variable as a generator. The pdf and cdf of the PL distribution are, respectively:

$$f_{\text{PL}}(x) = \alpha\beta\lambda^{\alpha}x^{\beta-1}\left(\lambda+x^{\beta}\right)^{-\alpha-1} \quad \text{for} \quad x>0, \quad \alpha,\beta,\lambda>0, \tag{4}$$

$$F_{\text{PL}}(x) = 1-\lambda^{\alpha}\left(x^{\beta}+\lambda\right)^{-\alpha}. \tag{5}$$

From Equation (2) to the cdf of the T-X family with T represents the pdf of a PL random variable as Equation (4) with the function $W[G(x;\boldsymbol{\xi})] = -\log(1-G(x;\boldsymbol{\xi}))$. The cdf of the PL-G family of distributions can be expressed as:

$$F_{\text{PL-G}}(x) = \int_0^{-\log\left(1-G(x;\boldsymbol{\xi})\right)}\left\{\alpha\beta\lambda^{\alpha}t^{\beta-1}\left(\lambda+t^{\beta}\right)^{-\alpha-1}\right\}dt \tag{6}$$

$$= 1-\lambda^{\alpha}\left[\left(-\log\left(1-G(x;\boldsymbol{\xi})\right)\right)^{\beta}+\lambda\right]^{-\alpha}, \tag{7}$$

where $\alpha>0$, $\beta>0$ and $\lambda>0$. Its corresponding pdf is given by

$$f_{\text{PL-G}}(x) = \alpha\beta\lambda^{\alpha}\frac{g(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}\left(-\log\left(1-G(x;\boldsymbol{\xi})\right)\right)^{\beta-1} \times\left[\left(\lambda+\left(-\log\left(1-G(x;\boldsymbol{\xi})\right)\right)\right)^{\beta}\right]^{-\alpha-1}. \tag{8}$$

3. Statistical Properties

In this section, we will present various properties of the PL-G family of distributions, including quantile function, skewness, kurtosis, order statistics, and moments.

3.1. The quantile function

Since $F(x)$ is distributed a uniform (U) distribution on $(0,1)$, if $F(x)=U$ then the quantile function (qf) of the PL-G family of distributions is $Q(u)=F^{-1}(x)$. The qf for the PL-G family of distributions can be obtained by inverting Equation (7) as follows:

$$Q_{\text{PL-G}}(u) = G^{-1}\left[\exp\left\{\lambda^{\frac{1}{\beta}}\left\{(1-u)^{-\frac{1}{\alpha}}-1\right\}^{\frac{1}{\beta}}\right\}+1\right]. \tag{9}$$

The median can be derived from Equation (9) by setting $u=0.5$ as

$$\text{median} = G^{-1}\left[\exp\left\{\lambda^{\frac{1}{\beta}}\left\{(0.5)^{-\frac{1}{\alpha}}-1\right\}^{\frac{1}{\beta}}\right\}+1\right]. \tag{10}$$

3.2. The skewness and kurtosis

We consider into the impact of the shape parameters on skewness and kurtosis. To assess these effects, we utilize quantile measures, such as Bowley skewness, as suggested in (Kenny and Keeping, 1962).

From the qf in Equation (9), we have the Bowley skewness of the PL-G family of distributions as

$$B = \frac{Q_{\text{PL-G}}\left(\frac{3}{4}\right)+Q_{\text{PL-G}}\left(\frac{1}{4}\right)-2Q_{\text{PL-G}}\left(\frac{1}{2}\right)}{Q_{\text{PL-G}}\left(\frac{3}{4}\right)-Q_{\text{PL-G}}\left(\frac{1}{4}\right)}.$$

In addition, we also take into consideration the Moors kurtosis as another measure to analyze the distribution's kurtosis (Moors, 1998) for the kurtosis of the PL-G family as

$$M = \frac{Q_{\text{PL-G}}\left(\frac{3}{8}\right)+Q_{\text{PL-G}}\left(\frac{1}{8}\right)+Q_{\text{PL-G}}\left(\frac{7}{8}\right)-Q_{\text{PL-G}}\left(\frac{5}{8}\right)}{Q_{\text{PL-G}}\left(\frac{6}{8}\right)-Q_{\text{PL-G}}\left(\frac{2}{8}\right)}.$$

3.3. Order statistics

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ represent the order statistics of a random sample X_i, where $i = 1, 2, \ldots, n$, from the PL-G family of distributions with the cdf in Equation (7) and pdf in Equation (8). The pdf of $X_{(j)}$, where $j = 1, 2, \ldots, n$, is given by:

$$\begin{aligned}
f_{X_{(j)}}(x) &= \frac{\Gamma(n+1)}{\Gamma(j)\Gamma(n-j+1)} f(x)\big[F(x)\big]^{j-1}\big[1-F(x)\big]^{n-j} \\
&= \frac{\Gamma(n+1)}{\Gamma(j)\Gamma(n-j+1)} \frac{g(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}\alpha\beta\lambda^{\alpha} \\
&\quad \times\left[-\log\Big(1-G(x;\boldsymbol{\xi})\Big)\right]^{\beta-1}\left[\left(\lambda+\Big(-\log\big(1-G(x;\boldsymbol{\xi})\big)\Big)\right)^{\beta}\right]^{-\alpha-1} \\
&\quad \times\left[1-\lambda^{\alpha}\left[\Big(-\log\big(1-G(x;\boldsymbol{\xi})\big)\Big)^{\beta}+\lambda\right]^{-\alpha}\right]^{j-1} \\
&\quad \times\left[1-\left[1-\lambda^{\alpha}\left[\Big(-\log\big(1-G(x;\boldsymbol{\xi})\big)\Big)^{\beta}+\lambda\right]^{-\alpha}\right]\right]^{n-j}
\end{aligned}$$

where $\Gamma(\cdot)$ is a gamma functions.

3.4. Moments

We applied the following series, (see Gradshteyn and Ryzhik, 2007, p. 25)

$$(a+b)^c = \sum_{j=0}^{b}\binom{b}{j}a^{b-j}c^j \quad \text{and} \quad \exp(-w^{1/\beta}) = \sum_{k=0}^{\infty}(-1)^k\frac{w^{k/\beta}}{k!}$$

where $w = (-\log(1-G(x;\boldsymbol{\xi}))^{\beta}$, then the rth moment of the PL-G family of distributions can be determined as

$$\begin{aligned}
E(X^r) &= \int_{-\infty}^{\infty} x^r f(x)dx \\
&= \alpha\beta\lambda^{\alpha}\int_{-\infty}^{\infty} x^r\left[g(x;\boldsymbol{\xi})\left(\sum_{j=0}^{-\alpha-1}\sum_{k=0}^{\infty}\binom{-\alpha-1}{j}\lambda^{-\alpha-1-j}1^k\frac{w^{1+(\frac{-1+k}{\beta})+j}}{k!}\right)\right]dx \\
&= \alpha\beta\lambda^{\alpha}\sum_{j=0}^{-\alpha-1}\sum_{k=0}^{\infty}\phi_r[\lambda_i(j,k)],
\end{aligned}$$

where

$$\lambda_i(j,k) = \binom{-\alpha-1}{j}\lambda^{-\alpha-1-j}1^k\frac{w^{1+(\frac{-1+k}{\beta})+j}}{k!} \text{ and } \phi_r = \int_{-\infty}^{\infty} x^r g(x;\boldsymbol{\xi})dx.$$

Consequentially, the mean and variance of the PL-G family of distributions can be determined from the moments of X as follows:

$$\begin{aligned} E(X) &= \alpha\beta\lambda^{\alpha}\sum_{j=0}^{-\alpha-1}\sum_{k=0}^{\infty}\phi_1[\lambda_i(j,k)], \\ \text{and } Var(X) &= \alpha\beta\lambda^{\alpha}\left[\sum_{j=0}^{-\alpha-1}\sum_{k=0}^{\infty}\phi_2[\lambda_i(j,k)] - \alpha\beta\lambda^{\alpha}\left(\sum_{j=0}^{-\alpha-1}\sum_{k=0}^{\infty}\phi_1[\lambda_i(j,k)]\right)^2\right]. \end{aligned}$$

4. Special Models of PL-G Family of Distributions

In this section, we introduce some special cases of the PL-G family of distributions. These special cases are constructed by utilizing three baseline distributions: Normal, Weibull, and beta distributions.

4.1. The power Lomax-normal (PL-N) distribution

The PL-N distribution is characterized by its cdf and pdf respectively:

$$\begin{aligned} F_{\text{PL-N}}(x) &= 1-\lambda^{\alpha}\left[\left(-\log\left(\frac{1}{2}+\frac{1}{2}\text{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)\right)^{\beta}+\lambda\right]^{-\alpha}; -\infty<x,\mu<\infty \\ f_{\text{PL-N}}(x) &= \frac{\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{\frac{1}{2}-\frac{1}{2}\text{erf}\left(\frac{x-\mu}{\sigma}\right)}\alpha\beta\lambda^{\alpha}\left[-\log\left(\frac{1}{2}-\frac{1}{2}\text{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)\right]^{\beta-1} \\ &\quad\times\left[\lambda+\left(-\log\left(\frac{1}{2}-\frac{1}{2}\text{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)\right)^{\beta}\right]^{-\alpha-1}; \alpha,\beta,\lambda,\sigma>0. \end{aligned}$$

For the normal distribution, the qf is $Q_N(u)=\mu+\sigma\sqrt{2}\text{erf}^{-1}(2u-1)$, hence we have the qf of the PL-N distribution as

$$Q_{\text{PL-N}}(u)=\mu+\sigma\sqrt{2}\text{erf}^{-1}\Big(2\exp[\lambda^{\frac{1}{\beta}}[(1-u)^{-\frac{1}{\alpha}}-1]^{\frac{1}{\beta}}]+1\Big),$$

where the error function (erf) is defined as: $\text{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x\exp(-t^2)dt$ by Abramowitz and Stegun (1965).

4.2. The power Lomax-Weibull (PL-W) distribution

The PL-W distribution has the cdf and pdf respectively:

$$\begin{aligned} F_{\text{PL-W}}(x) &= 1-\lambda^{\alpha}\left[\left(\left(\frac{x}{\theta}\right)^k\right)^{\beta}+\lambda\right]^{-\alpha}; x>0 \\ f_{\text{PL-W}}(x) &= \frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1}\alpha\beta\lambda^{\alpha}\left[\left(\frac{x}{\theta}\right)^k\right]^{\beta-1}\left[\lambda+\left(\left(\frac{x}{\theta}\right)^k\right)^{\beta}\right]^{-\alpha-1}; \alpha,\beta,\lambda,k,\theta>0. \end{aligned}$$

For the Weibull distribution, the qf is $Q_{\text{W}}(u)=\theta[-\log(1-u)]^{\frac{1}{k}}$, hence we have the qf of the PL-W distribution as

$$Q_{\text{PL-W}}(u)=\theta\left[\left[\left(\frac{1-u}{\lambda^{\alpha}}\right)^{-\frac{1}{\alpha}}-\lambda\right]^{\frac{1}{\beta}}\right]^{\frac{1}{k}}.$$

4.3. The power Lomax-Beta (PL-B) distribution

The PL-B distribution has the cdf and pdf respectively:

$$\begin{aligned} F_{\text{PL-B}}(x) &= 1-\lambda^{\alpha}\left[\left(-\log\left(1-\left(I_x(\delta,\phi)\right)\right)\right)^{\beta}+\lambda\right]^{-\alpha}; 0<x<1 \\ f_{\text{PL-B}}(x) &= \frac{\frac{x^{\delta-1}(1-x)^{\phi-1}}{\text{B}(\delta,\phi)}}{1-I_x(\delta,\phi)}\alpha\beta\lambda^{\alpha}\left[-\log\left(1-I_x\Big(\delta,\phi\Big)\right)\right]^{-\beta-1} \\ &\quad\times\left[\lambda+\left(-\log\Big(1-I_x\big(\delta,\phi\big)\Big)\right)^{\beta}\right]^{-\alpha-1}; \alpha,\beta,\lambda,\delta,\phi>0. \end{aligned}$$

For the Beta (B) distribution, its cdf is $I_x(v;\delta,\phi)$ which is the regularized incomplete beta function by Johnson et al. (1994). The qf is $Q_{\text{B}}(v)=I_x^{-1}(v;\delta,\phi)$ where $0\leqslant v\leqslant 1$, we have the qf of the PL-B distribution as

$$Q_{\text{PL-B}}(u) = I_x^{-1}\Big(\big[\exp\big[\lambda^{\frac{1}{\beta}}[(1-u)^{-\frac{1}{\alpha}}-1]^{\frac{1}{\beta}}\big]+1\big];\delta,\phi\Big). \tag{11}$$

Figure 1 shows the pdf and cdf plots of the PL-G family of distributions. The PL-N distribution has a range of random variables X that are real numbers. Its pdf graph can be both symmetric and non-symmetric, depending on the parameters. Meanwhile, the PL-W distribution features a random variable X that takes only positive values. This distribution has a variety of shapes, including symmetric, right-skewed, and a decreasing function. Furthermore, the PL-B distribution, where the random variable X ranges from 0 to 1, exhibits various shapes such as symmetric, right-skewed, U-shaped, decreasing and increasing functions.

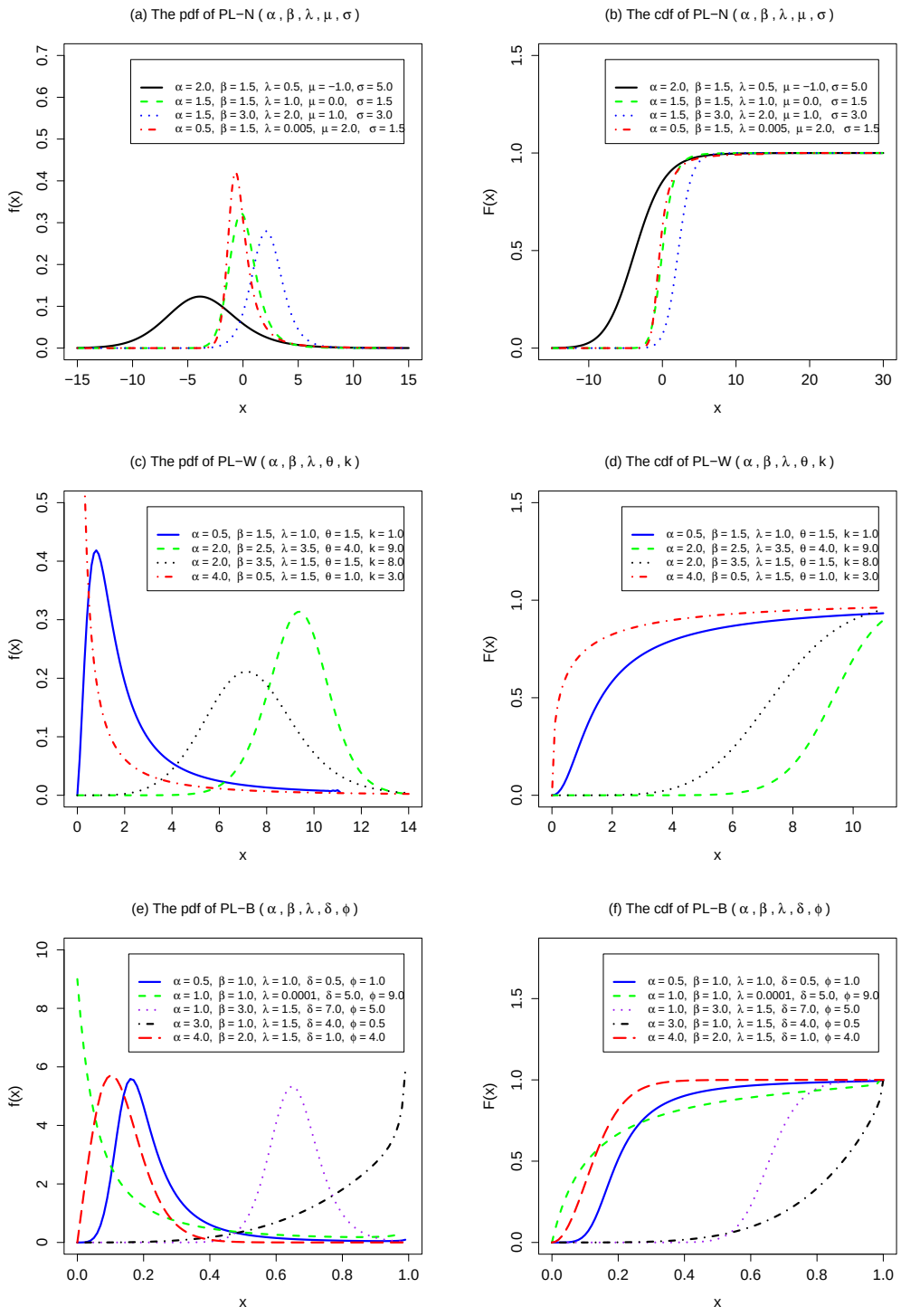


Figure 1 The pdf and cdf plots of the PL-N, PL-W and PL-B distributions

5. Parameter Estimation

In this section, we derive the maximum likelihood estimators (MLEs) for the model parameters $\boldsymbol{\Theta} = (\alpha, \beta, \lambda, \boldsymbol{\xi})$ of the PL-G family of distributions. Let $X_1, X_2, \ldots, X_n$ represent a random sample from the PL-G family of distributions with observed values $x_1, x_2, \ldots, x_n$. Consequently, the likelihood function can be expressed as

$$L(\boldsymbol{\Theta}) = (\alpha\beta\lambda^{\alpha})^n \prod_{i=1}^{n} \frac{g(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})} (\upsilon_i)^{\beta-1} \Big[\big(\lambda+\upsilon_i\big)^{\beta}\Big]^{-\alpha-1}$$

where $\upsilon_i = -\log(1-G(x;\boldsymbol{\xi}))$. The log-likelihood function of the pdf in Equation (8) is denoted as $\ell(\boldsymbol{\Theta}) = \log L_{\text{PL-G}}(\boldsymbol{\Theta})$, where:

$$\begin{aligned}\ell(\boldsymbol{\Theta}) &= n\log\alpha + n\log\beta + n\alpha\log\lambda + \sum_{i=1}^{n}\log\frac{g(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})} \\ &\quad +(\beta-1)\sum_{i=1}^{n}\log\upsilon_i + (-\alpha-1)\sum_{i=1}^{n}\log[\lambda+\upsilon_i]^{\beta}.\end{aligned}$$

The MLEs for the parameters $\boldsymbol{\Theta}$ denoted as $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$, and $\hat{\boldsymbol{\xi}}$ are calculated by equating the partial derivative of the log-likelihood function $\ell(\boldsymbol{\Theta})$ with respect to each parameter to zero. In other words, we have:

$$\frac{\partial\ell(\boldsymbol{\Theta})}{\partial\alpha} = 0, \frac{\partial\ell(\boldsymbol{\Theta})}{\partial\beta} = 0, \frac{\partial\ell(\boldsymbol{\Theta})}{\partial\lambda} = 0, \frac{\partial\ell(\boldsymbol{\Theta})}{\partial\boldsymbol{\xi}} = 0.$$

This system of equations is nonlinear. We concurrently solve them by employing a numerical procedure with the Newton-Raphson method (Verbeke and Cools, 1995). To find the MLEs, use the nlm function from the **stats** package in the R programming language (R Core Team, 2020).

6. Simulation Study

A Monte Carlo simulation study was undertaken to assess the performance of maximum likelihood estimators, focusing on bias and mean square error (MSE). The simulation study is performed using the statistical program R. The study involved varying sample sizes ($n = 60, 100, 200, 300$, and 500) and exploring different parameter values ($\alpha, \beta, \lambda, k$, and θ) with the PL-W distribution. Four sets of parameter configurations were considered

case 1: $\alpha = 0.5,\ \beta = 1.5,\ \lambda = 1,\ \theta = 1.5,\ k = 1,$
case 2: $\alpha = 0.8,\ \beta = 1.5,\ \lambda = 2.5,\ \theta = 3.5,\ k = 1.5,$
case 3: $\alpha = 1,\ \beta = 2.5,\ \lambda = 3.5,\ \theta = 0.5,\ k = 2$ and
case 4: $\alpha = 2,\ \beta = 3.5,\ \lambda = 1.5,\ \theta = 1.5,\ k = 0.5.$

Tables 2 - 5 provide the bias, standard error (SE), and MSE of the maximum likelihood estimators. The findings indicate that the MLE demonstrates asymptotic unbiasedness and consistency. In other words, SE decreases when the sample size increases. A simulation study demonstrates that the maximum likelihood method is appropriate for estimating the parameters of the PL-W distribution.

Table 2 Bias, SE, and MSE of the estimates under the method of MLE for the PL-W distribution with $\alpha = 0.5,\ \beta = 1.5,\ \lambda = 1,\ \theta = 1.5,\ k = 1$

	Bias					SE					MSE				
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{k}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{k}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{k}$
60	0.0005	-0.2953	0.3297	-0.2268	0.2902	0.0129	0.0095	0.0121	0.0284	0.0102	0.0100	0.0926	0.1175	0.0998	0.0904
100	0.0134	-0.3284	0.4059	-0.2460	0.2996	0.0083	0.0058	0.0092	0.0153	0.0055	0.0071	0.1113	0.1732	0.0840	0.0928
200	-0.0203	-0.3088	0.3645	-0.3140	0.3061	0.0037	0.0033	0.0041	0.0066	0.0031	0.0031	0.0975	0.1362	0.1072	0.0956
300	-0.0041	-0.3380	0.4073	-0.3298	0.2982	0.0027	0.0021	0.0031	0.0045	0.0019	0.0022	0.1156	0.1688	0.1148	0.0900
500	0.0072	-0.3426	0.4336	-0.2896	0.3027	0.0018	0.0013	0.0020	0.0028	0.0011	0.0017	0.1182	0.1899	0.0879	0.0922

Table 3 Bias, SE, and MSE of the estimates under the method of MLE for the PL-W distribution with $\alpha = 0.8,\ \beta = 1.5,\ \lambda = 2.5,\ \theta = 3.5,\ k = 1.5$

	Bias					SE					MSE				
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{k}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{k}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{k}$
60	0.0342	-0.2623	-0.2597	0.2795	0.3555	0.0266	0.0128	0.0189	0.0529	0.0113	0.0435	0.0787	0.0890	0.2458	0.1340
100	0.0202	-0.2826	0.0082	0.0311	0.3723	0.0162	0.0081	0.0136	0.0277	0.0061	0.0267	0.0864	0.0187	0.0775	0.1424
200	0.0041	-0.2852	-0.0069	-0.0046	0.3620	0.0082	0.0041	0.0072	0.0145	0.0030	0.0134	0.0847	0.0103	0.0423	0.1328
300	0.0135	-0.2981	0.1598	-0.0704	0.3710	0.0050	0.0026	0.0046	0.0086	0.0020	0.0077	0.0908	0.0319	0.0272	0.1388
500	0.0026	-0.2945	0.1481	-0.0922	0.3695	0.0031	0.0016	0.0027	0.0051	0.0012	0.0047	0.0880	0.0256	0.0214	0.1373

Table 4 Bias, SE, and MSE of the estimates under the method of MLE for the PL-W distribution with $\alpha = 1,\ \beta = 2.5,\ \lambda = 3.5,\ \theta = 0.5,\ k = 2$

	Bias					SE					MSE				
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{k}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{k}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{k}$
60	-0.0055	-0.2429	-2.4948	0.1412	0.2844	0.0295	0.0173	0.0020	0.0054	0.0204	0.0523	0.0769	6.2244	0.0217	0.1058
100	-0.0300	-0.2782	-2.7130	0.1633	0.3176	0.0186	0.0083	0.0017	0.0039	0.0086	0.0356	0.0842	7.3606	0.0282	0.1083
200	-0.0633	-0.2663	-2.7104	0.1600	0.3304	0.0078	0.0049	0.0010	0.0019	0.0052	0.0162	0.0758	7.3465	0.0263	0.1145
300	0.0116	-0.3483	-2.6143	0.1599	0.3148	0.0061	0.0026	0.0008	0.0012	0.0029	0.0112	0.1233	6.8348	0.0260	0.1016
500	0.0090	-0.3393	-2.6161	0.1604	0.3245	0.0032	0.0017	0.0005	0.0006	0.0019	0.0053	0.1166	6.8443	0.0259	0.1072

Table 5 Bias, SE, and MSE of the estimates under the method of MLE for the PL-W distribution with $\alpha = 2,\ \beta = 3.5,\ \lambda = 1.5,\ \theta = 1.5,\ k = 0.5$

	Bias					SE					MSE				
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{k}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{k}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{k}$
60	-0.4988	-2.0995	-0.2684	-0.1682	0.9067	0.0577	0.0121	0.0107	0.0368	0.0139	0.4485	4.4167	0.0789	0.1097	0.8337
100	-0.4140	-2.2064	-0.2324	-0.1373	0.9613	0.0364	0.0078	0.0091	0.0216	0.0063	0.3041	4.8744	0.0623	0.0654	0.9281
200	-0.0512	2.2743	-0.1427	0.1102	0.9308	0.0276	0.0030	0.0050	0.0139	0.0020	0.1552	5.1744	0.0253	0.0510	0.8672
300	-0.1753	-2.2929	-0.1576	-0.0031	1.0094	0.0169	0.0029	0.0036	0.0088	0.0015	0.1165	5.2600	0.0287	0.0231	1.0195
500	-0.0700	-2.3107	-0.1420	0.0436	1.0046	0.0118	0.0018	0.0025	0.0062	0.0007	0.0751	5.3412	0.0232	0.0209	1.0094

7. Applications

In this section, we present the flexibility of the proposed distribution over three real data sets. We compare the fits of its sub-models, namely the PL-B, PL-W, and PL-N distributions, which constitute a newly introduced PL-G family of distributions. The descriptions of the data are as follows:

The first data sets relates to the influence of soil fertility and the characterization of biological nitrogen fixation for the growth of Dimorphandra wilsonii Rizz. It was originally studied by Fonseca (2007) and is also featured in the work of Silva et al. (2013). This dataset consists of phosphorus concentration measurements in the leaves of 128 plants. The recorded values are as follows:

0.22, 0.17, 0.11, 0.10, 0.15, 0.06, 0.05, 0.07, 0.12, 0.09, 0.23, 0.25, 0.23, 0.24, 0.20, 0.08, 0.11, 0.12, 0.10, 0.06, 0.20, 0.17, 0.20, 0.11, 0.16, 0.09, 0.10, 0.12, 0.12, 0.10, 0.09, 0.17, 0.19, 0.21, 0.18, 0.26, 0.19, 0.17, 0.18, 0.20, 0.24, 0.19, 0.21, 0.22, 0.17, 0.08, 0.08, 0.06, 0.09, 0.22, 0.23, 0.22, 0.19, 0.27, 0.16, 0.28, 0.11, 0.10, 0.20, 0.12, 0.15, 0.08, 0.12, 0.09, 0.14, 0.07, 0.09, 0.05, 0.06, 0.11, 0.16, 0.20, 0.25, 0.16, 0.13, 0.11, 0.11, 0.11, 0.08, 0.22, 0.11, 0.13, 0.12, 0.15, 0.12, 0.11, 0.11, 0.15, 0.10, 0.15, 0.17, 0.14, 0.12, 0.18, 0.14, 0.18, 0.13, 0.12, 0.14, 0.09, 0.10, 0.13, 0.09, 0.11, 0.11, 0.14, 0.07, 0.07, 0.19, 0.17, 0.18, 0.16, 0.19, 0.15, 0.07, 0.09, 0.17, 0.10, 0.08, 0.15, 0.21, 0.16, 0.08, 0.10, 0.06, 0.08, 0.12, 0.13.

Table 6 provides some descriptive statistics for this data sets.

The second application involves a data sets related to remission times (in months) of a random sample of 128 bladder cancer patients, as documented in Lee and Wang (2003). The data are:

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Descriptive statistics for this data sets can be found in Table 6.

The following data sets represent the average outside temperature (in degrees Celsius) at his house in southeast England for 26 weeks before cavity-wall insulation had been installed. observed and reported by Hand et al. (1993). The data is as follows:

-0.8, -0.7, 0.4, 2.5, 2.9, 3.2, 3.6, 3.9, 4.2, 4.3, 5.4, 6.0, 6.0, 6.0, 6.2, 6.3, 6.9, 7.0, 7.4, 7.5, 7.5, 7.6, 8.0, 8.5, 9.1, 10.2.

Table 6 presents the descriptive statistics for this data sets.

Table 6 Descriptive statistics for data sets

Data sets	n	Min	Max	Mean	Variance
I: Phosphorus concentration in leaves	128	0.050	0.280	0.140	0.003
II: Bladder cancer patients	128	0.080	79.050	9.366	110.425
III: The average outside temperature in southeast England	26	-0.8	10.2	5.35	8.253

Table 7 LR tests for first real data

Hypothesis			Statistics w	p-value
H_0 : Beta	vs	H_1 : PL-B	0.4198	0.9361

From Table 7, the likelihood ratio test was conducted to compare the PL-B distribution with the Beta distribution as a sub-model. The test statistic was 0.4198 with a p-value of 0.9361. Since the p-value is considerably higher than the typical significance level, there is no evidence to reject the null hypothesis. Therefore, the beta distribution is adequate for describing the first real data, compared to the more general PL-B distribution.

Table 8 LR tests for second real data

Hypothesis			Statistics w	p-value
H_0 : Zubair-Weibull	vs	H_1 : PL-W	31.9386	0.0000
H_0 : Weibull-Weibull	vs	H_1 : PL-W	8.9698	0.0027
H_0 : Weibull	vs	H_1 : PL-W	8.6940	0.0336

From Table 8, test results from the LR test have indicated that the Power Lomax-Weibull (PL-W) distribution fits better as a model for the second real data set than do the Zubair-Weibull, Weibull-Weibull, or Weibull distribution. All p-values are found to be less than 0.05, which shows strong evidence against the null hypotheses. Thus, the PL-W model has a statistical advantage over the other models in fitting this real data.

Table 9 LR tests for third real data

Hypothesis			Statistics w	p-value
H_0 : Normal	vs	H_1 : PL-N	2.6198	0.4540
H_0 : Lomax-Normal	vs	H_1 : PL-N	2.2414	0.1343

From Table 9, the LR test were applied to evaluate whether the PL-N distribution provides a significantly better fit than its sub-models. The comparison between the PL-N and normal distributions yielded a test statistic of 2.6198 with a p-value of 0.4540, while the comparison with the Lomax-normal distribution resulted in a statistic of 2.2414 and a p-value of 0.1343. In both cases, the p-values are greater than 0.05, indicating that there is insufficient evidence to reject the null hypotheses. Hence, the normal and Lomax-normal distributions are considered adequate for describing the third real data.

We applied the PL-B distribution to the soil fertility data sets and subsequently compared it with the fits of the B distribution. Table 10 gives the maximum likelihood estimates of the model parameters and the measures of AIC and BIC, respectively. Based on the lowest AIC and BIC values, the PL-B distribution could be selected as the best model among the other distributions.

The PL-B distribution shows itself as the superior model when compared to other models, primarily due to its lowest AIC and BIC.

We employed the PL-W distribution to fit the data set of bladder cancer patients and conducted a comparison with fits from various other models, including Weibull-Weibull, Zubair-Weibull, and Weibull distribution. To evaluate the goodness of fit, we utilized the Kolmogorov-Smirnov (K-S) test from the R AdequacyModel package by Marinho et al. (2016). The results are presented in Table 11.

The PL-W distribution demonstrates the best performance with the lowest values of -2log$L$, K-S, AIC, and BIC. This indicates that the PL-W distribution provides the best fit for Data II compared to the other three models.

Table 10 Some statistic values of each models for Data I

	Distributions	
	B	PL-B
MLEs	-	$\hat{\alpha}$ = 25.2318
	-	$\hat{\beta}$ = 0.9662
	-	$\hat{\lambda}$ = 27.9103
	$\hat{\delta}$ = 5.7257	$\hat{\delta}$ = 6.6492
	$\hat{\phi}$ = 34.9442	$\hat{\phi}$ = 43.1024
-2log$L$	394.1786	393.7588
K-S	0.0971	0.0870
p-value	0.1792	0.2871
AIC	404.1786	403.7588
BIC	418.4388	418.0189

Table 11 Some statistic values of each models for Data II

	Distributions			
	Weibull	Zubair-Weibull	Weibull-Weibull	PL-W
MLEs	-	-	-	$\hat{\alpha}$ = 2.0702
	-	-	$\hat{\alpha}$ = 5.9868	$\hat{\beta}$ = 6.0152
	-	$\hat{\alpha}$ = 211.7543	$\hat{\beta}$ = 63.0845	$\hat{\lambda}$ = 5.1649
	$\hat{\theta}$ = 1.0478	$\hat{\theta}$ = 4.9929	$\hat{\theta}$ = 0.6608	$\hat{\theta}$ = 0.2373
	$\hat{k}$ = 9.5607	$\hat{k}$ = 0.1484	$\hat{k}$ = 0.0120	$\hat{k}$ = 3.8098
-2log$L$	828.1738	851.4184	828.4496	819.4798
K-S	0.0700	0.0968	0.0708	0.0351
p-value	0.557	0.1811	0.5416	0.9975
AIC	832.1738	857.4184	836.4496	829.4798
BIC	837.8778	865.9744	847.8577	843.7399

Table 12 Some statistic values of each models for Data III

		Distributions	
	Normal	Lomax-Normal	PL-N
MLEs	-	-	$\hat{\alpha}$ = 374.988
	-	$\hat{\beta}$ = 16806.37	$\hat{\beta}$ = 5.8493
	-	$\hat{\lambda}$ = 11825.13	$\hat{\lambda}$ = 5.4788
	$\hat{\mu}$ = 5.3499	$\hat{\mu}$ = 6.2848	$\hat{\mu}$ = 11.7225
	$\hat{\sigma}$ = 2.8170	$\hat{\sigma}$ = 3.0846	$\hat{\sigma}$ = 17.4584
-2log$L$	127.64	127.2616	125.0202
K-S	0.1682	0.1639	0.1142
p-value	0.454	0.4874	0.8867
AIC	131.64	135.2616	135.0202
BIC	134.1561	140.2939	141.3107

The PL-N distribution outperforms the normal distribution in fitting Data III, as evidenced by its lower -2log$L$ and K-S values. This suggests that the PL-N distribution is more suitable for explaining the characteristics of Data III. The results, as depicted in Figure 2, clearly demonstrate that the PL-G family of distributions outperforms the normal and Lomax-normal distributions.

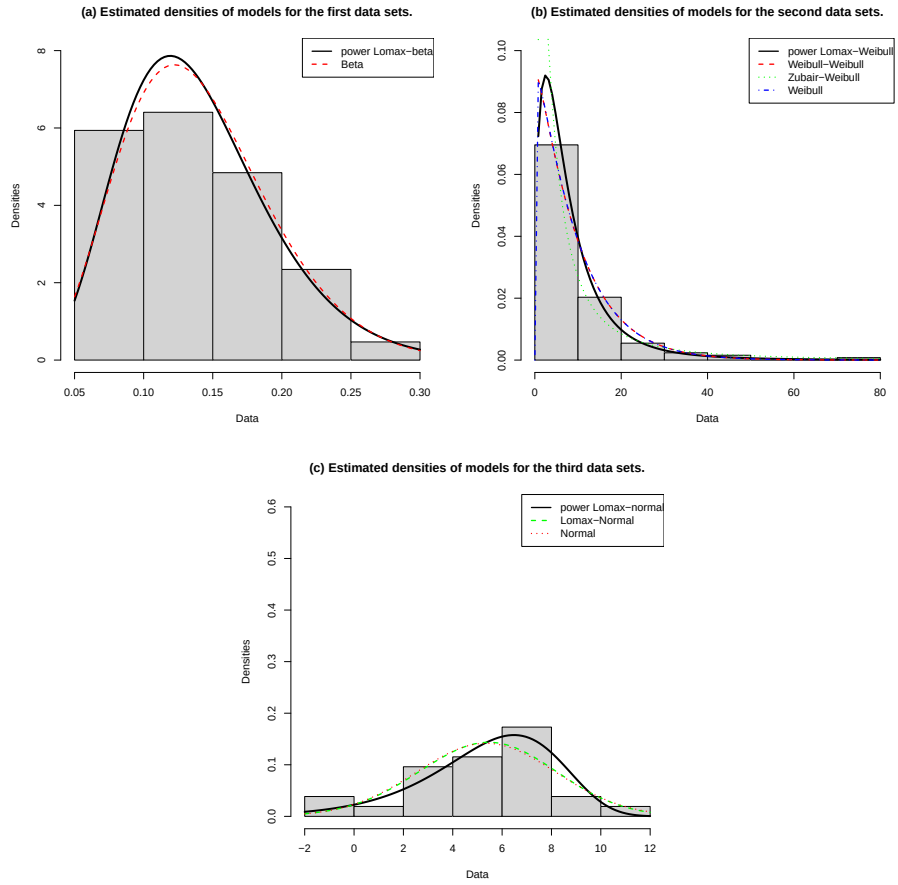


Figure 2 Estimated densities of models for the data sets

8. Discussion and Conclusion

In this paper, we introduce a new family of distributions known as the PL-G family of distributions. This family is constructed based on the T-X family of continuous distributions, utilizing the power Lomax distribution with positive parameters α, β, and λ as a generator. The advantage of adding parameters is that the newly constructed distributions can better fit the data than the existing distributions, but it affects the complexity of the function. The paper encompasses an extensive exploration of the statistical properties of the PL-G family of distributions, including the derivation and discussion of key elements such as the quantile function, skewness and kurtosis, order statistics, and moments. Furthermore, we present the maximum likelihood estimators of the model parameters. The efficiency and significance of this newly generated family are thoroughly examined using real data sets.

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Appendix

Proof: The rth moment of the PL-G family of distributions are given below.

$$\begin{aligned} E(X^r) &= \int_{-\infty}^{\infty} x^r f(x)dx \\ &= \int_{-\infty}^{\infty} x^r \left[\alpha\beta\lambda^{\alpha}\frac{g(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}\Big(-\log\Big(1-G(x;\boldsymbol{\xi})\Big)\Big)^{\beta-1}\right. \\ &\quad \left.\times\left[\Big(\lambda+\Big(-\log\big(1-G(x;\boldsymbol{\xi})\big)\Big)\Big)^{\beta}\right]^{-\alpha-1}\right]dx \end{aligned}$$

$$
\begin{aligned}
&= \alpha\beta\lambda^{\alpha}\int_{-\infty}^{\infty} x^{r}\left[\frac{g(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}\Big(-\log\big(1-G(x;\boldsymbol{\xi})\big)\Big)^{\beta-1}\right.\\
&\quad\left.\times\left[\Big(\lambda+\Big(-\log\big(1-G(x;\boldsymbol{\xi})\big)\Big)\Big)^{\beta}\right]^{-\alpha-1}\right]dx\\
&= \alpha\beta\lambda^{\alpha}\int_{-\infty}^{\infty} x^{r}\left[\frac{g(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}(w^{1/\beta})^{\beta-1}(\lambda+w)^{-\alpha-1}\right]dx\\
&= \alpha\beta\lambda^{\alpha}\int_{-\infty}^{\infty} x^{r}\left[\frac{g(x;\boldsymbol{\xi}}{1-G(x;\boldsymbol{\xi})}w^{1-1/\beta}\left(\sum_{j=0}^{-\alpha-1}\binom{-\alpha-1}{j}\lambda^{-\alpha-1-j}w^{j}\right)\right]dx\\
&= \alpha\beta\lambda^{\alpha}\int_{-\infty}^{\infty} x^{r}\left[\frac{g(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}\left(\sum_{j=0}^{-\alpha-1}\binom{-\alpha-1}{j}\lambda^{-\alpha-1-j}w^{1-\frac{1}{\beta}+j}\right)\right]dx\\
&= \alpha\beta\lambda^{\alpha}\int_{-\infty}^{\infty} x^{r}\left[\frac{g(x;\boldsymbol{\xi})}{e^{-w^{1/\beta}}}\left(\sum_{j=0}^{-\alpha-1}\binom{-\alpha-1}{j}\lambda^{-\alpha-1-j}w^{1-\frac{1}{\beta}+j}\right)\right]dx\\
&= \alpha\beta\lambda^{\alpha}\int_{-\infty}^{\infty} x^{r}\left[g(x;\boldsymbol{\xi})e^{w^{1/\beta}}\left(\sum_{j=0}^{-\alpha-1}\binom{-\alpha-1}{j}\lambda^{-\alpha-1-j}w^{1-\frac{1}{\beta}+j}\right)\right]dx\\
&= \alpha\beta\lambda^{\alpha}\int_{-\infty}^{\infty} x^{r}\left[g(x;\boldsymbol{\xi})\left(\sum_{j=0}^{-\alpha-1}\sum_{k=0}^{\infty}\binom{-\alpha-1}{j}\lambda^{-\alpha-1-j}1^{k}\frac{w^{1+(\frac{-1+k}{\beta})+j}}{k!}\right)\right]dx\\
E(X^{r}) &= \alpha\beta\lambda^{\alpha}\sum_{j=0}^{-\alpha-1}\sum_{k=0}^{\infty}\phi_{r}[\lambda_{i}(j,k)],
\end{aligned}
$$

$$
\text{where } \lambda_{i}(j,k) = \binom{-\alpha-1}{j}\lambda^{-\alpha-1-j}1^{k}\frac{w^{1+(\frac{-1+k}{\beta})+j}}{k!},\ \phi_{r}=\int_{-\infty}^{\infty}x^{r}g(x;\boldsymbol{\xi})dx
$$