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Adaptative Locally Asymptotically Optimal Test for Random Effects in GARMA Model

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Abstract

The main purpose of this paper is to develop a locally asymptotically optimal statistical *new* test for detecting the presence of random parameters in a Generalized Autoregessive Moving Average (GARMA) process. The purpose of the procedure study is to derive the *powerful* test for the hypothesis that the GARMA coefficients are constant overtime against the alternative that vary according to random effects. The asymptotic distribution and its tests properties are established under the Local Asymptotic Normality. It is shown that the proposed test statistic is; consistent, locally asymptotically optimal, performs better than the competing tests available in the literature, and constitutes a powerful technical tool for detecting the random effects in GARMA models. A simulation study was carried out to investigate the performance of this procedure. In fact, Monte Carlo method shows that the test has very good power for all cases considered. Additionally, a real data analysis is conducted to examine the performance of this procedure.

Keywords: Long memory time series, GARMA model, local asymptotic normality, pseudo-Gaussian test, rank test.

1. Introduction

In recent years, the field of statistics has witnessed a significant surge of interest in the area of time series modeling with changing frequency components. This trend has been particularly pronounced in the context of financial data analysis. Standard techniques are insufficient for unveiling latent features in time series data, as evidenced by the findings of (Thulasyammal Ramiah Pillai, 2015). The application of these models is not limited to specific fields; they have been employed in diverse contexts, including chemistry (Student, 1927), hydrology (Hurst, 1951), astronomical physics (Graf, 1983) and economics (Diebold and Rudebusch, 1989; Hassett, 1990; Porter-Hudak, 1990; Sowell, 1992).

Practical examples of GARMA models and their applications to analyze the volatility of a financial series with k-factors can be found in Henry et al. (1989). Generalized linear autoregressive moving average (GLARMA) models have also been considered of generalized state-space models for non-Gaussian time series, and the process of fitting these models using the R *glarma* package is described in Dunsmuir and Scott (2015). He present work focuses on the non-standard problem of

building a new test to discern between an ordinary GARMA-model and a random coefficient GARMA model (RCGARMA). Our primary interest lies in deriving asymptotically optimal tests for the aforementioned testing problems. The impetus for constructing these tests stems from the recognition that classical tests, such as the likelihood ratio (LR), and selection tools, including the Akaike (AIC) (Stoica and Selen, 2004) and Bayesian information criteria (BIC) (Schwarz, 1978), are asymptotic and may not be applicable to all scenarios.

Subsequently, we leverage the LAN property (Le Cam, 1986; Le Cam and Yang, 2000) by proposing an approach that involves randomizing the coefficient of GARMA models. The Random Coefficient Autoregressive (RCA) model has garnered significant attention in the extant literature, primarily due to its flexibility and analytical tractability. A notable contribution is the development of an efficient test for randomness in an RCA(p) model for $p \geq 1$ by Akharif and Hallin (2003). Our study is, to a certain extent, complementary to this work. We aim to derive both parametric and non-parametric tests, leveraging the Local Asymptotic Normality (LAN) framework.

In this paper, we focus on the GARMA(1,0) model with known parameters, defined as follows:

$$(1-\phi L)(1-2\eta L+L^2)^d X_t = \varepsilon_t, \tag{1}$$

where $(1-2\eta L+L^2)^d$ represents the long-memory Gegenbauer component, d represents integer differencing. X_t represents the observed process, L represents the Backshift operator, defined by $LX_t = X_{t-1}$ and ε_t represents the white noise with mean zero, finite variance and density represented by f. Alternatively, the model (1) can be expressed in the following form:

$$X_t = \sum_{n=1}^{\infty}[\phi C_{n-1}^{(-d)}(\eta) - C_n^{(-d)}(\eta)]X_{t-n} + \varepsilon_t, \tag{2}$$

where $(1-2\eta L+L^2)^d = \sum\limits_{n=0}^{\infty} C_n^{-d}(\eta)L^n$ and $C_n^{-d}(\eta) = \sum\limits_{k=0}^{[\frac{n}{2}]}(-1)^k \frac{\Gamma(n-k-d)}{\Gamma(-d)k!(n-2k)!}(2\eta)^{n-2k}$.

Under the alternative hypothesis, the model (2) is extended to the RCGARMA(1,0) model which takes the form:

$$X_t = \sum_{n=1}^{\infty}[(\phi+\sigma_\phi\phi_t)C_{n-1}^{(-d)}(\eta) - C_n^{(-d)}(\eta)]X_{t-n} + \varepsilon_t, \tag{3}$$

with ϕ_t is an i.i.d. sequence of random coefficients with density g, $\sigma_\phi \in \mathbb{R}^+$, $\phi \in \mathbb{R}$ is the autoregressive parameter and ϕ_t and ε_t are independent.

Our main contribution is to develop a test for the following hypothesis testing framework:

$$\begin{cases} \mathcal{H}_f^{(n)}(0) = \bigcup_f H_f^{(n)}(0) & : \sigma_\phi^2 = 0, \\ \\ \mathcal{H}_f^{(n)}(\sigma_\phi^2) = \bigcup_f H_f^{(n)}(\sigma_\phi^2) & : \sigma_\phi^2 > 0. \end{cases}$$

The present paper will commence with notations and certain technical assumptions that are required for the present study. In Section 3, the local asymptotic normality property will be established. Subsequently, in Sections 4 and 5, the optimal and pseudo-Gaussian tests will be examined. In Section 6, the focus will be on a correlogram-based version of LAN, while Section 7 will address a rank-based version. In conclusion, the results of the simulation will be presented in Section 8, and a real data analysis will be conducted in Section 9.

2. Assumptions and Notations

In order to develop the asymtotic theory, we need the following assumptions and notations. Denoted by $P_{f,\sigma_\phi}^{(n)}$ the probability distribution of a time serie $X^{(n)} = (X_1^{(n)}, ..., X_n^{(n)})$ generated by (3).

In order to prove the local asymptotic normality, some technical assumptions are necessary. The following assumptions are made on f and on the model parameters.

(A_1) The polynomial $\Phi(z) = 1 - \phi z$ does not have a root in the unit disk $D = \{z \in \mathbb{C} : |z| \leq 1\}$.

(A_2) $|\eta| < 1$ and $0 < d < \frac{1}{2}$.

(A_3) The innovation density f is such as:

(i) $f(x) > 0$, for $x \in \mathbb{R}$, $\int x f(x)\, dx = 0$ and $\int x^4 f(x)\, dx < \infty$.

(ii) f is absolutely continuous with the derivative $\dot{f}$ and a finite Fisher information:

$$l_\Phi := \int \phi_f^2 f(x)\, dx < \infty \text{ where } \Phi_f := -\frac{\dot{f}}{f}.$$

(iii) $\dot{f}$ is also absolutely continuous with its derivative $\ddot{f}$ continuous to the right, we take $\Psi_f := \frac{\ddot{f}}{f}$, suppose that $l_\Psi := \int \Psi_f^2 f(x)\, dx < \infty$.

Letting $\sigma^2 := \int x^2 f(x)\, dx$ and $(m_3)_f := \int x^3 f(x)\, dx$. Note that under (A_2), $\sigma^2 < \infty$, $(m_3)_f < \infty$ and

$$\int \Psi_f(x) f(x) dx = \int \Psi_f(x) x f(x)\, dx = 0.$$

On the density g of random perturbations Φ_t, we created the following hypothesis:

(A_4) $\int v g(v)\, dv = 0$ and $\int v^2 g(v)\, dv = 1$.

3. Local Asymptotic Normality

In order to derive the asymptotic distribution of the proposed test statistic and establish its theoretical properties, we apply the Local Asymptotic Normality (LAN) framework. To this end, we analyze the behavior of the log-likelihood function under local perturbations of the null hypothesis, utilising the notations and assumptions defined in the previous section.

Assuming that X_t=0, $\forall t \leq 0$. Let $Z_t := (1 - \phi L)(1 - 2\eta L + L^2)^d X_t$ the standardised residual as outlined in the Appendix. Under $\mathcal{H}_f^{(n)}(0)$, Z_t coincides with ε_t.

Consider the local parameter sequences $\theta^{(n)}$ defined by:

$$\theta^{(n)} = \theta + n^{-\frac{1}{2}}\tau,$$

with $\theta = 0$ and $\tau \in \mathbb{R}^+$.

Denoted by $\Lambda_f^{(n)}(\delta)$ the logarithm of the likelihood ratio of $\mathcal{H}_f^{(n)}(n^{-\frac{1}{2}}\delta)$ to $\mathcal{H}_f^{(n)}(0)$, defined by:

$$\Lambda_f^{(n)}(\delta) = \sum_{t=1}^{n} log(\int f(Z_t - n^{-\frac{1}{4}}\sqrt{\delta} v L(1 - 2\eta L + L^2)^d X_t) g(v) dv)) - \sum_{t=1}^{n} log(f(Z_t)) + o_P(1) \quad (4)$$

We get the following proposition:

Proposition 1 *Let (A1)-(A4) hold, then under $\mathcal{H}_f^{(n)}(0)$ and for all τ we have:*

i)

$$\Lambda_f^{(n)}(\delta) = \tau \Delta_f^{(n)} - \frac{1}{2}\tau^2 \Gamma_f + o_p(1), \quad (5)$$

with central sequence:

$$\Delta_f^{(n)} = \frac{n^{-\frac{1}{2}}}{2} \sum_{t=1}^{n} \Psi_f(Z_t)(\sum_{k=1}^{t-1} C_{k-1}^{(-d)}(\eta) X_{t-k})^2. \quad (6)$$

ii) $\Delta_f^{(n)}$ *is asymptotically normal with zero mean and variance* Γ_f *given by:*

$$\Gamma_f = \frac{1}{4} l_\Psi((1-\phi^4)^{-1}((m_4)_f - \sigma^4) + 2(1-\phi^2)^{-2}\sigma^4). \tag{7}$$

Proof: The proof of the LAN result in the proposition is based on Lemma 1 of Swensen (1985). Let: $\zeta_t^{(n)} = -\frac{n^{-\frac{1}{2}}}{2}\frac{1}{2}\sqrt{\delta}\Psi_f(Z_t)\left(\sum_{k=1}^{\infty} C_{k-1}^{(-d)}(\eta)X_{t-k}\right)^2$ and $\eta_t^{(n)} = \left[f\left(Z_t^{\infty}(n^{-\frac{1}{2}}\delta)\right)/f(Z_t)\right]^{\frac{1}{2}} - 1$.

In order to establish the LAN property, it is necessary to verify that the log-likelihood ratio satisfies the structure required by the LAN framework under local alternatives. This is achieved by verifying that the six regularity conditions introduced by Swensen are satisfied in our model:

$C_1)$ $\lim\limits_{n\to+\infty} \sum\limits_{t=1}^{n} E[(\zeta_t^{(n)} - \eta_t^{(n)})^2] = 0.$

$C_2)$ $sup\sum\limits_{n}^{n} \sum\limits_{t=1}^{n} E[(\zeta_t^{(n)})^2] < \infty.$

$C_3)$ $\max\limits_{1\le t\le n} |.\zeta_t^{(n)}| = o_p(1).$

$C_4)$ $\sum\limits_{t=1}^{n} (\zeta_t^{(n)})^2 - \frac{(\tau^n)^2}{4} = o_p(1).$

$C_5)$ $\sum\limits_{t=1}^{n} E[(\zeta_t^{(n)})^2 I[|\zeta^{(n)} > \frac{1}{2}|]|\mathcal{A}_t^{(n)}] = o_p(1).$

$C_6)$ $\sum\limits_{t=1}^{n} E[\zeta_t^{(n)}|\mathcal{A}_{t-1}^{(n)}] = 0$ p.s.

See Appendix.

4. Optimal Parametric Tests

According to LAN Property, the following statistic is considered the most powerful test based on $\Delta_f^{(n)}$.

$$T_f^{(n)} := (\Gamma_f)^{-\frac{1}{2}} \Delta_f^{(n)}. \tag{8}$$

So

$$T_f^{(n)} = n^{-\frac{1}{2}}(l_\Psi((1-\phi^4)^{-1}((m_4)_f - \sigma^4) + 2(1-\phi^2)^{-2}\sigma^4))^{-\frac{1}{2}} \sum_{t=1}^{n} \Psi_f(Z_t)(\sum_{k=1}^{t-1} C_{k-1}^{(-d)}(\eta)X_{t-k})^2. \tag{9}$$

Then, we have the following proposition:

Proposition 2 *Let (A1)-(A5) hold. Then for all* τ *:*

i) $T_f^{(n)}$ *is asymptotically normal, with a mean of zero under* $\mathcal{H}_f^{(n)}(0)$*, mean* $(\Gamma_f)^{\frac{1}{2}}\tau$ *under* $\mathcal{H}_f^{(n)}(n^{-\frac{1}{2}}\delta)$ *and variance one under both.*

ii) The sequence of the test which rejects the null hypothesis $\mathcal{H}_f^{(n)}(0)$ *when:*

$$T_f^{(n)} > z_{1-\alpha}$$

is locally asymptotically the most powerful at the α *level against the alternative hypothesis.*

5. Pseudo-Gaussian Tests

In this part, we will look for another version of the central sequence based on the Gaussian hypothesis and extend the validity of this optimal Gaussian test at densities g in a large density class; It's the pseudo-Gaussian test.

In the context of a Gaussian density, denoted by $f = f_{\mathcal{N}}$ the central sequence is specified as follows:

$$\Delta_{f_{\mathcal{N}}}^{(n)} = \frac{n^{-\frac{1}{2}}}{2} \sum_{t=1}^{n} (Z_t^2 - 1)(\sum_{k=1}^{t-1} C_{k-1}^{(-d)}(\eta) X_{t-k})^2. \tag{10}$$

We define

$$\Delta_g^{(n)} = \frac{n^{-\frac{1}{2}}}{2} \sum_{t=1}^{n} (Z_t^2 - m_2^{(n)} - (m_1^{(n)})^2)(\sum_{k=1}^{\infty} (\frac{1}{\phi})^{k-1}(Z_{t-k} - m_1^{(n)}))^2 \tag{11}$$

with $m_1^{(n)} = \frac{1}{n}\sum_{t=1}^{n} Z_t$ and $m_2^{(n)} = \frac{1}{n}\sum_{t=1}^{n}(Z_t - m_1^{(n)})^2$ are, respectively, estimators of $\mu_1(g) := \int_{\mathbb{R}} z g(z) dz$ and $\mu_2(g) := \int_{\mathbb{R}} (z - \mu_1(g))^2 g(z) dz$.

Decomposing $Z_{t-k} - m_1^{(n)}$ into $(Z_{t-k} - \mu_1^{(n)}(g)) + (\mu_1(g) - m_1)$ and $(Z_t^2 - m_2 - (m_1)^2)$ into $(Z_t^2 - \mu_2(g) - (\mu_1(g))^2) + (\mu_2(g) - m_2) + ((\mu_1(g))^2 - (m_1)^2))$.

Then as $n \to +\infty$, we have:

$$\Delta_g^{*(n)} = \frac{n^{-\frac{1}{2}}}{2} \sum_{t=1}^{n} (Z_t^2 - \mu_2(g) - (\mu_1(g))^2)(\sum_{k=1}^{\infty} (\frac{1}{\phi})^{k-1}(Z_{t-k} - \mu_1(g)))^2 + o_P(1). \tag{12}$$

So under $\mathcal{H}_g^{(n)}(0)$, $\Delta_g^{(n)}$ is asymptotically normal, with a mean zero and variance:

$$\Gamma_g = \frac{\sigma_2^2}{4}((1-\phi^4)^{-1})(\mu_4 - (\mu_2)^2)) + 2(1-\phi^2)^{-2}(\mu_2)^2)) \tag{13}$$

with $\mu_4 = \int_{\mathbb{R}} (z - \mu_1(g))^4 g(z) dz$ and $\sigma_2^2 = \int_{\mathbb{R}} (z^2 - \mu_2(g) - (\mu_1(g))^2)^2 g(z) dz$.

So the statistic of the pseudo-Gaussian test is given by:

$$T_g^{*(n)} = (\Gamma_g)^{-\frac{1}{2}} \Delta_g^{*(n)} = \frac{\Delta_g^{*(n)}}{[\sigma_2^2(1-\phi^4)^{-1}(\mu_4 - \mu_2^2) + 2(1-\phi^2)^{-2}\mu_2^2]^{1/2}}. \tag{14}$$

Practically, the pseudo-Gaussian test is based on:

$$T_g^{(n)} = \frac{\Delta_g^{(n)}}{(S_2^2((1-\phi^4)^{-1})(m_4^{(n)} - (m_2^{(n)})^2) + 2(1-\phi^2)^{-2}(m_2^{(n)})^2))^{\frac{1}{2}}} \tag{15}$$

with

- $S_2^2 = \frac{1}{n}\sum_{t=1}^{n}(Z_t^2 - m_2^{(n)} - (m_1^{(n)})^2)^2$ is the variance of $(Z_t^2 - m_2^{(n)} - (m_1^{(n)})^2)$.
- $m_4^{(n)} = \frac{1}{n}\sum_{t=1}^{(n)}(Z_t - m_1^{(n)})^4$ is the 4th central moment.

Then, under $\mathcal{H}_g^{(n)}(0)$ we have $T_g^{*(n)} - T_g^{(n)} = o_P(1)$ when $n \to \infty$. We then have the following result.

Proposition 3 *Let (A1)-(A5) hold for g. Then for all τ,*

i) $T_g^{(n)}$ is asymptotically normal, with a mean of zero under $\mathcal{H}_g^{(n)}(0)$, mean $(\Gamma_g)^{\frac{1}{2}}\tau$ under $\mathcal{H}_g^{(n)}(n^{-\frac{1}{2}}\delta)$ and variance one under both.

ii) The sequence of the test which rejects the null hypothesis $\mathcal{H}_g^{(n)}(0)$ when $T_g^{(n)} > z_{1-\alpha}$ is locally asymptotically the most powerful at the α level against the alternative hypothesis.

6. A Correlogram-Based Version of LAN.

This LAN result is sufficient for several cases, but generally, it is insufficient because the central sequence $\Delta_f^{(n)}$ transmits little intuitive content, so it is difficult to interpret.

Let us determine another LAN property formulation with another center sequence $\underline{\Delta}_f^{(n)}$ which is unlike $\Delta_f^{(n)}$ is measurable with respect to residuals.

This sequence is based on a correlogram. It's based on the residual autocorrelation coefficient which is defined as:

$$r_{f;\,k}^{(n)} := (n-k)^{-1} \sum_{t=k+1}^{n} \Phi_f(Z_t) Z_{t-k} (\sigma^2 l_\Phi(f)). \tag{16}$$

Autocorrelation coefficients are not sufficient for the detection of nonlinear series characteristics. So, we use a third-order generalization of the classical correlograms, namely, the cubic autocorrelations, which are defined by:

$$q_{f;\,kl} = \begin{cases} (n-k)^{-1} \sum\limits_{t=k+1}^{n} \Psi_f(Z_t) Z_{t-k}^2 [l_\Psi(f) m_4(f)]^{-\frac{1}{2}}, & k = l \\ (n-l)^{-1} \sum\limits_{t=l+1}^{n} \Psi_f(Z_t) Z_{t-k} Z_{t-l} [\sigma^4 l_\Psi(f)]^{-\frac{1}{2}}, & k < l \end{cases} \tag{17}$$

So we have the central sequence is given by:

$$\underline{\Delta}_f^{(n)} = \frac{1}{2} \sum_{k=1}^{n-1} (\frac{1}{\phi})^{2k-2} (n-k)^{\frac{1}{2}} q_{f;\,kk}^{(n)} [l_\Psi(f) m_4(f)]^{\frac{1}{2}} + \sum_{1 \le k < j \le n-1} (\frac{1}{\phi})^{k+j-2} (n-j)^{\frac{1}{2}} q_{f;\,kj}^{(n)} [\sigma^4 l_\Psi(f)]^{\frac{1}{2}}.$$

The LAN result of the proposition always remains valid for $\mathcal{H}_f^{(n)}(0)$ only and not for $\mathcal{H}_f^{(n)}(n^{-\frac{1}{2}}\delta)$more general as planned.

In fact, $\underline{\Delta}^{(n)}$ has a mean zero under the innovation density f, but in general, under $g \neq f$ is no longer zero and depends on g.

Also, the pseudo-Gaussian methods are valid under the general null hypothesis $\mathcal{H}^{(n)}$, but not reaching optimality, and in this case, we cannot reach parametric optimality, but there is another way to bypass the unknown of the underlying density and achieve semi-parametric efficiency; it is the semi-parametrically efficient pseudo-Gaussian method based on rank.

7. Rank-Based Methods

Rank-based methods are widely recognised for their robustness and efficiency in statistical inference, particularly when the underlying distribution differs from the Gaussian assumption.
In the context of testing for random effects in GARMA models, we propose a rank-based approach that preserves optimality properties across a broad range of distributions while improving robustness to model misspecification.

7.1. A rank-based version of central sequence

To construct a rank-based version of the central sequence, we denote by $R_t^{(n)}(\theta)$ the rank of $Z_t^{(n)}$. The conditional expectation of the central sequence $\Delta_f^{(n)}$ at these ranks gives a semi-parametrically efficient version of the central sequence:

$$\overline{\Delta}_f^{(n)} := E[\Delta_f^{(n)} / R_t^{(n)}]. \tag{18}$$

Then we have:

$$\overline{\Delta}_f^{(n)} = \frac{n^{-\frac{1}{2}}}{2} \sum_{t=1}^{n} (\Psi_f(F^{-1}(\frac{R_t^{(n)}}{n+1}) - \overline{\Psi}_f^{(n)}) (\sum_{k=1}^{t-1} C_{k-1}^{(-d)}(\eta) X_{t-k})^2, \tag{19}$$

with $\overline{\Psi}_f^{(n)} := \frac{1}{n}\sum_{t=1}^{n}\Psi_f(F^{-1}(\frac{t}{n+1}))$. Let $s_f^{2(n)} = \overline{\Psi_f^2}^{(n)} - (\overline{\Psi}_f^{(n)})^2$ where $\overline{\Psi_f^2}^{(n)} = \frac{1}{n}\sum_{t=1}^{n}\Psi_f^2(F^{-1}(\frac{t}{n+1}))$. therefore

$$\Gamma_f^{(n)} = \frac{1}{4}s_f^{2(n)}((1-\phi^4)^{-1}((m_4)_f - \sigma^4) + 2(1-\phi^2)^{-2}\sigma^4). \tag{20}$$

7.2. Local asymptotic linearity

The parameter θ plays the role of a nuisance, and must therefore be estimated under the null hypothesis $\mathcal{H}^{(n)}(0)$.

To control the effects of the substitution of an estimated value $\widehat{\theta}$ by the exact value in the central sequence based on the rank $\overline{\Delta}_f$, we need a local asymptotic linearity result.

Let's define the cross information coefficients:

$$I_\Psi(f,g) := \int_0^1 \Psi_f(F^{-1}(u))\Psi_g(G^{-1}(u))\,\mathrm{d}u,$$

and

$$\sigma(f^a, g^b) := \int_0^1 (F^{-1}(v))^a (G^{-1}(v))^b\,\mathrm{d}v,$$

with G is the distribution function associated with g.

Then the variance Γ_{fg} is given by

$$\Gamma_{fg} = \frac{1}{4}I_\Psi(f,g)((1-\phi^4)^{-1}(\sigma(f^2,g^2) - \sigma^2(f,g)) + 2(1-\phi^2)^{-2}\sigma^2(f,g)). \tag{21}$$

Remark 1 For $f = g$, $I_\Psi(f,f) = l_\Psi(f)$, $\sigma(f,f) = \sigma^2$ and $\sigma(f^2,f^2) = (m_4)_f$.

So we have for $\overline{\Delta}_f^{(n)}$ the following proposition:

Proposition 4 *Suppose the assumptions (A2) and (A3) are checked for f and g, then for all τ we have:*

i) Under $\mathcal{H}_g^{(n)}(0)$, as $n \to \infty$:

$$\overline{\Delta}_f^{(n)} = E[\Delta_f^{(n)}/R_1^{(n)}, ..., R_n^{(n)}] + o_{L^2}(1) = \Delta_{f,g}^{*(n)} + o_{L^2}(1) \tag{22}$$

with

$$\Delta_{f,g}^{*(n)} := \frac{n^{-\frac{1}{2}}}{2}\sum_{t=1}^{n}\Psi_f(F^{-1}(G(Z_t))(\sum_{k=1}^{t-1}C_{k-1}^{(-d)}(\eta)X_{t-k})^2. \tag{23}$$

ii) Under $\mathcal{H}_g^{(n)}(0)$, $\overline{\Delta}_f^{(n)}$ has mean zero and variance

$$\Gamma_f^{(n)} = \frac{1}{4}s_f^{2(n)}((1-\phi^4)^{-1}((m_4)_f - \sigma^4) + 2(1-\phi^2)^{-2}\sigma^4) = \Gamma_f + o(1). \tag{24}$$

iii) $\overline{\Delta}_f^{(n)}$ is asymptotically normal with mean zero under $\mathcal{H}_g^{(n)}(0)$, mean $\Gamma_{f,g}\tau$ under $\mathcal{H}_g^{(n)}(n^{-\frac{1}{2}}\delta)$ and variance Γ_f under both.

7.3. Optimal rank test

To enhance robustness, we construct a rank-based version of the test statistic, defined as follows:

$$\overline{T}_f := (\Gamma_f^{(n)})^{-\frac{1}{2}}\overline{\Delta}_f^{(n)}. \tag{25}$$

So we have the following result:

Proposition 5 *Suppose the assumptions (A1)-(A4) are verified, then for any τ we have:*

i) $\overline{T}_f^{(n)}$ is asymptotically normal, with a mean zero under $\mathcal{H}_g^{(n)}(0)$, mean $\tau\Gamma_{fg}(\Gamma_f)^{-\frac{1}{2}}$ under $\mathcal{H}_g^{(n)}(n^{-\frac{1}{2}}\delta)$ and variance one under both.

ii) The sequence of the test which rejects the null hypothesis $\mathcal{H}_f^{(n)}(0)$ when $\overline{T}_f^{(n)} > z_{1-\alpha}$ is locally asymptotically the most powerful at the α level against the alternative hypothesis.

7.4. The Wilcoxon, van der Waerden and Laplace test statistics

The rank test can be applied to several test statistics. In this study, we focus on four standard cases: the Wilcoxon test with logistic density, the van der Waerden test with normal density, the Laplace test with double-exponential density, and the Pseudo-Gaussian test. The selection of these statistics was made on the basis that they correspond to well-established score functions in the LAN framework, thereby enabling a meaningful comparison to be made between parametric and rank-based optimal tests under different innovation densities. The Pseudo-Gaussian test serves as a reference test, while the three rank-based tests are semi-parametric and robust under non-Gaussian innovations.

7.4.1. The Wilcoxon test

For the Wilcoxon test that uses logistic density, we have: $\Psi(F^{-1}(u)) = 6u(u-1)+1$, then the central sequence of this test written under the following form:

$$\overline{\Delta}_W := \frac{n^{-\frac{1}{2}}}{2}\sum_{t=1}^{n}(6(\frac{R_t^{(n)}}{n+1})(\frac{R_t^{(n)}}{n+1}-1)-\overline{\Psi}_W^{(n)})(\sum_{k=1}^{t-1}C_{k-1}^{(-d)}(\eta)X_{t-k})^2,$$

with

$$\overline{\Psi}_W^{(n)} = \frac{1}{n}\sum_{t=1}^{n}(6(\frac{t}{n+1})(\frac{t}{n+1}-1)).$$

Then we have the Wilcoxon test statistic is given by:

$$\overline{T}_W^{(n)} := \frac{2\overline{\Delta}_W^{(n)}}{(S_W^2((1-\phi^4)^{-1}((m_4)_f-\sigma^4)+2(1-\phi^2)^{-2}\sigma^4))^{\frac{1}{2}}},$$

with

$$S_W^2 = \frac{1}{n}\sum_{k=1}^{n}(6(\frac{k}{n+1})(\frac{k}{n+1}-1)+1)^2-\frac{1}{n^2}(\sum_{k=1}^{n}(6(\frac{k}{n+1})(\frac{k}{n+1}-1)+1))^2.$$

7.4.2. The van der Waerden test

The van der Waerden test uses the standard normal density. So we have $\Psi(F^{-1}(u)) = (\Phi^{-1}(u))^2 - 1$, where Φ^{-1} is the standard normal distribution function. Therfore the van der Waerden test statistic is given by:

$$\overline{T}_{vdw}^{(n)} := 2\frac{\overline{\Delta}_{vdw}^{(n)}}{(S_{vdw}^2((1-\phi^4)^{-1}((m_4)_f-\sigma^4)+2(1-\phi^2)^{-2}\sigma^4))^{\frac{1}{2}}}$$

where $S_{vdw}^2 = \frac{1}{n}\sum_{k=1}^{n}((\Phi^{-1}(\frac{k}{n+1}))^2-1)^2-\frac{1}{n^2}(\sum_{k=1}^{n}(\Phi^{-1}(\frac{k}{n+1}))^2-1))^2$, and

$$\overline{\Delta}_{vdw} := \frac{n^{-\frac{1}{2}}}{2}\sum_{t=1}^{n}((\Phi^{-1}(\frac{R_t^{(n)}}{n+1}))^2-\overline{\Psi}_{vdw}^{(n)})(\sum_{k=1}^{t-1}C_{k-1}^{(-d)}(\eta)X_{t-k})^2,$$

with

$$\overline{\Psi}_{vdw}^{(n)} = \frac{1}{n}\sum_{t=1}^{n}\Phi^{-1}(\frac{t}{n+1})^2.$$

7.4.3. The Laplace test

For Laplace test (with Double exponential density (De)), we have:

$$\Psi(u) = 2e^{|u|} \text{ and } F^{-1}(u) = log(2u)\mathbb{1}_{]0,\frac{1}{2}]} - log(2-2u)\mathbb{1}_{[\frac{1}{2},1[}.$$

So the statistic of Laplace test is given by

$$\overline{T}_L^{(n)} := \frac{2\overline{\Delta}_L^{(n)}}{(S_L^2((1-\phi^4)^{-1}((m_4)_f - \sigma^4) + 2(1-\phi^2)^{-2}\sigma^4))^{\frac{1}{2}}},$$

where

$$S_L^2 = \frac{1}{n}\sum_{k=1}^{n}\left(2e^{|F^{-1}\left(\frac{k}{n+1}\right)|}\right)^2 - \frac{1}{n^2}\left(\sum_{k=1}^{n}2e^{|F^{-1}\left(\frac{k}{n+1}\right)|}\right)^2,$$

and

$$\overline{\Delta}_L := n^{-\frac{1}{2}}\sum_{t=1}^{n}\left(e^{|F^{-1}\left(\frac{R_t^{(n)}}{n+1}\right)|} - \overline{\Psi}_L^{(n)}\right)\left(\sum_{k=1}^{t-1}C_{k-1}^{(-d)}(\eta)\,X_{t-k}\right)^2,$$

with

$$\overline{\Psi}_L^{(n)} = \frac{1}{n}\sum_{t=1}^{n}e^{|F^{-1}\left(\frac{t}{n+1}\right)|}.$$

8. Simulation

To examine the validity of our test we have performed a simulation study in R for the following model:

$$X_t = \sum_{n=1}^{\infty}((\phi + \sigma_\phi\phi_t)C_{n-1}^{(-d)}(\eta) - C_n^{(-d)}(\eta))X_{t-n} + \varepsilon_t, t = 1, ...n$$

where:

- $\phi = \frac{1}{2}$, $\eta = \frac{1}{4}$ and $d = \frac{1}{3}$.
- The ϕ_t's are i.i.d. Gaussian with mean zero and variance one.
- $\sigma_\phi = 0$ for null hypothesis and $\sigma_\phi = 0.2, 0.4, ..., 1$ for increasing alternative.
- The (ε_t)'s are iid with a density-Gaussian ($\mathcal{N}$), Logistic(l), Double Exponential(De), Student(t_5), the skew-Normal (s$\mathcal{N}(\delta)$) and skew-Student $st_5(\delta)$ densities.

These values were selected to satisfy the models stationarity and invertibility conditions and to represent a classical long-memory setting. They also provide a relevant scenario for evaluating random effects in GARMA processes.

We generated N=3000 independent samples of size n=100 for each combination of σ_ϕ and a distribution for the ε_t's.

For each of 6 densities we simulated at level α=0.05, the Pseudo-Gaussian test, the Van der Waerden test, the Wilcoxon test, and Laplace test. Rejection frequencies are reported in Figure 1.

As shown in Figure 1, all four tests; the Pseudo-Gaussian (PG), van der Waerden (VdW), Wilcoxon (W), and Laplace (L), maintain the nominal level under the null hypothesis ($\sigma_\phi = 0$), with rejection frequencies close to the significance level $\alpha = 0.05$ across all considered error distributions.

As σ_ϕ increases, indicating stronger departures from the null hypothesis, rejection rates rise for all tests, confirming their consistency. The proposed Pseudo-Gaussian test performs comparably to the rank-based tests under normal and light-tailed distributions, while the rank-based tests (VdW, Wilcoxon, and Laplace) outperform under heavy-tailed or skewed error distributions such as the double-exponential, Student-t_5, skew-normal, and skew-Student.

Overall, these results demonstrate that the proposed Pseudo-Gaussian test has strong power and robustness, and its performance is competitive with established rank-based procedures across a variety of realistic scenarios.

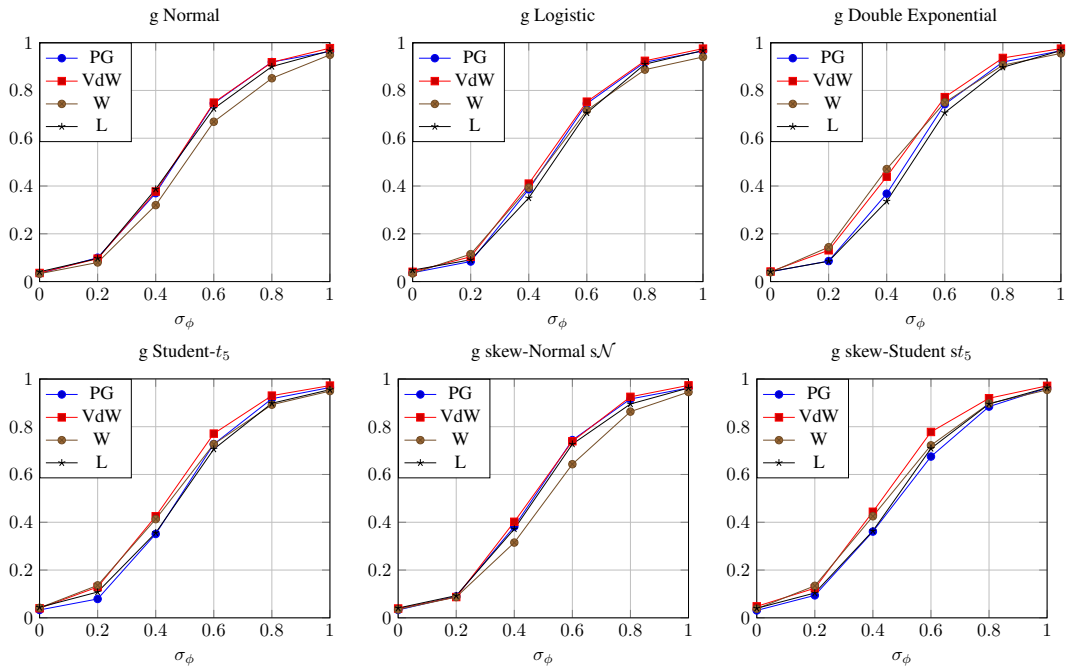


Figure 1 rejection frequencies (out of 3000 replications), at asymptotic levels α = 0.05, for σ_ϕ= 0 (null hypothesis), 0.2, 0.4, 0.6, 0.8, 1 (alternative hypotheses), with error density g that is Normal, Logistic ,Double Exponential, Student-t_5, skew-Normal and skew-Student of the Pseudo-Gaussian test, the Van der Waerden test ,the Wilcoxon test and Laplace test , for n = 100.

9. Real Data Analysis

To assess the practical relevance and robustness of the proposed Pseudo-Gaussian, Van der Waerden, and Wilcoxon tests, we apply them to a real-world time series dataset: the "Yearly Sunspot Data", available in the `R` statistical environment.

This dataset, originally compiled by Tong (1996), records the annual number of sunspots observed from 1749 to 1924. Sunspot activity is known to exhibit cyclic and nonlinear patterns, making it a classical benchmark in time series analysis, particularly in the study of long-range dependence and nonlinear dynamics.

The dataset contains 176 observations and is characterized by pronounced temporal dependence, making it suitable for evaluating the performance of tests designed to detect randomness or nonlinearity in the regression structure.

Before applying our tests, we perform a preliminary descriptive analysis of the series. A summary of its descriptive statistics is provided below:

Min 1	1st quantile	median	mean	3rd quantile	Max
0.00	14.12	40.55	45.03	64.28	154.40

Table 1 Descriptive statistics of the Sunspot data

In this section, we consider the GARMA(1,0) random coefficient model given by:

$$X_t = \sum_{n=1}^{\infty}((\phi + \sigma_\phi \phi_t)C_{n-1}^{(-d)}(\eta) - C_n^{(-d)}(\eta))X_{t-n} + \varepsilon_t \tag{26}$$

with:

- $\phi = 0.9$, $\eta = 0.8$ and $d = 0.4$ are the parameters of the model 26.
- $\phi_t \sim \mathcal{N}(0, 1)$.
- $\varepsilon_t \sim \mathcal{N}(0, 1)$.

To test the null hypothesis $\mathcal{H}_f^{(n)}(0) : \sigma_\phi^2 = 0$ against the alternative $\mathcal{H}_f^{(n)} : \sigma_\phi^2 > 0$, we will calculate the values of the Pseudo-Gaussian test, the Van der Wearden and the Wilcoxon test. The following table presents the result:

Test	calculated value	critical value
Pseudo-Gaussian	10.15	$z_{1-\alpha}$=1.644854
Van der Wearden	10.18	$z_{1-\alpha}$=1.644854
Wilcoxon	10.37	$z_{1-\alpha}$=1.644854

Table 2 Results of the Pseudo-Gaussian, Van der Waerden, and Wilcoxon tests for the Sunspot dataset

As we can see, all the calculated values are superior to the critical value $z_{1-\alpha}$. We can conclude that the null hypothesis should be rejected at the usual significance level ($\alpha = 0.05$).

These findings are consistent with the simulation study, as the analysis of real data also confirms the ability of the proposed test to detect random effects. In the Sunspot dataset, all three test statistics (Pseudo-Gaussian, van der Waerden and Wilcoxon) significantly exceeded the critical value, resulting in the rejection of the null hypothesis. This is consistent with the simulation results, in which the proposed test demonstrated high power to detect random effects under different innovation distributions. Therefore, the real data analysis reinforces the robustness and validity of our method.

10. Conclusion

The objective of this study was to detect randomness in the coefficients of GARMA models by testing the null hypothesis of constant parameters against the alternative of random effects. The development of locally asymptotically optimal tests based on the LAN property was undertaken, incorporating both pseudo-Gaussian and rank-based procedures. The simulation results and the real data analysis confirm the good performance of the proposed tests, which consistently show high power and robustness across different innovation distributions.

However, two limitations should be acknowledged. First, our analysis was restricted to the GARMA(1,0) model in order to simplify the theoretical framework and ensure tractability. Second, the simulation study was conducted under a single parameter configuration, mainly due to the heavy computational burden of Monte Carlo replications for multiple scenarios and distributions. These restrictions allowed us to present a clear proof of concept, but they also highlight opportunities for future work.

The innovation of this work lies in the construction of locally asymptotically optimal tests for detecting random effects in GARMA processes, which span both parametric and semi-parametric frameworks. In contrast to classical likelihood-based procedures or information criteria, our approach provides asymptotic optimality and robustness under a broad class of distributions. Potential applications of this methodology include financial econometrics, climatology, hydrology, and other areas of research where long-memory time series with random effects naturally arise.

Future work will focus on extending the approach to higher-order and multivariate GARMA models, developing efficient computational strategies to reduce the cost of large-scale simulations, and creating practical tools for data analysis.

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Appendix

We know that:

$$(1-2\eta L+L^2)^d=\sum_{n=0}^{\infty}C_n^{-d}(\eta)L^n, \tag{27}$$

with

$$C_n^{-d}(\eta)=\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\frac{\Gamma(n-k-d)}{\Gamma(-d)\,k!(n-2k)!}(2\eta)^{n-2k} \tag{28}$$

So,

$$(1-\phi L)(1-2\eta L+L^2)^dX_t=X_t+\sum_{n=1}^{\infty}C_n^{-d}(\eta)X_{t-n}-\phi\sum_{n=1}^{\infty}C_{n-1}^{-d}(\eta)X_{t-n}.$$

Letting:

$$\begin{aligned}Z_t^{\infty}&=(1-(\phi+n^{-1/4}\sqrt{\delta}v)L)(1-2\eta L+L^2)^dX_t\\&=Z_t-n^{-1/4}\sqrt{\delta}vL(1-2\eta L+L^2)^dX_t\end{aligned}$$

To determine the central sequence, we use the following formula:

$$\Delta_f^{(n)}=\frac{1}{2}\sum_{t=1}^{n}\frac{\partial^2_{\sqrt{\delta}}(\int f(Z_t^{\infty})g(v)dv)}{(\int f(Z_t^{\infty})g(v)dv)}|_{\tau=0}. \tag{29}$$

Proof of the proposition 1

Let's show the C_1-C_6 conditions:

Let:

$$U_t^{(n)}=n^{-\frac{1}{4}}(Z_t-Z_t^{\infty}(n^{-\frac{1}{2}}\delta))=n^{-\frac{1}{2}}\delta^{\frac{1}{2}}\sum_{k=1}^{\infty}C_{k-1}^{(-d)}(\eta)X_{t-k}.$$

And show that:

$$\lim_{n\to+\infty}\sum_{t=1}^{n}E[(U_t^{(n)})^2]<\infty.$$

We have:

$$\sum_{t=1}^{n}E[(U_t^{(n)})^2]=\sum_{t=1}^{n}n^{-1}\delta\sum_{k=1}^{\infty}\sum_{k'=1}^{\infty}C_{k-1}^{(-d)}(\eta)C_{k'-1}^{(-d)}(\eta)E(X_{t-k'}X_{t-k}),$$

$$\Rightarrow\sum_{t=1}^{n}E[(U_t^{(n)})^2]=\delta\sum_{k=1}^{\infty}\sum_{k'=1}^{\infty}C_{k-1}^{(-d)}(\eta)C_{k'-1}^{(-d)}(\eta)\mathrm{cov}(X_{t-k'},X_{t-k}),$$

with

$$\mathrm{cov}(X_{t-k'},X_{t-k})=(\Gamma_X)_{k,k'}=\sigma^2\sum_{u=|k-k'|}^{\infty}h_uh_{u-|k-k'|}.$$

$$\Rightarrow\lim_{n\to+\infty}\sum_{t=1}^{n}E[(U_t^{(n)})^2]<\infty$$

Let's move on to the demonstration of the 6 conditions:

C_1) We have:

$$\sum_{t=1}^{n} E[(\zeta_t^{(n)} - \eta_t^{(n)})^2] \leq 2\sum_{t=1}^{n} E\left[\left(\zeta_t^{(n)} + \frac{1}{2}\Phi_f(Z_t)U_t^{(n)}\right)^2\right] + 2\sum_{t=1}^{n} E\left[\left(\eta_t^{(n)} + \frac{1}{2}\Phi_f(Z_t)U_t^{(n)}\right)^2\right].$$

For all $c > 0$,

$$\Sigma_2^{(n)} = \sum_{t=1}^{n} E\left[\left(\eta_t^{(n)} + \frac{1}{2}\Phi_f(Z_t)U_t^{(n)}\right)^2\right] \Rightarrow \Sigma_2^{(n)} = E_1^{(n)} + E_2^{(n)},$$

where

$$E_1^{(n)} = \sum_{t=1}^{n} E\left[I\left[n^{-\frac{1}{2}}|U_t^{(n)}| \leq c\right]\left(\eta_t^{(n)} + \frac{1}{2}\Phi_f(Z_t)U_t^{(n)}\right)^2\right],$$

$$E_2^{(n)} = \sum_{t=1}^{n} E\left[I\left[n^{-\frac{1}{2}}|U_t^{(n)}| > c\right]\left(\eta_t^{(n)} + \frac{1}{2}\Phi_f(Z_t)U_t^{(n)}\right)^2\right].$$

From (ii) of Lemma 2.2 of Garel and Hallin (1995), we have:

$$E_1^{(n)} \leq E[(U_t^{(n)})^2]o_c(1) \quad \text{where } \lim_{n\to+\infty} o_c(1) = 0 \quad \Rightarrow \lim_{n\to+\infty} E_1^{(n)} = 0$$

And from (i) of Lemma 2.2 of Garel and Hallin (1995), we have for all $c > 0$:

$$E_2^{(n)} \leq \frac{1}{n}\sum_{t=1}^{n} E\left[I\left[n^{-\frac{1}{2}}|U_t^{(n)}| > c\right] n(U_t^{(n)})^2 l_\phi\right] \quad \text{and as } c \to \infty, I[\cdots] = 0 \quad \Rightarrow \lim_{n\to+\infty} E_2^{(n)} = 0.$$

Hence:

$$\lim_{n\to+\infty} \Sigma_2^{(n)} = 0 \quad \text{and similarly} \quad \lim_{n\to+\infty} \Sigma_1^{(n)} = 0 \quad \Rightarrow \lim_{n\to+\infty} \sum_{t=1}^{n} E[(\zeta_t^{(n)} - \eta_t^{(n)})^2] = 0$$

C_2) Let's show condition C_2.

$$\sum_{t=1}^{n} \mathbb{E}[(U_t^{(n)} - \eta_t^{(n)})^2] \leq \sum_{t=1}^{n} \mathbb{E}\left[\left(\zeta_t^{(n)} + \frac{1}{2}\Phi_f(Z_t)U_t^{(n)}\right)^2\right] + \sum_{t=1}^{n} \mathbb{E}\left[\left(\eta_t^{(n)} + \frac{1}{2}\Phi_f(Z_t)U_t^{(n)}\right)^2\right]$$

Similarly to C_1, split into parts:

$$\Sigma_2^{(n)} = \sum_{t=1}^{n} \mathbb{E}\left[I[n^{-1/2}|U_t^{(n)}| \leq c]\left(\eta_t^{(n)} + \frac{1}{2}\Phi_f(Z_t)U_t^{(n)}\right)^2\right]$$

$$+ \sum_{t=1}^{n} \mathbb{E}\left[I[n^{-1/2}|U_t^{(n)}| > c]\left(\eta_t^{(n)} + \frac{1}{2}\Phi_f(Z_t)U_t^{(n)}\right)^2\right].$$

Then using Lemma 2.2:

$$\lim_{n\to\infty} \Sigma_2^{(n)} = 0.$$

Thus:

$$\lim_{n\to\infty} \sum_{t=1}^{n} \mathbb{E}[(U_t^{(n)})^2] = 0.$$

C_3) Now we demonstrate condition C_3.

We have

$$\mathbb{E}[U_t^{(n)} \mid X] = 0 \quad \text{for all } t.$$

Let's compute the covariance:

$$\sum_{t=1}^{n} \mathbb{E}[U_t^{(n)} U_{t-k}^{(n)}] = 0 \quad \text{for all } k.$$

This implies that

$$\text{cov}(U_t^{(n)}, U_{t-k}^{(n)}) = 0.$$

This completes the proof for C_3.

C_4) For condition C_4, we need to show that:

$$\sum_{t=1}^{n} \mathbb{E}[(U_t^{(n)})^2] \leq C n^{-1/2},$$

where C is some constant.

This follows from the fact that:

$$\mathbb{E}[(U_t^{(n)})^2] \leq \frac{1}{n^{1/2}} \sum_{t=1}^{n} \mathbb{E}[U_t^{(n)}].$$

Thus, the condition is satisfied.

C_5) For condition C_5, we use the following:

$$\sum_{t=1}^{n} \mathbb{E}[(U_t^{(n)})^2] \leq \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}[(X_t)^2].$$

This ensures that

$$\lim_{n\to\infty} \sum_{t=1}^{n} \mathbb{E}[(U_t^{(n)})^2] < \infty.$$

This completes the proof for C_5.

C_6) Finally, for condition C_6, we need to show:

$$\sum_{t=1}^{n} \mathbb{E}[(U_t^{(n)})^2] = O(n^{-1/2}).$$

This condition is satisfied by the fact that

$$\mathbb{E}[(U_t^{(n)})^2] \leq n^{-1/2} \sum_{t=1}^{n} \mathbb{E}[U_t^{(n)}].$$

This proves that:

$$\lim_{n\to\infty} \sum_{t=1}^{n} \mathbb{E}[(U_t^{(n)})^2] = O(n^{-1/2}).$$

Thus, the proof for condition C_6 is completed.