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## A New Gamma Odd Lindley Generalized-G Family of Distributions with Applications

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### Abstract

In this study, we present a new and generalized family of distributions referred to as the Gamma Odd Lindley Generalized-G (GOLG-G) distribution. Some structural properties of the new family of distributions including hazard rate function, quantile function, moments, incomplete moments, distribution of the order statistics and Rényi entropy are derived. The parameters of the new family of distributions are estimated via the method of maximum likelihood. A simulation study to examine the bias and mean square error of the maximum likelihood estimates and applications to real data sets to illustrate the usefulness and applicability of the generalized family of distributions are given.

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**Keywords:** Generalized-G distribution, gamma-generator, family of distributions, maximum likelihood estimation, goodness-of-fit statistics.

### 1. Introduction

The usefulness and importance of new generalized and extended distributions in statistical modeling cannot be overemphasized. The literature on extended and generalized distributions, particularly in the modeling of monotone and non-monotone hazard rates functions, especially in reliability and survival analysis is growing. A useful family of distributions called the generalized-G (G-G) distribution have been used to extend several well known classical distributions including Weibull, Fréchet, log-logistic and Lomax distributions.

There are several well-known families of distributions in the statistical literature including the beta-G by Eugene et al. (2002), Kumaraswamy-G (Kw-G) by Cordeiro and Lemonte (2011), McDonald-G (Mc-G) by Alexander et al. (2012), transformed-transformer (T-X) by Alzaatreh et al. (2013), exponential generalized-G by Cordeiro et al. (2013), Kumaraswamy odd log-logistic family by Alizadeh et al. (2015), Kumaraswamy Marshall-Olkin family by Alizadeh et al. (2015), odd Lindley-G by Gomes-Silva et al. (2017), new power generalized Weibull-G by Oluyede et al. (2020), and odd-power generalized Weibull-G by Moakofi et al. (2021) to mention just a few.

Recent literature has shown growing interest in the development of flexible distributional frameworks aimed at modeling asymmetric, heavy-tailed, and complex data structures. Many of these contributions are based on generator mechanisms, entropy transformations, or parameter extensions of classical distributions, with applications spanning reliability analysis, health sciences, and engineering (see Colak et al. (2023), Shafiq et al. (2024), Sindhu et al. (2024), Sindhu et al. (2025)).

Collectively, these studies highlight the importance of constructing adaptable probabilistic models capable of capturing diverse data behaviors that are inadequately described by traditional distributions.

The  $T - X$  family of distributions (see Alzaatreh et al. (2013) for additional details) is given by

$$F(x) = \int_a^{W(G(x))} r(t) dt, \quad (1)$$

where  $r(t)$  is a probability density function (pdf) and  $W(G(x))$  satisfies the following conditions: (1)  $W(G(x)) \in [a, b]$ , for  $-\infty \leq a < x < b < \infty$ , (2)  $W(G(x))$  is differentiable and monotonically non-decreasing, and (3)  $W(G(x)) \rightarrow a$  as  $x \rightarrow -\infty$  and  $W(G(x)) \rightarrow b$  as  $x \rightarrow \infty$ .

The transformation  $W(G(x; \xi)) = \frac{1 - \bar{G}^\alpha(x; \xi)}{\bar{G}^\alpha(x; \xi)}$  by Aljarrah et al. (2014) is used in this paper for the development and extension of the gamma operator via the odd Lindley generalized-G distribution by introducing additional shape parameters to obtain a new generalized family of distributions called the gamma odd Lindley generalized-G (GOLG-G) family of distributions. We compare the special case of this new family of distributions with several other distributions in the literature via several goodness-of-fit statistics in order to establish the superiority, the effectiveness and usefulness of this new family of distributions.

Gomes-Silva et al. (2017) were the first to propose the odd Lindley-G distribution family. The odd Lindley-G family is novel in that it extends the classical Lindley distribution through an odd transformation mechanism, enabling the resulting models to capture substantial skewness and a broad range of hazard rate shapes, including increasing, decreasing, bathtub, and unimodal forms, which are not simultaneously attainable under many standard generalized families. Several new generalizations of odd Lindley-G families have been developed in the statistical literature, and they include Topp Leone odd Lindley-G family by Reyad et al. (2018), transmuted odd Lindley-G family by Reyad et al. (2018), generalized odd Lindley-G family by Afify et al. (2019), Marshall-Olkin odd Lindley-G family by Jamal et al. (2019), Kumaraswamy odd Lindley-G family by Chipepa et al. (2019), and exponentiated half-logistic odd Lindley-G family by Sengweni et al. (2021), just a few to mention.

The GOLG-G family of distributions is constructed and developed with the following motivations and contributions:

- to provide a new and flexible extension of the odd Lindley-G and odd Lindley generalized-G families by incorporating the gamma generator, thereby yielding a strict generalization of these existing models;
- to enhance the ability of the resulting family to model asymmetric data, including heavy-tailed, left-skewed, and reverse-J shaped distributions, while also containing several special models that may offer improved fits compared to the corresponding base distributions;
- to allow for a wide variety of hazard rate behaviors, including both monotone and non-monotone forms, and to exploit the versatility and flexibility of generalized distributions in modeling lifetime data and other scientific data sets.

### 1.1. Generalized families of distributions

Consider the family of distributions with the probability density function (pdf)  $f(x)$  and cumulative distribution function (cdf)  $F(x)$  given as:

$$f(x) = \frac{1}{\Gamma(\delta)\psi^\delta} [-\log(1 - G(x))]^{\delta-1} (1 - G(x))^{\frac{1}{\psi}-1} g(x), x \in R, \delta > 0, \quad (2)$$

and

$$F(x) = \frac{1}{\Gamma(\delta)\psi^\delta} \int_0^{-\log(1-G(x))} t^{\delta-1} e^{-t/\psi} dt = \frac{\gamma(\delta, -\psi^{-1} \log(1 - G(x)))}{\Gamma(\delta)}, \quad (3)$$

respectively, where  $\gamma(x, \delta) = \int_0^x t^{\delta-1} e^{-t} dt$  is the incomplete gamma function. When  $\psi=1$ , this distribution is referred to as the ZB-G family of distributions (Zografos and Balakrishnan, 2009). An

alternative gamma-generator was proposed by Ristić and Balakrishnan (2012). The cdf and pdf of the alternative generator are defined by

$$F_2(x) = 1 - \frac{1}{\Gamma(\delta)\psi^\delta} \int_0^{-\log(G(x))} t^{\delta-1} e^{t/\psi} dt, x \in R, \delta > 0, \quad (4)$$

and

$$f_2(x) = \frac{1}{\Gamma(\delta)\psi^\delta} [-\log(G(x))]^{\delta-1} G(x)^{1/\psi-1} g(x), \quad (5)$$

respectively. When  $\psi=1$ , this distribution is referred to as the Ristić and Balakrishnan-G (RB-G) family of distributions.

These families of distributions are generated by gamma random variables, with an extra positive shape parameter. Gamma generator techniques enhance distributional flexibility by introducing an additional shape parameter, allowing for improved control over tail behavior and the accommodation of diverse hazard rate forms. Pinho et al. (2012) introduced and studied the gamma exponentiated Weibull distribution. The authors showed that the gamma exponentiated Weibull distribution perform better than the parent distribution.

In this paper an extension or a generalization of the odd Lindley generalized-G family of distributions motivated by the results of Ristić and Balakrishnan family of distributions in Equation (4) when  $\psi = 1$  is developed and presented. The presentation is a generalization of the new odd Lindley generalized-G family of distributions that stems from the Ristić and Balakrishnan (2012) family of distributions. Statistical properties of the family of distributions are derived, the maximum likelihood estimates and applications of the model are presented. In addition, it is also the case that this generalization of the odd Lindley generalized-G family of distributions via the gamma-generator establishes the relationship between the distributions in Equations (3) and (4), and weighted distributions in general. See Oluyede et al. (2015) for additional details.

The results in this paper are organized in the following manner. In Section 2, the new GOLG-G family of distributions and its sub-families. Section 3 contains results on the expansion of the density, hazard rate function, quantile function, moments, the distribution of order statistics and Rényi entropy. In Section 4, some special cases of the GOLG-G family are examined. Estimation of the parameters of the GOLG-G family of distributions via the method of maximum likelihood are given in Section 5 followed by a Monte Carlo simulation study to examine the bias and mean square error of the maximum likelihood estimators in Section 6. Some applications to real data sets are given in Section 7. Concluding remarks follow in Section 8.

## 2. The New Family and its Sub-Families

This section is concerned with the development of the gamma odd Lindley generalized-G (GOLG-G) family of distributions and its sub-families.

### 2.1. The new family

First, we consider the family of the odd Lindley generalized-G distribution (Oluyede, 2019) with cdf and pdf given by

$$\begin{aligned} F_{OLG-G}(x; \lambda, \alpha, \xi) &= \int_0^{\frac{1-\overline{G}^\alpha(x; \xi)}{\overline{G}^\alpha(x; \xi)}} \frac{\lambda^2}{1+\lambda} (1+t) e^{-\lambda t} dt \\ &= 1 - \frac{\lambda + \overline{G}^\alpha(x; \xi)}{(1+\lambda)\overline{G}^\alpha(x; \xi)} \exp\left(-\lambda \left(\frac{1-\overline{G}^\alpha(x; \xi)}{\overline{G}^\alpha(x; \xi)}\right)\right) \end{aligned} \quad (6)$$

and

$$f_{OLG-G}(x; \lambda, \alpha, \xi) = \frac{\lambda^2 \alpha}{(1+\lambda)} \frac{g(x; \xi)}{[\overline{G}(x; \xi)]^{2\alpha+1}} \exp\left(-\lambda \left[\frac{1-\overline{G}^\alpha(x; \xi)}{\overline{G}^\alpha(x; \xi)}\right]\right), \quad (7)$$

respectively, for  $\lambda, \alpha > 0$  and parameter vector  $\xi$ . Now, if  $T$  is a random variable denoting the lifetime of a system with the generalized-G distribution function given by  $K(x; \xi) = 1 - \bar{G}^\alpha(x; \xi)$  and the random variable  $X$  is the odds ratio, then the risk that the system with lifetime  $T$  will fail at time  $x$  is given by  $\frac{K(x; \xi)}{1-K(x; \xi)} = \frac{1-\bar{G}^\alpha(x; \xi)}{\bar{G}^\alpha(x; \xi)}$ . Suppose we model the randomness of the odds ratio via the Lindley distribution with scale parameter  $\lambda > 0$ , that is,  $r(t; \lambda) = \frac{\lambda^2(1+t)}{(1+\lambda)}e^{-\lambda t}$ , for  $t > 0$ , then the cdf of the random variable  $X$  is the new odd Lindley generalized-G family of distributions with cdf given by Equation (6), that is,

$$P(X \leq x) = R\left(\frac{1 - \bar{G}^\alpha(x; \xi)}{\bar{G}^\alpha(x; \xi)}\right) = F_{OLG-G}(x; \lambda, \alpha, \xi). \quad (8)$$

We obtain the odd Lindley-G (OL-G) family of distributions (Gomes-Silva et al., 2017) when  $\alpha = 1$ . We also observe that different baseline distribution functions  $G(x; \xi)$ , lead to several new generalized distributions as sub-models.

The cdf and pdf of the proposed gamma odd Lindley generalized-G (GOLG-G) family of distributions are given by

$$\begin{aligned} F(x; \alpha, \lambda, \delta, \xi) &= 1 - \frac{1}{\Gamma(\delta)} \int_0^{-\log\left(1 - \frac{\lambda + \bar{G}^\alpha(x; \xi)}{(1+\lambda)\bar{G}^\alpha(x; \xi)} e^{-\lambda\left(\frac{1-\bar{G}^\alpha(x; \xi)}{\bar{G}^\alpha(x; \xi)}\right)}\right)} t^{\delta-1} e^{-t} dt \\ &= 1 - \frac{\gamma\left(-\log\left(1 - \frac{\lambda + \bar{G}^\alpha(x; \xi)}{(1+\lambda)\bar{G}^\alpha(x; \xi)} e^{-\lambda\left(\frac{1-\bar{G}^\alpha(x; \xi)}{\bar{G}^\alpha(x; \xi)}\right)}\right), \delta\right)}{\Gamma(\delta)} \end{aligned} \quad (9)$$

and

$$\begin{aligned} f(x; \alpha, \lambda, \delta, \xi) &= \frac{1}{\Gamma(\delta)} \left[ -\log\left(1 - \frac{\lambda + \bar{G}^\alpha(x; \xi)}{(1+\lambda)\bar{G}^\alpha(x; \xi)} e^{-\lambda\left(\frac{1-\bar{G}^\alpha(x; \xi)}{\bar{G}^\alpha(x; \xi)}\right)}\right) \right]^{\delta-1} \\ &\times \frac{\lambda^2 \alpha}{(1+\lambda)} \frac{g(x; \xi)}{[\bar{G}(x; \xi)]^{2\alpha+1}} e^{-\lambda\left[\frac{1-\bar{G}^\alpha(x; \xi)}{\bar{G}^\alpha(x; \xi)}\right]}, \end{aligned} \quad (10)$$

respectively, for  $\alpha, \lambda, \delta > 0$  and parameter vector  $\xi$ . The motivation for this combination is that the odd Lindley generalized-G family provides flexible skewness and hazard rate shapes, while the gamma generator introduces an additional shape parameter that significantly enhances tail behavior and overall distributional flexibility.

## 2.2. Sub-families of the GOLG-G family of distributions

Sub-families of the GOLG-G family of distributions are presented in this section.

- If  $\alpha = 1$ , we obtain the gamma odd Lindley-G (GOL-G) family of distributions with the pdf

$$\begin{aligned} f(x; \lambda, \delta, \xi) &= \frac{1}{\Gamma(\delta)} \left[ -\log\left(1 - \frac{\lambda + \bar{G}(x; \xi)}{(1+\lambda)\bar{G}(x; \xi)} e^{-\lambda\left(\frac{\bar{G}(x; \xi)}{\bar{G}(x; \xi)}\right)}\right) \right]^{\delta-1} \\ &\times \frac{\lambda^2 \alpha}{(1+\lambda)} \frac{g(x; \xi)}{[\bar{G}(x; \xi)]^3} e^{-\lambda\left[\frac{\bar{G}(x; \xi)}{\bar{G}(x; \xi)}\right]} \end{aligned}$$

for  $\lambda, \delta > 0$  and parameter vector  $\xi$ . This distribution is a new family of distributions.

- If  $\delta = 1$ , we obtain the OLG-G family of distributions with the pdf

$$f(x; \alpha, \lambda, \xi) = \frac{\lambda^2 \alpha}{(1+\lambda)} \frac{g(x; \xi)}{[\bar{G}(x; \xi)]^{2\alpha+1}} e^{-\lambda\left[\frac{1-\bar{G}^\alpha(x; \xi)}{\bar{G}^\alpha(x; \xi)}\right]},$$

for  $\alpha, \lambda > 0$ , and parameter vector  $\xi$ . This distribution is a new family of distributions.

- If  $\alpha = \delta = 1$ , we obtain the odd Lindley-G family of distributions (Gomes-Silva et al., 2017) with the pdf

$$f(x; \lambda, \xi) = \frac{\lambda^2}{(1 + \lambda)} \frac{g(x; \xi)}{[\bar{G}(x; \xi)]^3} e^{\left(-\lambda \left[\frac{G(x; \xi)}{\bar{G}(x; \xi)}\right]\right)},$$

for  $\lambda > 0$  and parameter vector  $\xi$ .

Note that different choices of the baseline cdf  $G(x; \xi)$  eventually produces several new gamma odd Lindley generalized distributions as sub-models. Table 1 presents a number of sub-models by changing the baseline cdf  $G(x; \xi)$  with the corresponding parameter vector  $\xi$ . It lists 14 baseline distributions, including Uniform, Exponential, Weibull, and Normal, and shows how each can be transformed into a corresponding GOLG distribution. The key transformation relies on the ratio  $G(x; \xi)/\bar{G}(x; \xi)$ , where  $G(x; \xi)$  is the cumulative distribution function and  $\bar{G}(x; \xi)$  is the survival function of the base distribution. By applying the GOLG framework with additional shape parameters  $(\alpha, \lambda, \delta)$  together with the original parameters  $(\xi)$ , researchers can construct extended distributions such as GOLG-Uniform, GOLG-Normal, or GOLG-Weibull, which provide greater flexibility than the base distributions.

**Table 1** Distributions and corresponding  $G(x; \xi)/\bar{G}(x; \xi)$  functions

| Distribution         | $G(x; \xi)/\bar{G}(x; \xi)$                             | $\Psi = (\alpha, \lambda, \delta, \xi)$    | GOLG-G                    |
|----------------------|---------------------------------------------------------|--------------------------------------------|---------------------------|
| Uniform              | $x/(\theta - x)$                                        | $(\alpha, \lambda, \delta, \theta)$        | GOLG-Uniform              |
| Exponential          | $e^{\beta x} - 1$                                       | $(\alpha, \lambda, \delta, \beta)$         | GOLG-Exponential          |
| Weibull              | $e^{\beta x^\gamma} - 1$                                | $(\alpha, \lambda, \delta, \beta, \gamma)$ | GOLG-Weibull              |
| Fréchet              | $\left(e^{\beta x^\gamma} - 1\right)^{-1}$              | $(\alpha, \lambda, \delta, \beta, \gamma)$ | GOLG-Fréchet              |
| Half-logistic        | $(e^x - 1)/2$                                           | $(\alpha, \lambda, \delta)$                | GOLG-Half-logistic        |
| Power function       | $[(\theta x)^{-k} - 1]^{-1}$                            | $(\alpha, \lambda, \delta, \theta, k)$     | GOLG-Power function       |
| Pareto               | $(x/\theta)^k - 1$                                      | $(\alpha, \lambda, \delta, \theta, k)$     | GOLG-Pareto               |
| Burr XII             | $[1 - x^c]^k - 1$                                       | $(\alpha, \lambda, \delta, c, k)$          | GOLG-Burr XII             |
| Log-logistic         | $[1 - x^c] - 1$                                         | $(\alpha, \lambda, \delta, c)$             | GOLG-Log-logistic         |
| Lomax                | $[1 - x]^k - 1$                                         | $(\alpha, \lambda, \delta, k)$             | GOLG-Lomax                |
| Gumbel               | $\left[e^{e^{-\frac{x-\mu}{\sigma}}} - 1\right]^{-1}$   | $(\alpha, \lambda, \delta, \mu, \sigma)$   | GOLG-Gumbel               |
| Normal               | $\frac{\Phi((x-\mu)/\sigma)}{[1-\Phi((x-\mu)/\sigma)]}$ | $(\alpha, \lambda, \delta, \mu, \sigma)$   | GOLG-Normal               |
| Kumaraswamy          | $(1 - x^a)^{-b} - 1$                                    | $(\alpha, \lambda, \delta, a, b)$          | GOLG-Kumaraswamy          |
| Modified Exponential | $e^{\theta x e^{\beta x}} - 1$                          | $(\alpha, \lambda, \delta, \theta, \beta)$ | GOLG-Modified Exponential |

### 3. Some Statistical Properties

In this section, series representation of the density function, hazard rate function, moments, incomplete moments, distribution of the order statistics, Rényi entropy and stochastic ordering for the GOLG-G family of distributions are presented. These results not only establish the theoretical foundation of the proposed model but also have direct practical implications: they facilitate reliability analysis through hazard rate characterization, support survival modeling via order statistics and moment-based summaries, and enable risk assessment using tail behavior and entropy measures. Furthermore, more detailed derivations of these properties are provided in the Appendix to ensure full transparency and reproducibility.

### 3.1. Series representation of the density function

Series representation of the GOLG-G density function is presented in this subsection. Let  $y = \frac{\lambda + \overline{G}^\alpha(x; \xi)}{(1 + \lambda)\overline{G}^\alpha(x; \xi)} e^{-\lambda \left( \frac{1 - \overline{G}^\alpha(x; \xi)}{\overline{G}^\alpha(x; \xi)} \right)}$ ,  $0 < y < 1$ ,  $\alpha, \lambda > 0$ , and parameter vector  $\xi$ , then using the results of Gradshteyn and Ryzhik (2000), and the exponential and generalized binomial series expansions, the series representation of the GOLG-G density function can be written as

$$f(x) = \sum_{j,t=0}^{\infty} \omega(j, t) g_{j+t+1}^*(x; \xi), \quad (11)$$

where  $g_{j+t+1}^*(x; \xi) = (j + t + 1)[G(x; \xi)]^{j+t} g(x; \xi)$  is the exponentiated-G pdf with the power or exponentiation parameter  $j + t + 1$  and parameter vector  $\xi$ , and  $\omega(j, t)$  are the weights given by

$$\begin{aligned} \omega(j, t) &= \sum_{m,s,q,k,p=0}^{\infty} (-1)^{k+p+t} \binom{\delta-1}{m} \binom{m+s+\delta-1}{q} \binom{k}{p} \\ &\times \frac{[\lambda(m+s+\delta)]^k b_{s,m} \lambda^2 \alpha \lambda^{m+s+\delta-1-q}}{(1+\lambda)^{m+s+\delta} k! \Gamma(\delta)} \frac{\lambda^{m+s+\delta-1-q}}{(j+t+1)} \binom{\alpha(p+q)}{t} \\ &\times \binom{-\alpha(m+s+\delta+k+1)-1}{j}. \end{aligned} \quad (12)$$

The appendix clarifies the notation  $b_{s,m}$  with detailed derivations of the series representation of GOLG-G density function.

### 3.2. Hazard rate and quantile functions

The hazard rate function (hrf)  $h_F(x; \alpha, \lambda, \delta, \xi) = h_F(x)$  of the GOLG-G family of distributions is given by

$$\begin{aligned} h_F(x) &= \left[ -\log \left( 1 - \frac{\lambda + \overline{G}^\alpha(x; \xi)}{(1 + \lambda)\overline{G}^\alpha(x; \xi)} e^{-\lambda \left( \frac{1 - \overline{G}^\alpha(x; \xi)}{\overline{G}^\alpha(x; \xi)} \right)} \right) \right]^{\delta-1} \\ &\times \frac{\lambda^2 \alpha}{(1 + \lambda)} \frac{g(x; \xi)}{[\overline{G}(x; \xi)]^{2\alpha+1}} e^{-\lambda \left[ \frac{1 - \overline{G}^\alpha(x; \xi)}{\overline{G}^\alpha(x; \xi)} \right]} \\ &\times \left( \gamma \left( -\log \left( 1 - \frac{\lambda + \overline{G}^\alpha(x; \xi)}{(1 + \lambda)\overline{G}^\alpha(x; \xi)} e^{-\lambda \left( \frac{1 - \overline{G}^\alpha(x; \xi)}{\overline{G}^\alpha(x; \xi)} \right)} \right), \delta \right) \right)^{-1}. \end{aligned} \quad (13)$$

The quantile function  $Q(u)$  of the GOLG-G family of distributions can be obtained by solving the nonlinear equation:

$$\begin{aligned} F(Q(u)) &= 1 - \frac{\gamma \left( -\log \left( 1 - \frac{\lambda + \overline{G}^\alpha(Q(u))}{(1 + \lambda)\overline{G}^\alpha(Q(u))} e^{-\lambda \left( \frac{1 - \overline{G}^\alpha(Q(u))}{\overline{G}^\alpha(Q(u))} \right)} \right), \delta \right)}{\Gamma(\delta)} \\ &= u, \end{aligned} \quad (14)$$

for  $0 \leq u \leq 1$ .

By following the detailed derivations in the Appendix section, the quantile function of the GOLG-G family of distributions is given by

$$Q(u) = G^{-1} \left[ 1 - \left( \frac{-\lambda}{1 + W_{-1} \left( (1 + \lambda) \left[ e^{-\gamma^{-1}((1-u)\Gamma(\delta), \delta)} - 1 \right] e^{-(1+\lambda)} \right)} \right)^{1/\alpha} \right], \quad (15)$$

where  $W_{-1}$  is the lower branch of the Lambert W function. It follows therefore that random numbers can be generated based on Equation (15) for specified baseline cdf  $G$ .

### 3.3. Moments

Let  $Y_{k+1} \sim \text{Exponentiated} - G(k+1, \xi)$ , then the  $r^{\text{th}}$  raw moment,  $\mu'_r$  of the GOLG-G family of distributions is given by:

$$\begin{aligned}\mu'_r = E(X^k) &= \sum_{k=0}^{\infty} C_{k+1} E(Y_{k+1}^r) \\ &= \sum_{k=0}^{\infty} C_{k+1} \int_{-\infty}^{\infty} y^r g_{k+1}^*(y; \xi) dy \\ &= \sum_{k=0}^{\infty} C_{k+1} (k+1) \int_0^1 [Q_G(u; \xi)]^r u^k du.\end{aligned}\quad (16)$$

The coefficients of variation (CV), Skewness (CS) and Kurtosis (CK) can be readily obtained. Since the model can exhibit heavy-tailed behavior depending on the baseline distribution  $G$  and shape parameters, the  $r^{\text{th}}$  moment exists if and only if  $\int_{-\infty}^{\infty} x^r f(x; \alpha, \lambda, \delta, \xi) dx < \infty$ , which imposes explicit constraints on  $r$  relative to the tail behavior of the baseline distribution  $G$  and the parameters. For many heavy-tailed distributions, such as Pareto or certain Lomax-type distributions, moments of order  $r$  greater than or equal to the tail index of the baseline distribution diverge. Therefore, when analyzing moments for heavy-tailed models, it is essential to explicitly specify the range of  $r$  for which the moments are finite. In particular, when the baseline distribution  $G$  has finite moments of all orders (e.g., exponential or log-logistic), all moments of the GOLG-G distribution exist for any  $r > 0$ .

### 3.4. Incomplete moments

It is important to observe that conditional and incomplete moments are particularly useful for the analysis of lifetime models and in measuring inequality. The  $r^{\text{th}}$  incomplete moment of the GOLG-G family of distributions is given by

$$m_r(t) = \int_t^{\infty} x^r f(x; \alpha, \lambda, \delta, \xi) dx = \sum_{k=0}^{\infty} C_{k+1} E\left(Y_{k+1}^r I_{\{Y_{k+1}^r \geq t\}}\right), \quad (17)$$

where

$$\begin{aligned}E\left(Y_{k+1}^r I_{\{Y_{k+1}^r \geq t\}}\right) &= \int_t^{\infty} y^r g_{k+1}^*(y; \xi) dy \\ &= (k+1) \int_{G(u; \xi)}^1 [Q_G(u; \xi)]^r u^k du,\end{aligned}\quad (18)$$

for  $\alpha, \lambda, \delta > 0$ , and parameter vector  $\xi$ . The incomplete moments of order  $r$  exist under the same conditions as the corresponding regular moments, since they involve integration over a subset of the distributions support. The mean residual life function is given by  $E(X - t | X > t) = E(X | X > t) - t = V_F(t) - t$ , where  $V_F(t)$  is referred to as the vitality function of the distribution function  $F$ . Also, the mean deviations, Bonferroni and Lorenz curves can be readily computed from the first incomplete moment.

### 3.5. Order statistics and Rényi entropy

Order statistics and entropy are particularly useful in reliability, lifetime data analysis and information theory. In this subsection, the distribution of the  $i^{\text{th}}$  order statistics and Rényi entropy for the GOLG-G family of distributions are presented.

### 3.5.1. Order statistics

Let  $X_1, X_2, \dots, X_n$  be independent and identically distributed GOLG-G random variables. The pdf of the  $i^{th}$  order statistic from the GOLG-G pdf  $f(x; \alpha, \lambda, \delta, \xi) = f(x)$  can be written as

$$f_{i:n}(x) = \sum_{m=0}^{n-i} \omega(i, m) f_{m+i}(x; \alpha, \lambda, \delta, \xi), \quad (19)$$

where  $f_{m+i}(x; \alpha, \lambda, \delta, \xi)$  is the exponentiated gamma odd Lindley generalized-G pdf with exponentiated parameter  $m + i > 0$  and weights  $\omega(i, m)$  given by

$$\begin{aligned} \omega(i, m) &= \frac{1}{B(i, n-i+1)} \frac{(-1)^m}{m+i} \binom{n-i}{m} \\ &= (-1)^m \binom{n}{m+i} \binom{m+i-1}{m}. \end{aligned} \quad (20)$$

Detailed derivations of the pdf of the  $i^{th}$  order statistic are provided in the Appendix.

### 3.5.2. Rényi entropy

Rényi entropy (Rényi, 1960) is an extension of Shannon entropy. This provides a generalized measure of uncertainty that captures not only the spread of the distribution but also the relative weight of its tails, making it particularly useful for assessing variability in reliability, survival, and risk modeling contexts. By evaluating Rényi entropy across different parameter settings, one can gain insights into how the distributions shape, tail behavior, and concentration of probability mass change, thereby offering a deeper understanding of the information encoded by the model. Detailed derivations of the Rényi entropy are also provided in the Appendix to ensure transparency and reproducibility. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v} \log \left( \int_0^\infty [f(x; \alpha, \lambda, \delta, \xi)]^v dx \right), v \neq 1, v > 0. \quad (21)$$

It is well known that Rényi entropy tends to Shannon entropy as  $v \rightarrow 1$ .

By following the detailed derivations in the Appendix section, Rényi entropy for the GOLG-G family of distributions is given by

$$I_R(v) = (1-v)^{-1} \log \left[ \sum_{j,t=0}^{\infty} W_{j,t} e^{(1-v)I_{REG}} \right], \quad (22)$$

where  $I_{REG} = (1-v)^{-1} \log \left( \int_0^\infty \left[ \left( \frac{j+t}{v} + 1 \right) [G(x; \xi)]^{\frac{j+t}{v}} g(x; \xi) \right]^v dx \right)$  is the Rényi entropy of exponentiated-G distribution with power parameter  $\frac{j+t}{v} + 1$  and

$$\begin{aligned} W_{j,t} &= \sum_{m,s,k,p,q=0}^{\infty} \binom{v\delta-v}{m} \binom{\alpha(p+q)}{t} \binom{k}{p} \binom{m+s+\delta-1}{q} \\ &\times \frac{b_{s,m} \lambda^{2v} \alpha^v}{\left[ \frac{j+t}{v} + 1 \right]^v (1+\lambda)^{v+m+s+\delta-1}} \frac{(-1)^{k+p+j} [v\lambda(m+s+\delta)]^k}{(\Gamma(\delta))^v k!} \\ &\times \lambda^{m+s+\delta-1-q} \binom{-\alpha(k+m+s+\delta+2v+1)-v}{j}. \end{aligned}$$

### 3.6. Stochastic ordering

In this section, we present stochastic orders for the GOLG-G family of distributions. Consider  $X_1$  and  $X_2$  as two independent random variables following  $\text{GOLG-G}(\alpha, \lambda, \delta_1, \xi)$  and  $\text{GOLG-G}(\alpha, \lambda, \delta_2, \xi)$  distributions. Then, the pdfs of  $X_1$  and  $X_2$  are

$$f_1(x) = f_1(x; \alpha, \lambda, \delta_1, \xi) = \frac{1}{\Gamma(\delta_1)} \left[ -\log \left( 1 - \frac{\lambda + \bar{G}^\alpha(x; \xi)}{(1 + \lambda)\bar{G}^\alpha(x; \xi)} e^{-\lambda \left( \frac{1 - \bar{G}^\alpha(x; \xi)}{\bar{G}^\alpha(x; \xi)} \right)} \right) \right]^{\delta_1 - 1} \\ \times \frac{\lambda^2 \alpha}{(1 + \lambda)} \frac{g(x; \xi)}{[\bar{G}(x; \xi)]^{2\alpha + 1}} e^{-\lambda \left[ \frac{1 - \bar{G}^\alpha(x; \xi)}{\bar{G}^\alpha(x; \xi)} \right]},$$

and

$$f_2(x) = f_2(x; \alpha, \lambda, \delta_2, \xi) = \frac{1}{\Gamma(\delta_2)} \left[ -\log \left( 1 - \frac{\lambda + \bar{G}^\alpha(x; \xi)}{(1 + \lambda)\bar{G}^\alpha(x; \xi)} e^{-\lambda \left( \frac{1 - \bar{G}^\alpha(x; \xi)}{\bar{G}^\alpha(x; \xi)} \right)} \right) \right]^{\delta_2 - 1} \\ \times \frac{\lambda^2 \alpha}{(1 + \lambda)} \frac{g(x; \xi)}{[\bar{G}(x; \xi)]^{2\alpha + 1}} e^{-\lambda \left[ \frac{1 - \bar{G}^\alpha(x; \xi)}{\bar{G}^\alpha(x; \xi)} \right]}.$$

We say  $X_1$  is stochastically smaller than  $X_2$  in likelihood ratio order if  $\frac{f_1(x)}{f_2(x)}$  is an decreasing function of  $x$ . It is well known that likelihood ratio order implies hazard rate order which in turn implies stochastic order (see Shaked and Shanthikumar (1994) for additional details).

The ratio,  $\frac{f_1(x)}{f_2(x)}$  takes the form

$$\frac{f_1(x)}{f_2(x)} = \frac{\Gamma(\delta_2)}{\Gamma(\delta_1)} \left[ -\log \left( 1 - \frac{\lambda + \bar{G}^\alpha(x; \xi)}{(1 + \lambda)\bar{G}^\alpha(x; \xi)} e^{-\lambda \left( \frac{1 - \bar{G}^\alpha(x; \xi)}{\bar{G}^\alpha(x; \xi)} \right)} \right) \right]^{\delta_1 - \delta_2}. \quad (23)$$

If we differentiate Equation (23) with respect to  $x$ , we get

$$\frac{d}{dx} \left( \frac{f_1(x)}{f_2(x)} \right) = (\delta_1 - \delta_2) \frac{\Gamma(\delta_2)}{\Gamma(\delta_1)} \left[ -\log \left( 1 - \frac{\lambda + \bar{G}^\alpha(x; \xi)}{(1 + \lambda)\bar{G}^\alpha(x; \xi)} e^{-\lambda \left( \frac{1 - \bar{G}^\alpha(x; \xi)}{\bar{G}^\alpha(x; \xi)} \right)} \right) \right]^{\delta_1 - \delta_2 - 1} \\ \times \frac{-f_{\text{OLG-G}}(x; \lambda, \alpha, \xi)}{F_{\text{OLG-G}}(x; \lambda, \alpha, \xi)},$$

where  $f_{\text{OLG-G}}(x; \lambda, \alpha, \xi)$  and  $F_{\text{OLG-G}}(x; \lambda, \alpha, \xi)$  are given by Equations (7) and (6), respectively.

Finally, if  $\delta_2 < \delta_1$ , then  $\frac{d}{dx} \left( \frac{f_1(x)}{f_2(x)} \right) < 0$ . Therefore, the random variables  $X_1$  and  $X_2$  are stochastically ordered if  $\delta_2 < \delta_1$ .

## 4. Some Special Cases

Some special cases of the GOLG-G family of distributions, specifically when the baseline distribution function  $G(x; \xi)$  are uniform, log-logistic, logistic and Weibull distributions, respectively are presented in this section.

### 4.1. GOLG-uniform distribution

When the cdf and pdf of the baseline distribution are given by  $G(x; \theta) = \frac{x}{\theta}$  and  $g(x; \theta) = \frac{1}{\theta}$  for  $0 < x < \theta < \infty$ , then the resulting new GOLG-uniform distribution has cdf and pdf given by

$$F(x; \alpha, \lambda, \delta, \theta) = 1 - \frac{\gamma \left( -\log \left[ 1 - \frac{\lambda + \left(1 - \frac{x}{\theta}\right)^\alpha}{(1 + \lambda) \left(1 - \frac{x}{\theta}\right)^\alpha} \exp \left( -\lambda \left[ \left(1 - \frac{x}{\theta}\right)^{-\alpha} - 1 \right] \right) \right], \delta \right)}{\Gamma(\delta)},$$

and

$$f(x; \alpha, \lambda, \delta, \theta) = \frac{1}{\Gamma(\delta)} \left( -\log \left[ 1 - \frac{\lambda + (1 - \frac{x}{\theta})^\alpha}{(1 + \lambda)(1 - \frac{x}{\theta})^\alpha} e^{-\lambda[(1 - \frac{x}{\theta})^{-\alpha} - 1]} \right] \right)^{\delta-1} \\ \times \frac{\lambda^2 \alpha}{(1 + \lambda)} \frac{1}{\theta (1 - \frac{x}{\theta})^{2\alpha+1}} e^{-\lambda[(1 - \frac{x}{\theta})^{-\alpha} - 1]},$$

respectively, for  $\alpha, \lambda, \delta > 0$ , and  $0 < x < \theta < \infty$ . The hrf is given by

$$h_F(x) = \left( -\log \left[ 1 - \frac{\lambda + (1 - \frac{x}{\theta})^\alpha}{(1 + \lambda)(1 - \frac{x}{\theta})^\alpha} e^{-\lambda[(1 - \frac{x}{\theta})^{-\alpha} - 1]} \right] \right)^{\delta-1} \\ \times \frac{\lambda^2 \alpha}{(1 + \lambda)} \frac{1}{\theta (1 - \frac{x}{\theta})^{2\alpha+1}} e^{-\lambda[(1 - \frac{x}{\theta})^{-\alpha} - 1]} \\ \times \left( \gamma \left( -\log \left[ 1 - \frac{\lambda + (1 - \frac{x}{\theta})^\alpha}{(1 + \lambda)(1 - \frac{x}{\theta})^\alpha} e^{-\lambda[(1 - \frac{x}{\theta})^{-\alpha} - 1]} \right], \delta \right) \right)^{-1},$$

for  $\alpha, \lambda, \delta, \theta > 0$  and  $x > 0$ .

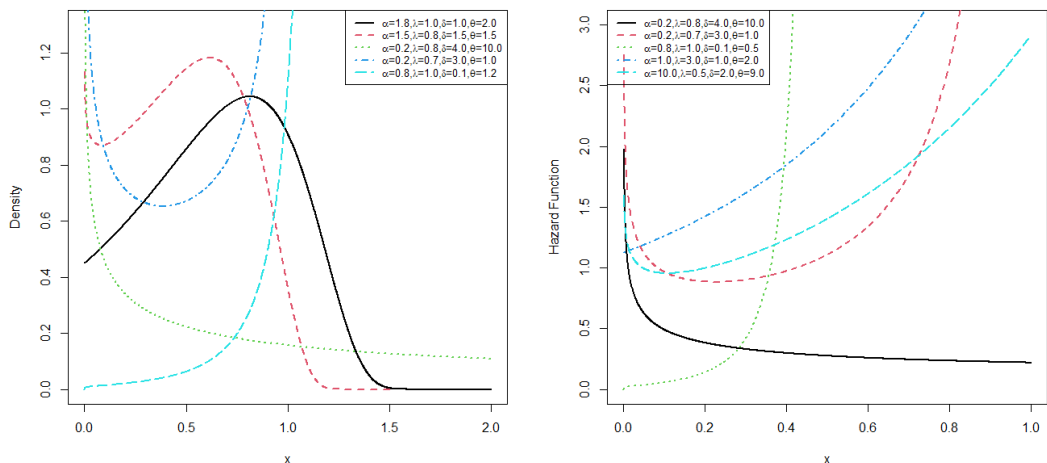


**Figure 1** Plots of the pdf and hrf for the GOLG-Uniform distribution

The shapes of the pdfs include uni-modal, left-skewed, J and reverse-J shapes. The plots of the hrf exhibit bathtub, increasing and decreasing shapes.

#### 4.2. GOLG-Log-logistic distribution

Let the cdf and pdf of the baseline distribution be given by  $G(x; c) = 1 - (1 + x^c)^{-1}$  and  $g(x; c) = cx^{c-1}(1 + x^c)^{-2}$  for  $c > 0$  and  $x > 0$ , then the cdf and pdf of the new GOLG-Log-logistic distribution are given by

$$F(x; \alpha, \lambda, \delta, c) = 1 - \frac{\gamma \left( -\log \left[ 1 - \frac{\lambda + (1 + x^c)^{-\alpha}}{(1 + \lambda)(1 + x^c)^{-\alpha}} e^{-\lambda[(1 + x^c)^\alpha - 1]} \right], \delta \right)}{\Gamma(\delta)},$$

and

$$f(x; \alpha, \lambda, \delta, c) = \frac{1}{\Gamma(\delta)} \left( -\log \left[ 1 - \frac{\lambda + (1+x^c)^{-\alpha}}{(1+\lambda)(1+x^c)^{-\alpha}} e^{(-\lambda[(1+x^c)^\alpha - 1])} \right] \right)^{\delta-1} \\ \times \frac{\lambda^2 \alpha}{(1+\lambda)} \frac{cx^{c-1}(1+x^c)^{-2}}{(1+x^c)^{-2\alpha-1}} \exp(-\lambda[(1+x^c)^\alpha - 1]),$$

respectively, for  $\alpha, \lambda, \delta, c > 0$ , and  $x > 0$ . The corresponding hrf is given by

$$h_F(x) = \left( -\log \left[ 1 - \frac{\lambda + (1+x^c)^{-\alpha}}{(1+\lambda)(1+x^c)^{-\alpha}} e^{(-\lambda[(1+x^c)^\alpha - 1])} \right] \right)^{\delta-1} \\ \times \frac{\lambda^2 \alpha}{(1+\lambda)} \frac{cx^{c-1}(1+x^c)^{-2}}{(1+x^c)^{-2\alpha-1}} \exp(-\lambda[(1+x^c)^\alpha - 1]), \\ \times \left( \gamma \left( -\log \left[ 1 - \frac{\lambda + (1+x^c)^{-\alpha}}{(1+\lambda)(1+x^c)^{-\alpha}} e^{(-\lambda[(1+x^c)^\alpha - 1])} \right], \delta \right) \right)^{-1}.$$

Figure 2 illustrates the graphs of the pdf and hrf of the GOLG-Log-logistic distribution using several combinations of the parameters  $\alpha, \lambda, \delta$ , and  $c$ .

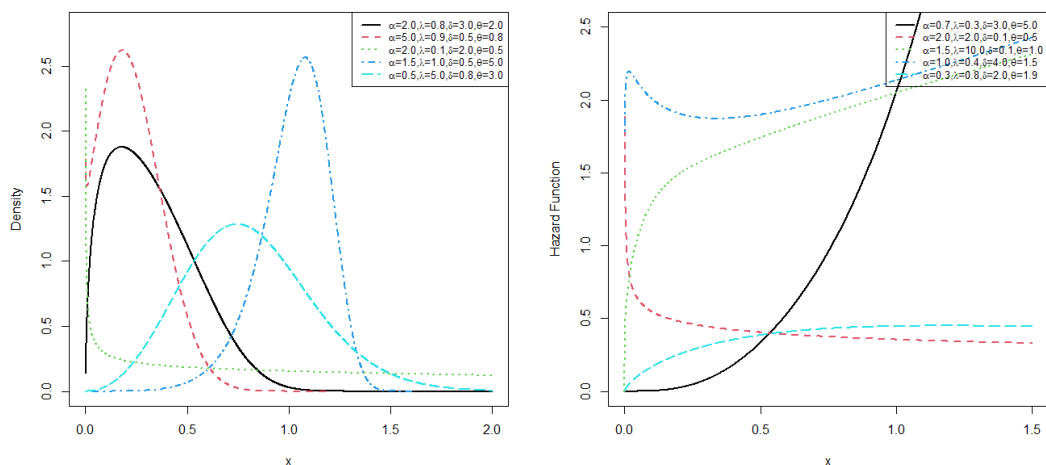


**Figure 2** Plots of the pdf and hrf for the GOLG-Log-logistic distribution

The shapes of the pdf include uni-modal, left- or right-skewed and reverse-J. The plots of the hrf shows increasing, decreasing, and upside-down bathtub followed by bathtub shapes. We present in Figures 3 and 4, 3D plots of skewness and kurtosis of the GOLG-Log-logistic distribution. We note that

- GOLG-Log-logistic skewness and kurtosis increase as  $\delta$  increases when  $\alpha, \lambda$ , and  $c$  are all fixed.
- When  $\delta$  and  $\lambda$  are fixed, GOLG-Log-logistic skewness decreases as  $c$  or  $\alpha$  is increased.
- When  $\delta, \alpha$  and  $c$  are fixed, GOLG-Log-logistic skewness increases as  $\lambda$  is increased.

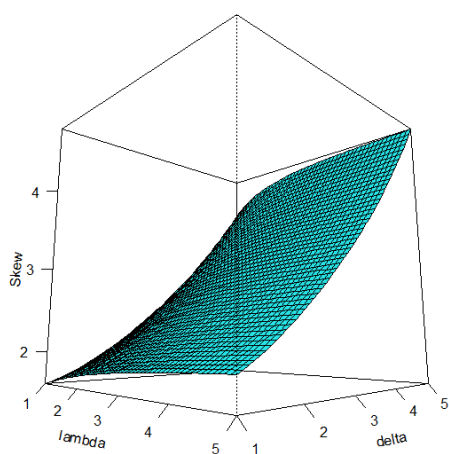
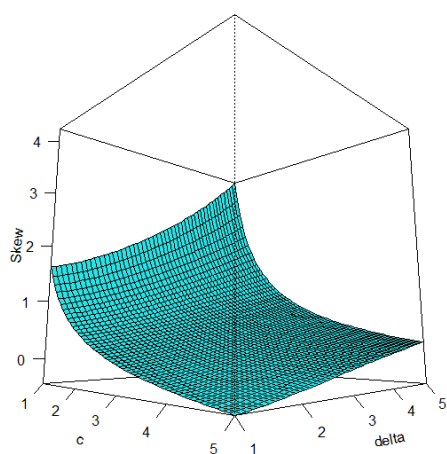
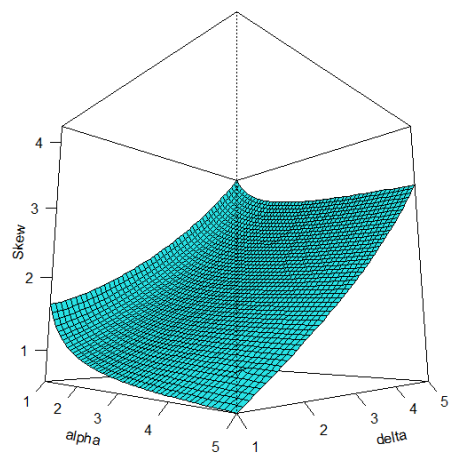
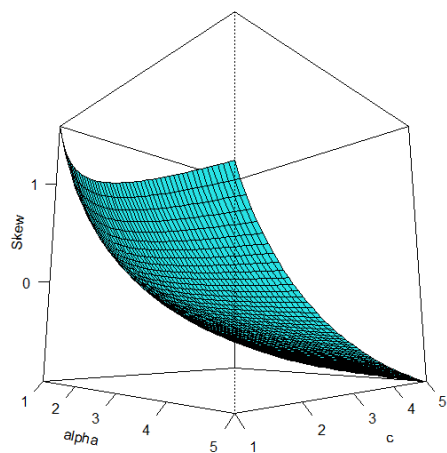
GOLG-LLoG(1.0,  $\lambda$ ,  $\delta$ , 1.0)GOLG-LLoG(1.0, 1.0,  $\delta$ ,  $c$ )GOLG-LLoG( $\alpha$ , 1.0,  $\delta$ , 1.0)GOLG-LLoG( $\alpha$ , 1.0, 1.0,  $c$ )

**Figure 3** 3D plots of the skewness for the GOLG-Log-logistic distribution for some selected parameter values

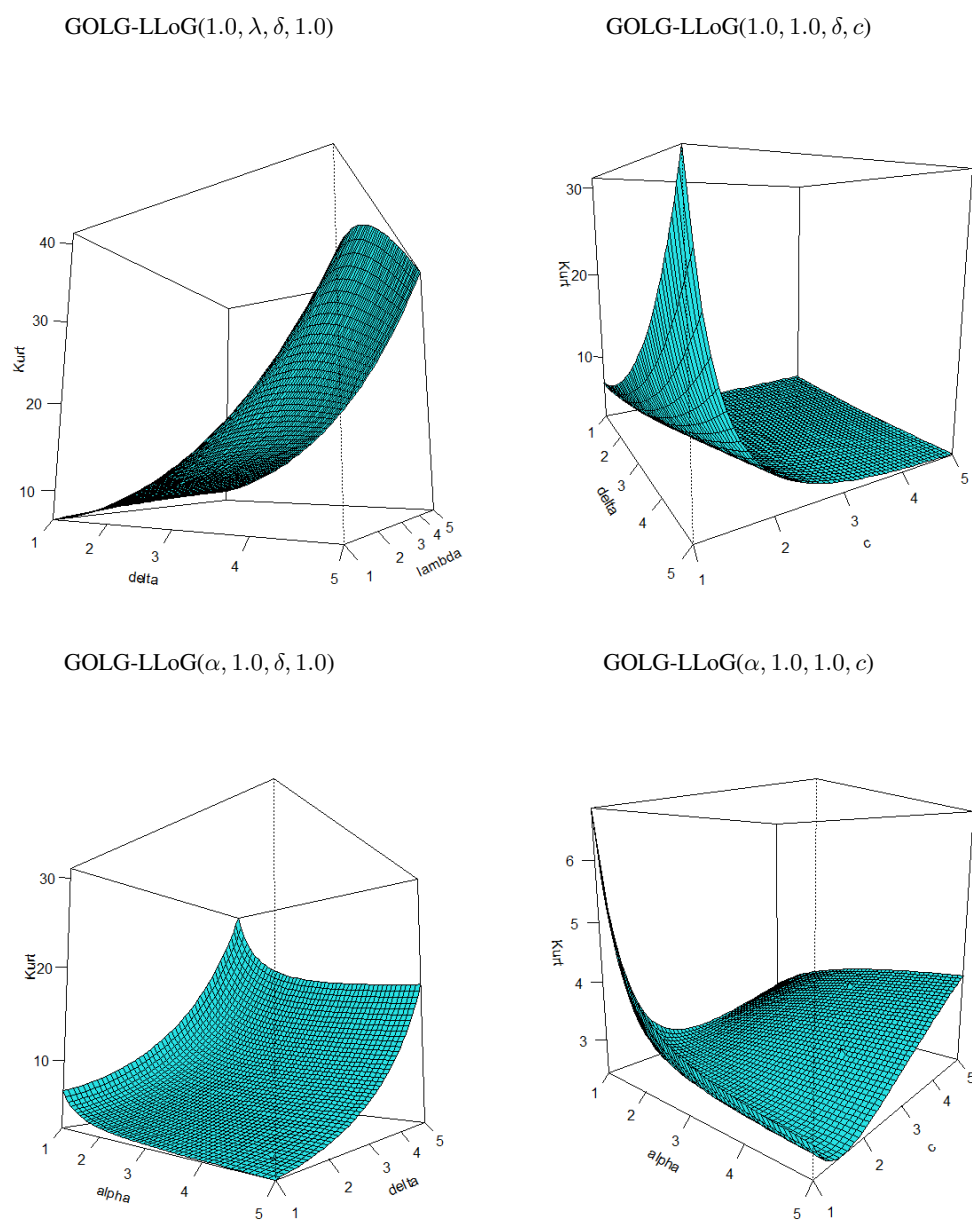


**Figure 4** 3D plots of the kurtosis (Kurt) for the GOLG-Log-logistic distribution for some selected parameter values

### 4.3. GOLG-logistic distribution

Suppose the baseline distribution has the cdf and pdf given by  $G(x; \sigma) = (1 + \exp(-x/\sigma))^{-1}$  and  $g(x; \sigma) = \frac{\exp(-x/\sigma)}{\sigma} (1 + \exp(-x/\sigma))^{-2}$  for  $\sigma > 0$ , respectively, then the new GOLG-logistic distribution has cdf and pdf given by

$$F(x; \alpha, \lambda, \delta, \sigma) = 1 - \frac{\gamma \left( -\log \left[ 1 - \frac{\lambda + [1 - (1 + e^{(-x/\sigma)})^{-1}]^\alpha}{(1 + \lambda)[1 - (1 + e^{(-x/\sigma)})^{-1}]^\alpha} e^{-\lambda([1 - (1 + e^{(-x/\sigma)})^{-1}]^{-\alpha} - 1)} \right], \delta \right)}{\Gamma(\delta)},$$

and

$$\begin{aligned} f(x; \alpha, \lambda, \delta, \sigma) &= \left( -\log \left[ 1 - \frac{\lambda + [1 - (1 + e^{(-x/\sigma)})^{-1}]^\alpha}{(1 + \lambda)[1 - (1 + e^{(-x/\sigma)})^{-1}]^\alpha} e^{-\lambda([1 - (1 + e^{(-x/\sigma)})^{-1}]^{-\alpha} - 1)} \right] \right)^{\delta-1} \\ &\times \frac{1}{\Gamma(\delta)} \frac{\lambda^2 \alpha}{(1 + \lambda)} \frac{e^{(-x/\sigma)} (1 + e^{(-x/\sigma)})^{-2}}{\sigma [1 - (1 + e^{(-x/\sigma)})^{-1}]^{2\alpha+1}} e^{-\lambda([1 - (1 + e^{(-x/\sigma)})^{-1}]^{-\alpha} - 1)}, \end{aligned}$$

respectively, for  $\alpha, \lambda, \delta, \sigma > 0$ . The hrf is given by

$$\begin{aligned} h_F(x) &= \left( -\log \left[ 1 - \frac{\lambda + [1 - (1 + e^{(-x/\sigma)})^{-1}]^\alpha}{(1 + \lambda)[1 - (1 + e^{(-x/\sigma)})^{-1}]^\alpha} e^{-\lambda([1 - (1 + e^{(-x/\sigma)})^{-1}]^{-\alpha} - 1)} \right] \right)^{\delta-1} \\ &\times \frac{\lambda^2 \alpha}{(1 + \lambda)} \frac{e^{(-x/\sigma)} (1 + e^{(-x/\sigma)})^{-2}}{\sigma [1 - (1 + e^{(-x/\sigma)})^{-1}]^{2\alpha+1}} e^{-\lambda([1 - (1 + e^{(-x/\sigma)})^{-1}]^{-\alpha} - 1)} \\ &\times \left( \gamma \left( -\log \left[ 1 - \frac{(\lambda + [1 - (1 + e^{(-x/\sigma)})^{-1}]^\alpha) e^{-\lambda([1 - (1 + e^{(-x/\sigma)})^{-1}]^{-\alpha} - 1)}}{(1 + \lambda)[1 - (1 + e^{(-x/\sigma)})^{-1}]^\alpha} \right], \delta \right) \right)^{-1}. \end{aligned}$$

Figure 5 shows graphs of pdf and hrf for different combinations of the parameters of the GOLG-logistic distribution. The shapes of the graphs of the pdf of the GOLG-logistic distribution include

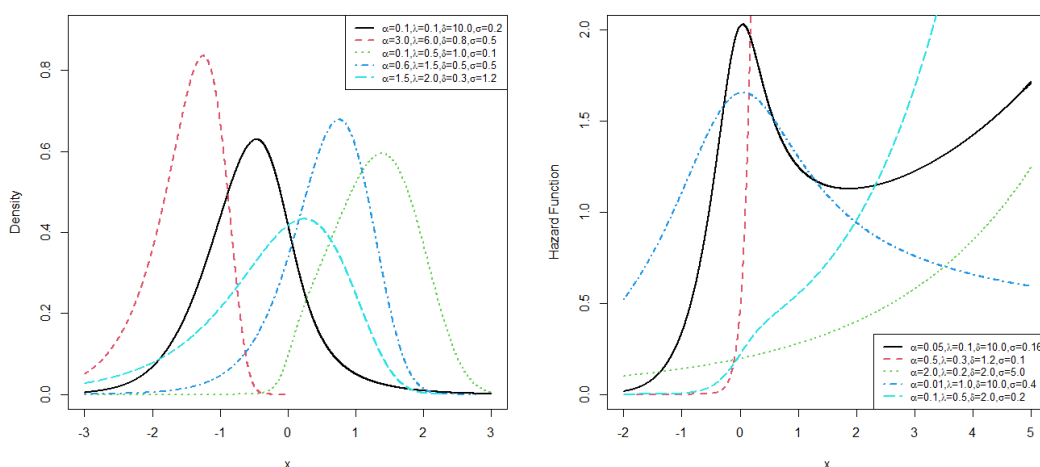


**Figure 5** Plots of the pdf and hrf for the GOLG-logistic distribution

uni-modal and left- or right-skewed. Plots of the hrf show increasing, upside-down bathtub and upside-down bathtub followed by bathtub shapes.

#### 4.4. GOLG-Weibull distribution

Consider the Weibull distribution as the baseline distribution with the cdf and pdf given by  $G(x; \beta) = 1 - \exp(-x^\beta)$  and  $g(x; \beta) = \beta x^{\beta-1} \exp(-x^\beta)$  for  $\beta > 0$  and  $x > 0$ , respectively, then the resulting new GOLG-Weibull distribution has cdf and pdf given by

$$F(x; \alpha, \lambda, \delta, \beta) = 1 - \frac{\gamma \left( -\log \left( 1 - \frac{\lambda + e^{-\alpha x^\beta}}{(1+\lambda)e^{-\alpha x^\beta}} e^{-\lambda(e^{\alpha x^\beta} - 1)} \right), \delta \right)}{\Gamma(\delta)},$$

and

$$\begin{aligned} f(x; \alpha, \lambda, \delta, \beta) &= \frac{1}{\Gamma(\delta)} \left( -\log \left( 1 - \frac{\lambda + e^{-\alpha x^\beta}}{(1+\lambda)e^{-\alpha x^\beta}} e^{-\lambda(e^{\alpha x^\beta} - 1)} \right) \right)^{\delta-1} \\ &\times \frac{\lambda^2 \alpha}{(1+\lambda)} \frac{\beta x^{\beta-1} e^{-x^\beta}}{e^{-(2\alpha+1)x^\beta}} \exp \left( -\lambda \left[ e^{\alpha x^\beta} - 1 \right] \right), \end{aligned}$$

respectively, for  $\alpha, \lambda, \delta, \beta > 0$ , and  $x > 0$ .

Observe that when  $\beta = 1$  and  $\beta = 2$ , we obtain the GOLG-exponential (GOLG-E) and GOLG-Rayleigh (GOLG-R) distributions, respectively. The GOLG-Weibull hrf is given by

$$\begin{aligned} h_F(x) &= \left( -\log \left( 1 - \frac{\lambda + e^{-\alpha x^\beta}}{(1+\lambda)e^{-\alpha x^\beta}} e^{-\lambda(e^{\alpha x^\beta} - 1)} \right) \right)^{\delta-1} \\ &\times \frac{\lambda^2 \alpha}{(1+\lambda)} \frac{\beta x^{\beta-1} \exp(-x^\beta)}{e^{-(2\alpha+1)x^\beta}} \exp \left( -\lambda \left[ e^{\alpha x^\beta} - 1 \right] \right) \\ &\times \left( \gamma \left( -\log \left( 1 - \frac{\lambda + e^{-\alpha x^\beta}}{(1+\lambda)e^{-\alpha x^\beta}} e^{-\lambda(e^{\alpha x^\beta} - 1)} \right), \delta \right) \right)^{-1}. \end{aligned}$$

Figure 6 shows the graphs of the pdf and hrf of the GOLG-Weibull distribution using several combinations of the parameters  $\alpha, \lambda, \delta$ , and  $\beta$ .

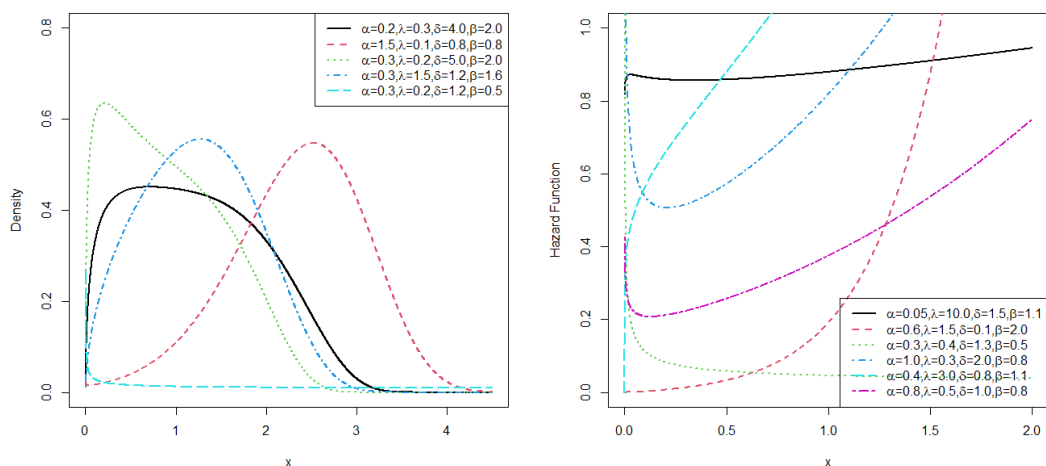


**Figure 6** Plots of the pdf and hrf for the GOLG-Weibull distribution

The pdf of the GOLG-Weibull distribution take various shapes that include uni-modal, right-skewed, left-skewed and reverse-J. Plots of the hrf for the GOLG-Weibull distribution show decreasing, increasing, bathtub and upside-down bathtub followed by bathtub shapes for the selected values of the model parameters. 3D plots of skewness and kurtosis of GOLG-Weibull distribution are presented in the Appendix.

### 5. Maximum Likelihood Estimation

Let  $X \sim \text{GOLG} - G(\alpha, \lambda, \delta, \xi)$  and  $\Delta = (\alpha, \lambda, \delta, \xi)^T$  be the vector of model parameters. The log-likelihood function  $\ell_n = \ell_n(\Delta)$  based on a random sample of size  $n$  from the GOLG-G family of distributions is given by

$$\begin{aligned} \ell_n(\Delta) &= -n \ln(\Gamma(\delta)) + (\delta - 1) \sum_{i=1}^n \ln \left( -\ln \left[ 1 - \frac{\lambda + \bar{G}^\alpha(x_i; \xi)}{(1 + \lambda) \bar{G}^\alpha(x_i; \xi)} e^{-\lambda \left( \frac{1 - \bar{G}^\alpha(x_i; \xi)}{\bar{G}^\alpha(x_i; \xi)} \right)} \right] \right) \\ &+ 2n \ln(\lambda) + n \ln(\alpha) - n \ln(1 + \lambda) + \sum_{i=1}^n \ln(g(x_i; \xi)) \\ &- (2\alpha + 1) \sum_{i=1}^n \ln(\bar{G}(x_i; \xi)) - \lambda \sum_{i=1}^n (\bar{G}^{-\alpha}(x_i; \xi) - 1). \end{aligned} \quad (24)$$

The first derivative of the log-likelihood function with respect to each component of the parameter vector  $\Delta = (\alpha, \lambda, \delta, \xi)^T$ , that is, elements of the score vector  $U(\Delta)$  are given in the Appendix. Let  $\hat{\Delta} = (\hat{\alpha}, \hat{\lambda}, \hat{\delta}, \hat{\xi})$  be the maximum likelihood estimate of  $\Delta = (\alpha, \lambda, \delta, \xi)$  which is obtained by solving the nonlinear equation  $(\frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \lambda}, \frac{\partial \ell_n}{\partial \delta}, \frac{\partial \ell_n}{\partial \xi_k})^T = \mathbf{0}$ , ( $k = 1, 2, \dots, q$ ) using a numerical method such as Newton-Raphson procedure. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have:  $\sqrt{n}(\hat{\Delta} - \Delta) \xrightarrow{d} N_{q+3}(\mathbf{0}, I^{-1}(\Delta))$ , where  $I(\Delta)$  is the expected Fisher information matrix. The asymptotic behavior is still valid if  $I(\Delta)$  is replaced by the observed information matrix evaluated at  $\hat{\Delta}$ , that is  $J(\hat{\Delta})$ . The multivariate normal distribution  $N_{q+3}(\mathbf{0}, (J(\hat{\Delta}))^{-1})$ , where the mean vector  $\mathbf{0} = (0, 0, 0, \mathbf{0})^T$ , can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions. That is, the approximate 100(1 -  $\eta$ )% two-sided confidence intervals for  $\alpha$ ,  $\lambda$ ,  $\delta$  and  $\xi_k$  are given by:

$$\hat{\alpha} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\alpha\alpha}^{-1}(\hat{\Delta})}, \quad \hat{\lambda} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\lambda\lambda}^{-1}(\hat{\Delta})}, \quad \hat{\delta} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\delta\delta}^{-1}(\hat{\Delta})}, \quad \text{and} \quad \hat{\xi}_k \pm Z_{\frac{\eta}{2}} \sqrt{I_{\xi_k \xi_k}^{-1}(\hat{\Delta})},$$

respectively, where  $I_{\alpha\alpha}^{-1}(\hat{\Delta})$ ,  $I_{\lambda\lambda}^{-1}(\hat{\Delta})$ ,  $I_{\delta\delta}^{-1}(\hat{\Delta})$ , and  $I_{\xi_k \xi_k}^{-1}(\hat{\Delta})$  are the diagonal elements of  $I_n^{-1}(\hat{\Delta})$ , and  $Z_{\frac{\eta}{2}}$  is the upper  $\frac{\eta}{2}$ th percentile of a standard normal distribution.

We can use the likelihood ratio (LR) test to compare the fit of the GOLG-G family of distributions with its sub-models for a given data set. For example, to test  $\delta = 1$ , the LR statistic is  $\omega = 2[\ln(L(\hat{\alpha}, \hat{\lambda}, \hat{\delta}, \hat{\xi}_k)) - \ln(L(\tilde{\alpha}, \tilde{\lambda}, 1, \tilde{\xi}_k))]$ , where  $\hat{\alpha}$ ,  $\hat{\lambda}$ ,  $\hat{\delta}$  and  $\hat{\xi}_k$  are the unrestricted estimates, and  $\tilde{\alpha}$ ,  $\tilde{\lambda}$ , and  $\tilde{\xi}_k$  are the restricted estimates. The LR test rejects the null hypothesis if  $\omega > \chi_e^2$ , where  $\chi_e^2$  denote the upper 100 $\epsilon$ % point of the  $\chi^2$  distribution with 1 degree of freedom.

### 6. Simulation Study

In this section, we conduct a simulation study to examine the bias and mean squared error of the maximum likelihood estimates of the GOLG-Weibull distribution.

The performance of the GOLG-Weibull distribution is examined by conducting various simulations for different sizes ( $n=25, 50, 100, 200, 400, 800, 1000$ ) via the R package. We simulate  $N = 1000$  samples for the true parameters values given in Table 2. The table lists the mean MLEs of

the model parameters along with the respective average bias and root mean squared errors (RMSEs). The average bias (ABias) and RMSE for the estimated parameter, say  $\hat{\theta}$  are given by:

$$ABias(\hat{\theta}) = \frac{\sum_{i=1}^N \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^N (\hat{\theta}_i - \theta)^2}{N}},$$

respectively. From the results, it is evident that the mean estimates of the parameters tend to approach the true values, as the average bias and RMSEs move toward their ideal values with increasing sample size  $n$ .

**Table 2** Monte Carlo simulation results for GOLG-W distribution: mean, ABias and RMSE

| Parameter | $n$  | $\alpha = 0.5, \lambda = 0.5, \delta = 2.0, \beta = 2.5$ |           |          | $\alpha = 1.5, \lambda = 1.0, \delta = 0.5, \beta = 1.0$ |           |          |
|-----------|------|----------------------------------------------------------|-----------|----------|----------------------------------------------------------|-----------|----------|
|           |      | Mean                                                     | ABias     | RMSE     | Mean                                                     | ABias     | RMSE     |
| $\alpha$  | 25   | 0.745910                                                 | 0.245910  | 0.767577 | 1.686528                                                 | 0.186528  | 1.052639 |
|           | 50   | 0.643004                                                 | 0.143004  | 0.616705 | 1.610593                                                 | 0.110593  | 0.967177 |
|           | 100  | 0.599075                                                 | 0.099075  | 0.521132 | 1.430423                                                 | -0.069577 | 0.852004 |
|           | 200  | 0.536043                                                 | 0.036043  | 0.368152 | 1.455868                                                 | -0.044132 | 0.768591 |
|           | 400  | 0.523801                                                 | 0.023801  | 0.238603 | 1.486223                                                 | -0.013777 | 0.624917 |
|           | 800  | 0.508089                                                 | 0.008089  | 0.163580 | 1.486661                                                 | -0.013339 | 0.435940 |
|           | 1000 | 0.500237                                                 | 0.000237  | 0.148485 | 1.499002                                                 | -0.000998 | 0.404797 |
| $\lambda$ | 25   | 1.279807                                                 | 0.779807  | 8.234771 | 0.646604                                                 | -0.353396 | 1.377716 |
|           | 50   | 0.577737                                                 | 0.077737  | 1.465193 | 0.684380                                                 | -0.315620 | 1.245598 |
|           | 100  | 0.443868                                                 | -0.056132 | 0.678799 | 0.747091                                                 | -0.252909 | 0.828852 |
|           | 200  | 0.460328                                                 | -0.039672 | 0.448663 | 0.860750                                                 | -0.139250 | 0.701976 |
|           | 400  | 0.469526                                                 | -0.030474 | 0.307298 | 0.907531                                                 | -0.092469 | 0.567369 |
|           | 800  | 0.486298                                                 | -0.013702 | 0.225054 | 0.941964                                                 | -0.058036 | 0.422941 |
|           | 1000 | 0.494729                                                 | -0.005271 | 0.196749 | 0.951830                                                 | -0.048170 | 0.380218 |
| $\delta$  | 25   | 4.325143                                                 | 2.325143  | 3.572806 | 3.352072                                                 | 2.852072  | 3.939103 |
|           | 50   | 3.916158                                                 | 1.916158  | 2.996916 | 2.619761                                                 | 2.119761  | 3.063822 |
|           | 100  | 3.444451                                                 | 1.444451  | 2.439067 | 2.100799                                                 | 1.600799  | 2.411058 |
|           | 200  | 2.930508                                                 | 0.930508  | 1.808170 | 1.415963                                                 | 0.915963  | 1.464846 |
|           | 400  | 2.483528                                                 | 0.483528  | 1.233943 | 0.997695                                                 | 0.497695  | 0.804726 |
|           | 800  | 2.251456                                                 | 0.251456  | 0.807606 | 0.708991                                                 | 0.208991  | 0.380382 |
|           | 1000 | 2.169479                                                 | 0.169479  | 0.717183 | 0.664314                                                 | 0.164314  | 0.328376 |
| $\beta$   | 25   | 3.312422                                                 | 0.812422  | 1.948732 | 1.746938                                                 | 0.746938  | 1.183072 |
|           | 50   | 3.165877                                                 | 0.665877  | 1.548747 | 1.608542                                                 | 0.608542  | 1.023895 |
|           | 100  | 3.004299                                                 | 0.504299  | 1.278089 | 1.590102                                                 | 0.590102  | 0.921936 |
|           | 200  | 2.858887                                                 | 0.358887  | 0.947857 | 1.409713                                                 | 0.409713  | 0.683080 |
|           | 400  | 2.677367                                                 | 0.177367  | 0.671524 | 1.285789                                                 | 0.285789  | 0.505886 |
|           | 800  | 2.600906                                                 | 0.100906  | 0.453862 | 1.192595                                                 | 0.192595  | 0.386737 |
|           | 1000 | 2.563456                                                 | 0.063456  | 0.419434 | 1.181161                                                 | 0.181161  | 0.375305 |

7. Applications

Examples to illustrate the flexibility and usefulness of the GOLG-Log-logistic distribution for data modeling are presented in this section. The GOLG-Log-logistic distribution is compared with OLG-Log-logistic, Kumaraswamy odd Lindley-Log-logistic (KOL-LLoG) (Chipepa et al., 2019), exponentiated half-logistic odd Lindley-Weibull (EHL-OL-W) (Sengweni et al., 2021) and transmuted odd Lindley-Lomax (TOL-Lx) (Reyad et al., 2018), beta generalized Lindley (BGL) (Oluyede and Yang, 2015), beta Weibull (BW) (Lee et al., 2007) and new modified Weibull (NMW) (Doostmoradi et al., 2014) distributions. The competing non-nested models are selected to provide a fair comparison with the proposed GOLG-Log-logistic distribution, focusing on equiparameter families of similar flexibility and complexity. These models represent a range of recently proposed generalized Lindley-type and Weibull-type families and are widely cited in the literature, serving as well-established benchmarks for evaluating goodness-of-fit and modeling flexibility. The pdfs of the KOL-LLoG, EHL-OL-W, TOL-Lx, BGL, BW and NMW distributions are given in the Appendix.

The NLMixed procedure in SAS was used to estimate the model parameters and the package AdequacyModel in R software for goodness-of-fit statistics. The estimated values of the parameters (standard error in parenthesis),  $-2\log\text{-likelihood statistic } (-2\ln(L))$ , Akaike Information Criterion ( $AIC = 2p - 2\ln(L)$ ), Bayesian Information Criterion ( $BIC = p\ln(n) - 2\ln(L)$ ) and Consistent Akaike Information Criterion ( $AICC = AIC + 2\frac{p(p+1)}{n-p-1}$ ), where  $L = L(\hat{\Delta})$  is the value of the likelihood function evaluated at the parameter estimates,  $n$  is the number of observations, and  $p$  is the number of estimated parameters are presented.

Also presented are the following goodness-of-fit statistics: Cramér-von Mises ( $W^*$ ) and Anderson-Darling Statistics ( $A^*$ ) described by Chen and Balakrishnan (1995), as well as Kolmogorov-Smirnov (K-S) statistic and its P-value. It is well known that for the value of the log-likelihood function at its maximum ( $\ell_n$ ), larger value is good and preferred, and for AIC, AICC, BIC, and the goodness-of-fit statistics  $W^*$ ,  $A^*$  and  $K - S$ , smaller values are preferred. The results are presented in Tables 5 and 4.

We present plots of the fitted densities, the histogram of the data and probability plots (Chambers et al., 1983) for each example to show how well our model fits the observed data sets. To obtain the probability plot, we plotted  $F_{GOLG-G}(x_{(j)}; \hat{\alpha}, \hat{\lambda}, \hat{\delta}, \hat{\xi})$  against  $\frac{j - 0.375}{n + 0.25}$ ,  $j = 1, 2, \dots, n$ , where  $x_{(j)}$  are the ordered values of the observed data. The measures of closeness are given by the sum of squares,  $SS = \sum_{j=1}^n \left[ F_{GOLG-G}(x_{(j)}; \hat{\alpha}, \hat{\lambda}, \hat{\delta}, \hat{\xi}) - \left( \frac{j - 0.375}{n + 0.25} \right) \right]^2$ . These plots are shown in Figures 11 and 9. The total time on test (TTT) scaled plots, estimated cdfs and K-M survival curves are presented in Figures 12 and 10, respectively for both data sets.

### 7.1. Active repair times data

The first data set consists of 40 active repair times (hours) for an airborne communication transceiver (see Chhikara and Folks (1977)). The data range from 0.50 to 24.50 hours, with a sample mean of 4.01 hours and a standard deviation of 5.17 hours, indicating substantial variability in repair times. The median is 2.10 hours, suggesting that most repairs are completed relatively quickly. The distribution exhibits strong right skewness, with a sample skewness of 2.616, reflecting the presence of several large repair times. In addition, the kurtosis value of 10.023 indicates a highly leptokurtic distribution with heavy tails, making this dataset particularly suitable for flexible lifetime models capable of capturing skewness and tail behavior. The data are as follows:

0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00, 24.50.

The estimated variance-covariance matrix for GOLG-Log-logistic model on active repair times data set is given by

$$\begin{pmatrix} 0.0022 & -0.0007 & 0.0115 & -0.049 \\ -0.0007 & 0.0037 & -0.1451 & -0.056 \\ 0.0115 & -0.1451 & 5.9770 & 2.512 \\ -0.0490 & -0.0560 & 2.5120 & 3.086 \end{pmatrix},$$

and the 95% two-sided asymptotic confidence intervals for  $\alpha, \lambda, \delta$  and  $c$  are given by  $0.1340 \pm 0.0915$ ,  $0.0428 \pm 0.1200$ ,  $6.6827 \pm 4.7918$ , and  $5.8237 \pm 3.4431$ , respectively.

Table 5 indicates that GOLG-Log-logistic distribution has the highest p-value for the K-S statistic and the lowest values for all goodness-of-fit statistics. The GOLG-Log-logistic distribution demonstrates superior performance when compared with other non-nested KOL-LLoG, EHLol-W, TOL-Lx, BGL, BW and NMW models for this data set. Further, Figure 7 shows that our model outperforms the competing non-nested models. To formally assess the improvement over the nested OLG-Log-logistic model, a likelihood ratio test was conducted. The test statistic, calculated as  $-2(\ell_{\text{OLG-Log-logistic}} - \ell_{\text{GOLG-Log-logistic}}) = -2(189.0 - 177.3) = 23.4$ , follows a chi-square distribu-

**Table 3** Parameter estimates and goodness-of-fit statistics for various models fitted for active repair times data

| Model             | Estimates                                                 |                                                                             |                                       |                                                     | Statistics  |       |        |       |        |        |         |         |
|-------------------|-----------------------------------------------------------|-----------------------------------------------------------------------------|---------------------------------------|-----------------------------------------------------|-------------|-------|--------|-------|--------|--------|---------|---------|
|                   | $\hat{\alpha}$                                            | $\hat{\lambda}$                                                             | $\hat{\delta}$                        | $\hat{c}$                                           | $-2 \log L$ | $AIC$ | $AICC$ | $BIC$ | $W^*$  | $A^*$  | $K - S$ | p-value |
| GOLG-Log-logistic | 0.1340<br>(0.0467)                                        | 0.0428<br>(0.0612)                                                          | 6.6827<br>(2.4448)                    | 5.8237<br>(1.7567)                                  | 177.3       | 185.3 | 186.4  | 192.1 | 0.0406 | 0.2846 | 0.0929  | 0.8803  |
| OLG-Log-logistic  | $\hat{\alpha}$<br>15.5340<br>( $4.4627 \times 10^{-05}$ ) | $\hat{\lambda}$<br>$1.8856 \times 10^{-05}$<br>( $3.0267 \times 10^{-06}$ ) | $\hat{\delta}$<br>1<br>-              | $\hat{c}$<br>0.0817<br>( $8.6072 \times 10^{-03}$ ) | 189.0       | 195.0 | 195.7  | 200.1 | 0.1270 | 0.9201 | 0.1273  | 0.5362  |
| KOL-LLoG          | $\hat{a}$<br>5.7423<br>(3.0222)                           | $\hat{b}$<br>0.1625<br>(0.0307)                                             | $\hat{\lambda}$<br>3.5464<br>(0.7555) | $\hat{c}$<br>0.7487<br>(0.0674)                     | 182.2       | 190.2 | 191.4  | 197   | 0.0705 | 0.5423 | 0.1300  | 0.5105  |
| EHLLO-W           | $\hat{\alpha}$<br>162.7400<br>(0.0016)                    | $\hat{\lambda}$<br>7.2969<br>(0.6123)                                       | $\hat{a}$<br>0.1385<br>(0.0186)       | $\hat{b}$<br>46.4550<br>(0.0203)                    | 179.7       | 187.7 | 188.8  | 194.4 | 0.0585 | 0.3985 | 0.1043  | 0.7767  |
| TOL-Lx            | $\hat{\alpha}$<br>0.0989<br>(0.0594)                      | $\hat{\lambda}$<br>0.5992<br>(0.3988)                                       | $\hat{\beta}$<br>14.2601<br>(11.1026) | $\hat{\theta}$<br>0.7059<br>(0.0825)                | 186.8       | 194.8 | 196.0  | 201.6 | 0.1084 | 0.7741 | 0.1280  | 0.5290  |
| BGL               | $\hat{\alpha}$<br>2.3077<br>(2.9060)                      | $\hat{\lambda}$<br>1.6205<br>(0.0422)                                       | $\hat{a}$<br>0.6328<br>(0.6203)       | $\hat{b}$<br>0.1750<br>(0.0338)                     | 191.0       | 199.0 | 200.1  | 205.7 | 0.1406 | 1.0202 | 0.1985  | 0.0856  |
| BW                | $\hat{a}$<br>0.0781<br>(0.0084)                           | $\hat{b}$<br>171.93<br>(118.31)                                             | $\hat{\lambda}$<br>200.22<br>(0.9169) | $\hat{c}$<br>51.0092<br>(1.1375)                    | 181.7       | 189.7 | 190.9  | 196.5 | 0.0697 | 0.4969 | 0.1226  | 0.5845  |
| NMW               | $\hat{\alpha}$<br>0.0766<br>(0.0496)                      | $\hat{\lambda}$<br>0.9424<br>(0.2060)                                       | $\hat{\gamma}$<br>1.6411<br>(0.3349)  | $\hat{\beta}$<br>0.1739<br>(0.0759)                 | 186.4       | 194.4 | 195.6  | 201.2 | 0.0774 | 0.6104 | 0.1164  | 0.6500  |

tion with one degree of freedom under the null hypothesis. With a corresponding p-value well below 0.001, the test strongly rejects the null hypothesis at 0.05 significance level, indicating that the additional parameter in the GOLG-Log-logistic provides a highly statistically significant improvement over the OLG-Log-logistic for this data set. This conclusion is further supported by the GOLG-Log-logistic’s superior performance across all model selection criteria, including information measures ( $AIC = 185.3$  vs.  $195.0$ ,  $AICC = 186.4$  vs.  $195.7$ ,  $BIC = 192.1$  vs.  $200.1$ ) and goodness-of-fit statistics ( $K - S = 0.0929$  vs.  $0.1273$ ,  $W^* = 0.0406$  vs.  $0.3270$ ,  $A^* = 0.2846$  vs.  $0.9201$ ). The substantial differences in both the likelihood ratio test and all goodness-of-fit measures demonstrate that the additional flexibility introduced by the gamma generator in the GOLG-Log-logistic translates to meaningful and statistically significant improvements in model performance for this data set. This result illustrates that while GOLG-Log-logistic is a generalization of OLG-Log-logistic, there are cases where the enhanced flexibility provides substantial benefits, making it the clearly preferred model for certain data structures.

In Figure 8, we see that the cdf line for the GOLG-Log-logistic distribution indicated by the blue line is closer to the empirical cdf while the survival function in blue is also close to the Kaplan-Meier(K-M) curve which indicate that our model is the best in explaining the active repair times data. The TTT plot for the active repair times data, along with the  $45^\circ$  reference line, indicates a unimodal hazard rate function. Specifically, the plot rises above the  $45^\circ$  line initially, reaches a peak, and then falls below it, suggesting that the failure rate first increases and then decreases. This behavior supports the suitability of our proposed model for these data.

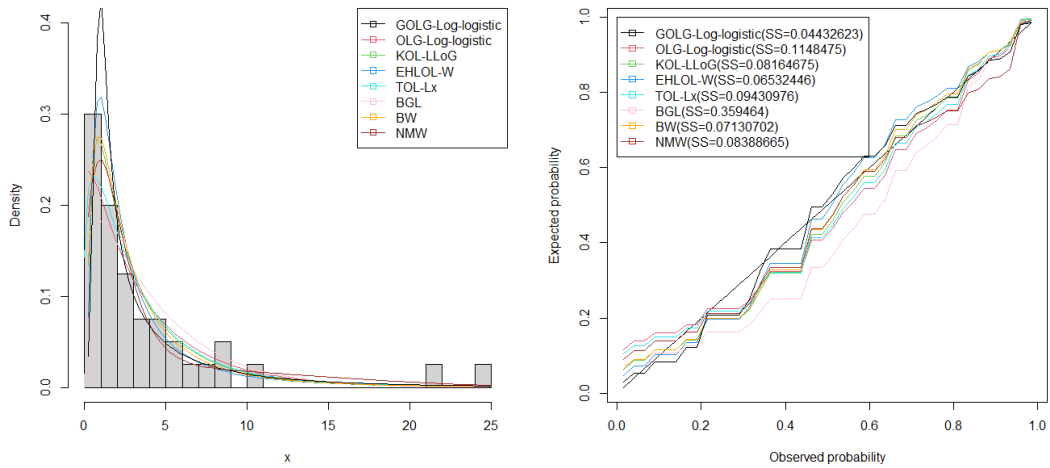


**Figure 7** Histogram, fitted density and probability plots for active repair times data

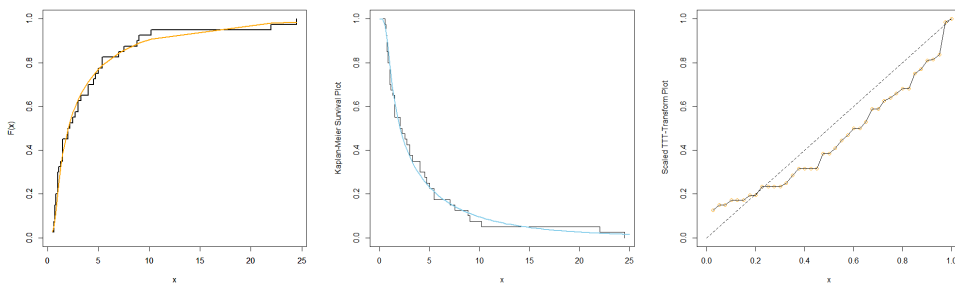


**Figure 8** Estimated cdf, Kaplan-Meier survival and scaled TTT-Transform plots for the GOLG-Log-logistic distribution for active repair times data

## 7.2. Level of mercury data

The second data set refers to the level of mercury in 34 albacore caught in the Eastern Mediterranean obtained from Mol et al. (2012). The mercury level data set consists of 34 observations ranging from 0.534 to 2.242, with a sample mean of 1.358 and a standard deviation of 0.441. The median value is 1.362, indicating that the distribution is well centered around its mean. The sample skewness is 0.209, suggesting mild right skewness, while the kurtosis value of 2.032 indicates a slightly flatter (platykurtic) distribution compared to the normal distribution. The observations are as follows:

1.007, 1.447, 0.763, 2.010, 1.346, 1.243, 1.586, 0.821, 1.735, 1.396, 1.109, 0.993, 2.007, 1.373, 2.242, 1.647, 1.350, 0.948, 1.501, 1.907, 1.952, 0.996, 1.433, 0.866, 1.049, 1.665, 2.139, 0.534, 1.027, 1.678, 1.214, 0.905, 1.525, 0.763.

The estimated variance-covariance matrix for GOLG-Log-logistic model on level of mercury data set is given by

$$\begin{pmatrix} 0.0414 & -0.0091 & 1.1249 & 0.0552 \\ -0.0091 & 0.0141 & -1.9137 & -1.0176 \\ 1.1249 & -1.9137 & 259.9800 & 139.5000 \\ 0.0552 & -1.0176 & 139.5000 & 84.5551 \end{pmatrix},$$

and the 95% two-sided asymptotic confidence intervals for  $\alpha, \lambda, \delta$  and  $c$  are given by  $0.4159 \pm 0.3865, 0.0127 \pm 0.1776, 7.6877 \pm 24.1051$ , and  $9.2200 \pm 14.3895$ , respectively.

**Table 4** Parameter estimates and goodness-of-fit statistics for various models fitted for level of mercury data

| Model             | Estimates                            |                                        |                                       |                                       | Statistics  |       |        |       |        |        |        |         |
|-------------------|--------------------------------------|----------------------------------------|---------------------------------------|---------------------------------------|-------------|-------|--------|-------|--------|--------|--------|---------|
|                   | $\hat{\alpha}$                       | $\hat{\lambda}$                        | $\hat{\delta}$                        | $\hat{c}$                             | $-2 \log L$ | $AIC$ | $AICC$ | $BIC$ | $W^*$  | $A^*$  | $K-S$  | p-value |
| GOLG-Log-logistic | 0.4159<br>(0.1972)                   | 0.0127<br>(0.0906)                     | 7.6877<br>(12.2985)                   | 9.2200<br>(7.3416)                    | 38.2        | 46.2  | 47.6   | 52.4  | 0.0346 | 0.2237 | 0.0973 | 0.9046  |
| OLG-Log-logistic  | $\hat{\alpha}$<br>0.4733<br>(0.3284) | $\hat{\lambda}$<br>1.0807<br>(0.6918)  | $\hat{\delta}$<br>1<br>(-)            | $\hat{c}$<br>4.3642<br>(1.6102)       | 38.9        | 44.9  | 45.7   | 49.5  | 0.0389 | 0.2576 | 0.1054 | 0.8446  |
| KOL-LLoG          | $\hat{a}$<br>1.3856<br>(1.4032)      | $\hat{b}$<br>0.0929<br>(0.0170)        | $\hat{\lambda}$<br>3.1200<br>(0.6820) | $\hat{c}$<br>3.0736<br>(0.2290)       | 39.4        | 47.4  | 48.8   | 53.5  | 0.0373 | 0.2558 | 0.1333 | 0.5814  |
| EHLol-W           | $\hat{\alpha}$<br>3.9549<br>(5.8514) | $\hat{\lambda}$<br>4.2385<br>(12.7045) | $\hat{a}$<br>1.0883<br>(1.4369)       | $\hat{b}$<br>2.4336<br>(4.7142)       | 39.0        | 47.0  | 48.4   | 53.1  | 0.0396 | 0.2606 | 0.1033 | 0.8611  |
| TOL-Lx            | $\hat{\alpha}$<br>0.1324<br>(0.1593) | $\hat{\lambda}$<br>0.3937<br>(0.5487)  | $\hat{\beta}$<br>0.0475<br>(0.6154)   | $\hat{\theta}$<br>37.1560<br>(447.95) | 41.3        | 49.3  | 50.7   | 55.5  | 0.0662 | 0.4630 | 0.1065 | 0.8352  |
| BGL               | $\hat{\alpha}$<br>1.2010<br>(0.2000) | $\hat{\lambda}$<br>0.2210<br>(0.0527)  | $\hat{a}$<br>4.3360<br>(0.3952)       | $\hat{b}$<br>83.7324<br>(2.6204)      | 38.9        | 46.9  | 48.3   | 53.0  | 0.0379 | 0.2491 | 0.1173 | 0.7377  |
| BW                | $\hat{k}$<br>3.0086<br>(0.6774)      | $\hat{\lambda}$<br>1.9538<br>(0.7491)  | $\hat{a}$<br>3.0473<br>(2.5778)       | $\hat{b}$<br>0.0507<br>(0.0892)       | 38.9        | 46.9  | 48.3   | 53.0  | 1.0154 | 5.6303 | 0.2059 | 0.1120  |
| NMW               | $\hat{\alpha}$<br>0.0410<br>(0.0419) | $\hat{\gamma}$<br>4.8836<br>(1.8504)   | $\hat{\lambda}$<br>4.4038<br>(0.9638) | $\hat{\beta}$<br>0.2485<br>(0.1033)   | 42.2        | 50.2  | 51.6   | 56.3  | 0.0506 | 0.3491 | 0.1061 | 0.8385  |

The results in Table 4 indicates that the GOLG-Log-logistic distribution demonstrates competitive performance when compared with other non-nested KOL-LLoG, EHLol-W, TOL-Lx, BGL, BW and NMW models for the level of mercury data set. In addition, Figure 9 shows that our model outperforms the competing non-nested models on level of mercury data. To formally assess the improvement over the nested OLG-Log-logistic model, a likelihood ratio test was conducted. The test statistic, calculated as  $-2(\ell_{\text{OLG-Log-logistic}} - \ell_{\text{GOLG-Log-logistic}}) = -2(38.9 - 38.2) = 1.4$ , follows a chi-square distribution with one degree of freedom under the null hypothesis. With a corresponding p-value of approximately 0.2367, the test fails to reject the null hypothesis at 0.05 significance level, indicating that the additional parameter in the GOLG-Log-logistic does not provide a statistically significant improvement over the OLG-Log-logistic for this data set. However, the results present a mixed picture: while the OLG-Log-logistic exhibits superior performance in information criteria ( $AIC = 44.9$  vs. 46.2,  $AICC = 45.7$  vs. 47.6,  $BIC = 49.5$  vs. 52.4), which penalize model complexity, the GOLG-Log-logistic demonstrates better goodness-of-fit across multiple measures ( $K-S = 0.0973$  vs. 0.1054,  $W^* = 0.0346$  vs. 0.0389,  $A^* = 0.2237$  vs. 0.2576). This divergence between information criteria and goodness-of-fit statistics suggests a trade-off between model parsimony and fit quality. Although the GOLG-Log-logistic offers enhanced flexibility through the gamma generator and achieves better empirical fit, the likelihood ratio test and information criteria indicate that the marginal improvement may not justify the additional complexity for this particular data set. This observation reinforces the interpretation that while GOLG-Log-logistic is a strict generalization of OLG-Log-logistic, the additional flexibility does not guarantee superior performance in all applications when model parsimony is considered.

In Figure 10, we see that the cdf curve for the GOLG-Log-logistic distribution indicated in blue is closer to the empirical cdf while the survival function in blue is also close to the Kaplan-Meier(K-M) curve which indicate that our model is the best in explaining the level of mercury data. The TTT plot for the level of mercury data, plotted against the 45° reference line, exhibits a concave pattern above the diagonal, indicating that the underlying hazard rate is increasing. This suggests that the probability of higher mercury levels occurring rises over time. Therefore, the data is consistent with an increasing hazard function, and our proposed model, which accommodates such hazard behavior, is appropriate for fitting this data set.

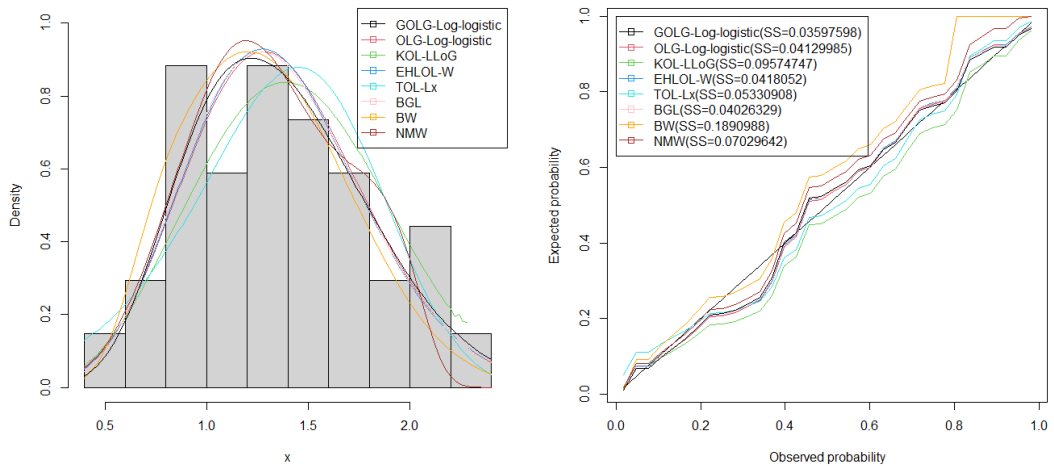


**Figure 9** Histogram, fitted density and probability plots for level of mercury data

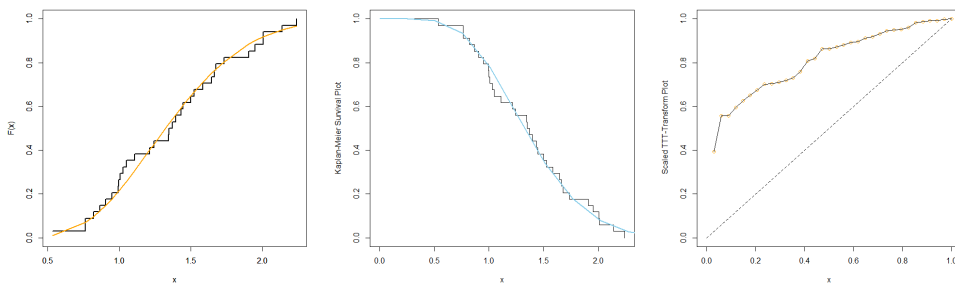


**Figure 10** Estimated cdf, Kaplan-Meier survival and scaled TTT-Transform plots for the GOLG-Log-logistic distribution for level of mercury data

### 7.3. Aircraft windshield data

The third data set consists of failure times of 84 aircraft windshields given in Table 16.11 of Murthy et al. (2004). The aircraft windshield data set contains 64 observations, with values ranging from 0.040 to 4.663. The sample mean is 2.557, the median is 2.355, and the standard deviation is 1.119, indicating moderate variability in the data. The skewness is 0.098, suggesting slight right skew, and the kurtosis is 2.292, indicating a distribution close to mesokurtic. The observations are as follows:

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

The estimated variance-covariance matrix for GOLG-Log-logistic model on aircraft windshield

data set is given by

$$\begin{pmatrix} 1.58 \times 10^{-3} & -5.27 \times 10^{-6} & -2.45 \times 10^{-4} & 3.24 \times 10^{-3} \\ -5.27 \times 10^{-6} & 2.23 \times 10^{-8} & 8.18 \times 10^{-7} & -1.08 \times 10^{-5} \\ -2.45 \times 10^{-4} & 8.18 \times 10^{-7} & 3.80 \times 10^{-5} & -5.02 \times 10^{-4} \\ 3.24 \times 10^{-3} & -1.08 \times 10^{-5} & -5.02 \times 10^{-4} & 6.64 \times 10^{-3} \end{pmatrix},$$

and the 95% two-sided asymptotic confidence intervals for  $\alpha, \lambda, \delta$  and  $c$  are given by  $2.5103 \pm 0.0778, 0.0005 \pm 0.0002, 8.1131 \pm 0.0122$ , and  $1.5391 \pm 0.1597$ , respectively.

**Table 5** Parameter estimates and goodness-of-fit statistics for various models fitted for aircraft windshield data

| Model             | Estimates                              |                                       |                                       |                                      | Statistics  |       |        |       |        |        |         |         |
|-------------------|----------------------------------------|---------------------------------------|---------------------------------------|--------------------------------------|-------------|-------|--------|-------|--------|--------|---------|---------|
|                   | $\hat{\alpha}$                         | $\hat{\lambda}$                       | $\hat{\delta}$                        | $\hat{c}$                            | $-2 \log L$ | $AIC$ | $AICC$ | $BIC$ | $W^*$  | $A^*$  | $K - S$ | p-value |
| GOLG-Log-logistic | 2.5103<br>(0.0397)                     | 0.0005<br>(0.0001)                    | 8.1131<br>(0.0062)                    | 1.5391<br>(0.08150)                  | 253.4       | 261.4 | 261.9  | 271.2 | 0.0532 | 0.4316 | 0.0662  | 0.8546  |
| OLG-Log-logistic  | $\hat{\alpha}$<br>2.2598<br>(1.2439)   | $\hat{\lambda}$<br>0.0962<br>(0.0910) | $\hat{\delta}$<br>1<br>-              | $\hat{c}$<br>1.0204<br>(0.4422)      | 254.9       | 260.9 | 261.2  | 268.2 | 0.0511 | 0.4472 | 0.0702  | 0.7732  |
| KOL-LLoG          | $\hat{a}$<br>0.2015<br>(0.0501)        | $\hat{b}$<br>3.0578<br>(4.7351)       | $\hat{\lambda}$<br>0.0003<br>(0.0010) | $\hat{c}$<br>5.2752<br>(0.2933)      | 256.8       | 264.8 | 265.3  | 274.5 | 0.1036 | 0.6650 | 0.1016  | 0.3510  |
| EHLLOL-W          | $\hat{\alpha}$<br>1.3605<br>(0.5349)   | $\hat{\lambda}$<br>0.5106<br>(1.2880) | $\hat{a}$<br>0.6720<br>(0.6095)       | $\hat{b}$<br>1.1495<br>(3.0353)      | 254.6       | 262.6 | 263.1  | 272.3 | 0.0883 | 0.6101 | 0.0861  | 0.5613  |
| TOL-Lx            | $\hat{\alpha}$<br>0.2061<br>(0.1703)   | $\hat{\lambda}$<br>0.2170<br>(0.4621) | $\hat{\beta}$<br>0.1958<br>(0.4235)   | $\hat{\theta}$<br>4.9654<br>(6.8228) | 253.6       | 261.6 | 262.2  | 271.4 | 0.0817 | 0.5698 | 0.0861  | 0.5612  |
| BGL               | $\hat{\alpha}$<br>39.8830<br>(26.7480) | $\hat{\lambda}$<br>0.6822<br>(0.1831) | $\hat{a}$<br>0.0487<br>(0.0446)       | $\hat{b}$<br>157.8600<br>(29.0346)   | 255.1       | 263.1 | 263.6  | 272.8 | 0.0995 | 0.6935 | 0.1026  | 0.3396  |
| BW                | $\hat{k}$<br>5.9092<br>(0.5880)        | $\hat{\lambda}$<br>0.2653<br>(0.0571) | $\hat{a}$<br>0.2715<br>(0.0474)       | $\hat{b}$<br>0.7834<br>(0.7805)      | 255.2       | 263.2 | 263.7  | 272.9 | 0.1330 | 0.8514 | 0.0862  | 0.5605  |
| NMW               | $\hat{\alpha}$<br>0.0938<br>(0.0223)   | $\hat{\gamma}$<br>1.8987<br>(0.4130)  | $\hat{\lambda}$<br>2.8630<br>(0.6011) | $\hat{\beta}$<br>0.0027<br>(0.0161)  | 255.8       | 267.8 | 268.3  | 277.5 | 0.1150 | 0.7573 | 0.0999  | 0.3707  |

According to the results in Table 5, the GOLG-Log-logistic distribution demonstrates superior performance when compared with other non-nested KOL-LLoG, EHLLOL-W, TOL-Lx, BGL, BW and NMW models across the aircraft windshield data set. Further, Figure 11 shows that GOLG-Log-logistic distribution outperforms the competing non-nested models. To formally assess the improvement over the nested OLG-Log-logistic model, a likelihood ratio test was conducted. The test statistic, calculated as  $-2(\ell_{\text{OLG-Log-logistic}} - \ell_{\text{GOLG-Log-logistic}}) = 3.0$ , follows a chi-square distribution with one degree of freedom under the null hypothesis. With a corresponding p-value of 0.0832, the test fails to reject the null hypothesis at 0.05 significance level, suggesting that the additional parameter in the GOLG-Log-logistic does not provide a statistically significant improvement over the OLG-Log-logistic for this particular data set. Nevertheless, the GOLG-Log-logistic exhibits better performance in terms of some goodness-of-fit measures ( $A^*=0.4316$  vs 0.4472,  $K - S = 0.0662$  vs. 0.0702), indicating practical improvements despite the marginal statistical significance. Although the GOLG-Log-logistic distribution offers enhanced flexibility through the gamma generator, the likelihood ratio test indicates that the marginal improvement over the simpler OLG-Log-logistic structure may not always be statistically compelling, suggesting that the parsimonious OLG-Log-logistic model can be sufficient for certain data sets. This observation supports the interpretation that GOLG-Log-logistic is a strict generalization of OLG-Log-logistic, offering enhanced flexibility rather than guaranteed dominance in all cases.

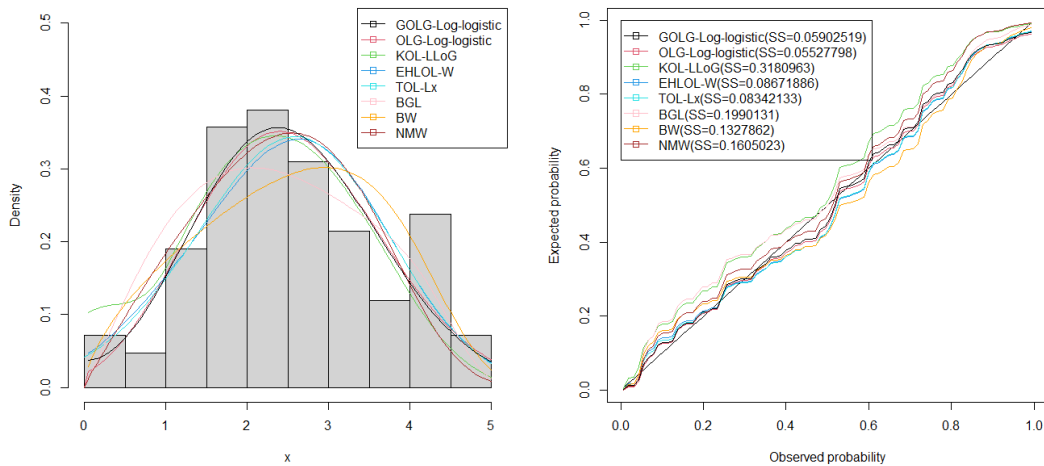


**Figure 11** Histogram, fitted density and probability plots for aircraft windshield data

In Figure 12, we see that the cdf line for the GOLG-Log-logistic distribution indicated by the blue line is closer to the empirical cdf while the survival function in blue is also close to the Kaplan-Meier(K-M) curve which indicate that our model is the best in explaining the aircraft windshield data. The hazard rate increases (decreases) if the scaled TTT-transform is concave (convex). In addition, for a distribution with bathtub (unimodal) failure rate the scaled TTT-transform is first convex (concave) and then concave (convex). The TTT plot for the aircraft windshield data exhibits a concave pattern above the  $45^\circ$  reference line, indicating that the underlying hazard rate is increasing.. This suggests that the likelihood of failure increases over time. Therefore, the data is consistent with an increasing hazard function, and our proposed model, which accommodates such hazard behavior, is appropriate for fitting this data set.

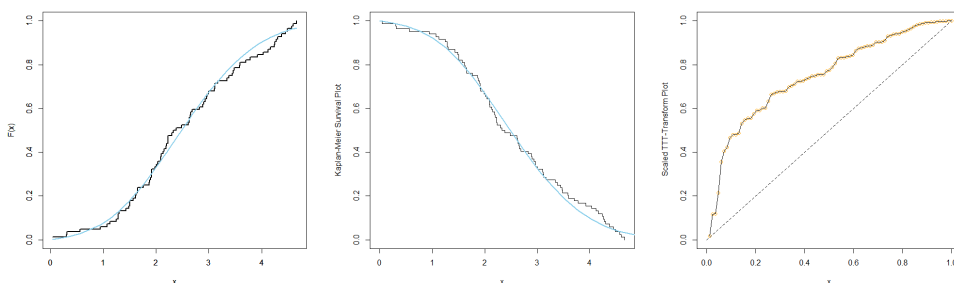


**Figure 12** Estimated cdf, Kaplan-Meier survival and scaled TTT-Transform plots for the GOLG-Log-logistic distribution for aircraft windshield data

## 8. Concluding Remarks

This paper introduces a new generalized distribution called the gamma odd Lindley generalized-G (GOLG-G) family of distributions, which significantly extends the flexibility of existing probability models for analyzing complex data. The GOLG-G family encompasses several new and known distributions as special cases or sub-models, providing researchers with a unified framework for dis-

tribution selection. A key strength of the proposed family is the remarkable flexibility of its hazard rate functions for specified baseline cdfs, which can accommodate increasing, decreasing, bathtub-shaped, and upside-down bathtub-shaped patterns, making it suitable for diverse real-world applications in reliability, survival analysis, and risk modeling.

We have derived closed-form expressions for important statistical properties including moments, distribution of order statistics, and Rényi entropy, which facilitate practical implementation and theoretical analysis. The maximum likelihood estimation technique was employed to estimate the model parameters, and extensive simulation studies demonstrated satisfactory performance across different sample sizes, with parameter estimates showing consistency and efficiency as sample size increases. The practical applicability and superiority of a special case of the GOLG-G family were illustrated through applications to three real data sets: active repair times data, level of mercury data, and aircraft windshield failure data. Across these diverse applications spanning maintenance systems, environmental monitoring, and reliability engineering, the proposed distribution consistently outperformed competing models based on goodness-of-fit criteria, demonstrating its versatility in handling different data characteristics.

While the GOLG-G family demonstrates strong performance, some limitations exist. The multi-parameter structure may challenge estimation with small or censored samples and requires careful numerical optimization. Additionally, this study employs only maximum likelihood estimation, and the univariate framework does not address multivariate dependencies or covariate effects common in practical applications. Several promising avenues emerge from this work. Future research should investigate alternative estimation methods including Bayesian estimation, least-squares methods (ordinary and weighted), Anderson-Darling estimation, Cramér-von Mises estimation, and maximum product of spacings estimation, with comparative studies examining their efficiency and robustness across different scenarios. Important extensions include: (i) developing regression models to incorporate covariate effects; (ii) constructing multivariate forms using copula-based or transformation methods; (iii) exploring applications in finance, actuarial science, hydrology, and economics; and (iv) developing user-friendly software packages to facilitate widespread adoption. We will continue investigating these aspects in future work and hope our study serves as a reference for further research in this area.

## References

- Afify AZ, Cordeiro GM, Maed ME, Alizadeh M, Al-Mofleh H, Nofal ZM. The generalized odd Lindley-G family: properties and applications. *An Acad Bras Cienc.* 2019; 91(3): e20180040.
- Aljarrah MA, Lee C, Famoye F. On generating T-X family of distributions using quantile functions. *J Stat Distrib Appl.* 2014; 1: 1-17.
- Alexander C, Cordeiro GM, Ortega EMM, Sarabia JM. Generalized beta-generated distributions. *Comput Stat Data An.* 2012; 56(6): 1880-1897.
- Alizadeh M, Emadi M, Doostparast M, Cordeiro GM, Ortega EMM, Pescim RR. A new family of distributions: the Kumaraswamy odd log-logistic, properties and applications. *Hacet J Math Stat.* 2015; 44(6): 1491-1512.
- Alizadeh M, Tahir MH, Cordeiro GM, Mansoor M, Zubair M, Hamedani GG. The Kumaraswamy Marshall-Olkin family of distributions. *J Egypt Math Soc.* 2015; 23(3): 546-557.
- Alzaatreh A, Lee C, Famoye F. A new method for generating families of continuous distributions. *Metron.* 2013; 71(1): 63-79.
- Colak AB, Sindhu TN, Lone SA, Shafiq A, Abushal TA. Reliability study of generalized Rayleigh distribution based on inverse power law using artificial neural network with Bayesian regularization. *Tribol Int.* 2023; 185: 108544.
- Cordeiro GM, Ortega EMM, da Cunha DCC. The exponentiated generalized class of distributions. *J Data Sci.* 2013; 11(1): 1-27.

- Eugene N, Lee C, Famoye F. Beta-normal distribution and its applications. *Commun Stat Theory*. 2002; 31(4): 497-512.
- Cordeiro GM, Lemonte AJ. On the Marshall-Olkin extended Weibull distribution. *Stat Pap*. 2011; 54: 333-353.
- Chambers J, Cleveland W, Kleiner B, Tukey J. Graphical methods for data analysis. New York: Chapman and Hall/CRC; 1983.
- Chen G, Balakrishnan N. A general purpose approximate goodness-of-fit test. *J Qual Technol*. 1995; 27(2): 154-161.
- Chhikara RS, Folks JL. The inverse Gaussian distribution as a lifetime model. *Technometrics*. 1977; 19(4): 461-468.
- Chipepa F, Oluyede B, Makubate B. A new generalized family of odd Lindley-G distributions with applications. *Int J Probab Stat*. 2019; 8(6): 1-22.
- Doostmoradi A, Zadkarami MR, Roshani Sheykhabad A. A new modified Weibull distribution and its applications. *J Stat Res Iran*. 2014; 11(1): 97-118.
- Gradshteyn IS, Ryzhik IM. Table of integrals, series and products. San Diego: Academic Press; 2000.
- Gomes-Silva F, Percontini A, de Brito E, Ramos MW, Venancio R, Cordeiro G. The odd Lindley-G family of distributions. *Austrian J Stat*. 2017; 46(1): 65-87.
- Hosseini B, Afshari M, Alizadeh M. The generalized odd-gamma-G family of distributions : properties and applications. *Austrian J Stat*. 2018; 47(2): 69-89.
- Jamal F, Reyad H, Chesneau C, Nasir MA, Othman S. The Marshall-Olkin odd Lindley-G family of distributions: theory and applications. *Punjab Univ J Math*. 2019; 51(7): 111-125.
- Lee C, Famoye F, Olumolade O. Beta-Weibull distribution: some properties and applications. *J Mod Appl Stat Methods*. 2007; 6(1): 173-186.
- Moakofi T, Oluyede B, Chipepa F, Makubate B. Odd power generalized Weibull-G family of distributions: model, properties and applications. *J Stat Model Theory Appl*. 2021; 2(1): 121-142.
- Mol S, Ozden O, Karakulak S. Levels of selected metals in albacore (*Thunnus alalunga*, Bonnaterre, 1788) from the Eastern Mediterranean. *J Aquat Food Prod Technol*. 2012; 21(2): 111-117.
- Murthy DP, Xie M, Jiang R. Weibull models. New Jersey: John Wiley and Sons; 2004.
- Oluyede BO. A generalized odd Lindley generalized-G family of distributions: model, properties and applications. submitted; 2019.
- Oluyede B, Moakofi T, Chipepa F, Makubate B. A new power generalized Weibull-G family of distributions: properties and applications. *J Stat Model Theory Appl*. 2020; 1(2): 167-191.
- Oluyede BO, Huang S, Yang T. A new class of generalized modified Weibull distribution with applications. *Austrian J Stat*. 2015; 44(3): 45-68.
- Oluyede BO, Yang T. A new class of generalized Lindley distribution with applications. *J Stat Comput Simul*. 2015; 85(10): 2072-2100.
- Pinho LGB, Cordeiro GM, Nobre JS. The gamma-exponentiated Weibull distribution. *J Stat Theory Appl*. 2012; 11(4): 379-395.
- Ristić MM, Balakrishnan N. The gamma-exponentiated exponential distribution. *J Stat Comput Simul*. 2012; 82(8): 1191-1206.
- Rényi A. On measures of entropy and information. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*. 1960; 1: 547-561.
- Reyad H, Jamal F, Othman S, Hamedani GG. The transmuted odd Lindley-G family of distributions. *Asian J Probab Stat*. 2018; 1(3): 1-25.
- Reyad H, Alizadeh M, Jamal F, Othman S. The Topp Leone odd Lindley-G family of distributions: Properties and applications. *J Stat Manag Syst*. 2018; 21(7): 1273-1297.
- Sengweni W, Oluyede B, Makubate B. The exponentiated half-logistic odd Lindley-G family of distributions with applications. *J Nonlinear Sci Appl*. 2021; 14(5): 287-309.
- Shaked M, Shanthikumar JG. Stochastic Orders and Their Applications, 1994. Academic Press, New York.

- Shafiq A, Sindhu TN, Riaz MB, Hassan MK, Abushal TA. A statistical framework for a new Kavya-Manoharan Bilal distribution using ranked set sampling and simple random sampling. *Heliyon*. 2024; 10(9): e30762.
- Sindhu TN, Abd Elgawad MA, Shafiq A, Abushal TA. Entropy-transformed teissier distribution: A modern statistical framework for engineering, pharmaceutical, and metrological applications. *J. Radiat. Res. Appl. Sci.* 2025; 18(3): 101773.
- Sindhu TN, Shafiq A, Riaz MB, Abushal TA, Ahmad H, Almetwally EM, Askar S. Introducing the new arcsine-generator distribution family: An in-depth exploration with an illustrative example of the inverse Weibull distribution for analyzing healthcare industry data. *J. Radiat. Res. Appl. Sci.* 2024; 17(2): 100879.
- Zografos K, Balakrishnan N. On families of beta- and generalized gamma-generated distribution and associated inference. *Stat Methodol.* 2009; 6(4): 344-362.

## Appendix

The link below contains the appendix

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