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The SAS Family of Distributions: Properties and Applications

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Abstract

In this study, our aim is to introduce a new transformation based on the cumulative distribution function (CDF), known as the SAS-transformation, an acronym derived from the names of its developers: Shantanu, Arun and Shubham. The proposed transformation technique is illustrated using the Toppe-Leone distribution as a baseline. We computed its various statistical properties, including the survival function, hazard rate function, moments, conditional moments, quantile function, mean deviation, order statistics, entropy, stochastic ordering and identifiability. We conduct a simulation study to assess the long-term performance of the model and estimator's under different classical estimation methods. Finally, to demonstrate the practical applicability of the proposed model, we apply it to three real datasets and assess its performance using the Akaike Information criterion (AIC), Corrected Akaike Information criterion (AICc), Bayesian Information criterion (BIC), and log-likelihood (LL) values.

Keywords: Transformation technique, Topp-Leone distribution, order statistics, classical method of estimation, simulation study.

1. Introduction

Probability distributions are foundational tools in statistical modeling, playing a critical role in analyzing lifetime data across diverse domains such as actuarial science, biomedical studies, economics, reliability engineering, and environmental sciences. The increasing demand for models with greater flexibility and improved goodness-of-fit has driven the development of generalized families of distributions, often constructed through functional transformations of baseline distributions. A classical example is the Exponentiated-family introduced by Mudholkar and Srivastava (1993), which modifies the CDF by raising it to a positive power. Another important development is the Quadratic Rank Transmutation Map (QRTM), proposed by Shaw and Buckley (2007), which transforms a baseline CDF via a quadratic map, offering enhanced shape adaptability with minimal added complexity. By combining the structural features of both the Expontiated-family introduced by Mudholkar and Srivastava (1993) and QRTM proposed by Shaw and Buckley (2007), Ahmad (2020) introduced a more generalized transformation. This transformation allows for greater flexibility in capturing a variety of distributional shapes by adjusting the model parameters. The DUS-transformation, introduced by Kumar et al. (2015) and its generalization by Maurya et al. (2017), also represents a significant advancement.

In addition to these developments, recent attention has been directed toward transformation based approaches that aim to enhance flexibility while maintaining analytical tractability. One such

approach is the alpha power transformation (APT) introduced by Mahdavi and Kundu (2017), which provides a simple yet effective mechanism for generating new families of distributions. However, the structural and mathematical properties of the APT family were not extensively explored in the original work. Subsequently, Mead et al. (2019) conducted a more detailed investigation of the APT family, deriving several statistical properties and demonstrating its applicability in diverse modeling scenarios. More recently, Agegnehu et al. (2024) proposed a new alpha power transformed beta distribution and studies its theoretical properties and practical applications, further highlighting the growing the growing interest in transformation-based distributional framework.

However, a notable limitation of many such transformation-based models lies in the trade-off between flexibility and analytical simplicity. While introducing additional parameters can enhances shape flexibility, it often increases the computational complexity and complicates statistical inference.

In this context, we propose a new transformation called the SAS-transformation, named after its developers Shantanu, Arun, and Shubham. This transformation offers a fresh perspective by retaining analytical simplicity while providing enhanced shape flexibility.

Let X be a random variable with a CDF $G(x;\xi)$, and a corresponding PDF $g(x;\xi)$, which serves as the baseline distribution. The SAS-transformation is defined as;

$$F_{SAS}(x;\xi) = G(x;\xi)\exp\{G(x;\xi)-1\}; \qquad \xi > 0, x \in \Re. \tag{1}$$

The function $F_{SAS}(x;\xi)$ satisfies the fundamental properties of a valid CDF. Specifically, since $0 \leq G(x;\xi) \leq 1$, it follows that $0 \leq F_{SAS}(x;\xi) \leq 1$. Moreover, as $G(x;\xi)$ is non-decreasing in x, the transformation preserves monotonicity, ensuring that $F_{SAS}(x;\xi) \leq F_{SAS}(y;\xi)$ for all $x \leq y$. This transformation modifies the baseline distribution by incorporating an exponential adjustment, thereby enhancing its flexibility for modeling diverse data behaviors.

To further characterize the distribution, differentiating Equation (1) with respect to x, we obtain the corresponding PDF as

$$f_{SAS}(x;\xi) = g(x;\xi)\{1+G(x;\xi)\}\exp\{G(x;\xi)-1\}; \qquad x \in \Re. \tag{2}$$

The SAS-transformation possesses several appealing features that make it a useful and tractable generalization technique. It enhances shape flexibility without introducing additional parameters, thereby preserving model parsimony.

Furthermore, the SAS-transformation can be viewed as a hybrid generalization mechanism. It incorporates characteristics of both the exponentiated family, through transformation of the baseline CDF, and quadratic rank transmutation maps, through functional modification that induces additional shape flexibility. In contrast to these approaches, however, the SAS-transformation maintains single-parameter simplicity while introducing nonlinear exponential tilting , thereby achieving a balanced trade-off between flexibility and analytical tractability.

The survival function (SF) and hazard rate function (HF) are obtained as

$$S_{SAS}(x;\xi) = 1 - G(x;\xi)\exp\{G(x;\xi)-1\}; \qquad x \in \Re \tag{3}$$

and

$$h_{SAS}(x;\xi) = \frac{g(x;\xi)\{1+G(x;\xi)\}\exp\{G(x;\xi)-1\}}{1-G(x;\xi)\exp\{G(x;\xi)-1\}}; \qquad x \in \Re \tag{4}$$

respectively. Owing to this structure, the SAS-family is capable of accommodating a wide range of hazard rate behaviors, including increasing and bathtub-shaped forms.

To illustrate the applicability of the proposed transformation, we now consider a specific sub-model from the SAS-family. In particular, we introduce the SAS Top-Leone distribution, denoted by SAS_{TL}. this model is highly flexible and capable of capturing divers hazard rate patterns.

The rest of the work of this manuscript is organized as follows: Section 2 formally introduces the proposed sub-model. Section 3 discusses the shapes of the probability density function and hazard

rate. In Section 4 we derive several statistical properties of the proposed model. Section 5 presents the methods for the parameter estimation. Section 6 reports the results of the simulation study, while Section 7 illustrates the practical utility of the model through real data applications. Finally, conclusions are provided in Section 8.

2. SAS-transformed Topp-leone Distribution

In this section, we introduce a specific member of SAS-family, namely SAS-transformed Topp-Leone distribution.

Let the CDF of baseline distribution is $G(x;\xi) = \{x(2-x)\}^{\xi}$ and the corresponding PDF is $g(x;\xi) = 2\xi(1-x)\{x(2-x)\}^{\xi-1}; \quad \xi > 0$ and $0 < x < 1$. Substituting into Equation (1), the CDF of the SAS Topp-Leone distribution (SAS_{TL}) is

$$F_{SAS_{TL}}(x;\xi) = \{x(2-x)\}^{\xi} \exp\left(\{x(2-x)\}^{\xi} - 1\right), \quad 0 < x < 1. \tag{5}$$

We denote it by $SAS_{TL}(x;\xi)$. The corresponding PDF is;

$$f_{SAS_{TL}}(x;\xi) = \begin{cases} 2\xi(1-x)\{x(2-x)\}^{\xi-1}\left[1+\{x(2-x)\}^{\xi}\right] \\ \qquad \times \exp\left(\{x(2-x)\}^{\xi} - 1\right), & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

The survival function and hazard rate function are obtained as,

$$S_{SAS_{TL}}(x;\xi) = 1 - \{x(2-x)\}^{\xi} \exp\left(\{x(2-x)\}^{\xi} - 1\right) \tag{7}$$

and

$$h_{SAS_{TL}}(x;\xi) = \frac{2\xi(1-x)\{x(2-x)\}^{\xi-1}\left[1+\{x(2-x)\}^{\xi}\right]\exp\left(\{x(2-x)\}^{\xi} - 1\right)}{1 - \{x(2-x)\}^{\xi}\exp\left(\{x(2-x)\}^{\xi} - 1\right)}. \tag{8}$$

3. Shape of the Distribution and Hazard Rate

Figure 1 displays the plots for PDF and CDF of $SAS_{TL}(x;\xi)$ distribution to demonstrate how different values of the parameter ξ influence their shapes.

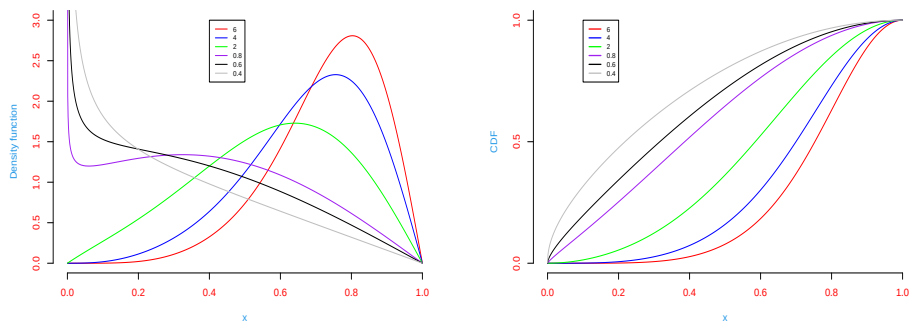


Figure 1 PDF and CDF plots of the $SAS_{TL}(x;\xi)$ distribution at different choices of parameter.

Clearly, we observe that the PDF of $SAS_{TL}(x;\xi)$ distribution accommodates decreasing and left-skewed shapes. The graphical representation of survival and hazard function displayed in Figure 2.

Figure 2 indicates that the $SAS_{TL}(x;\xi)$ distribution exhibits increasing and bathtub-shaped hazard.

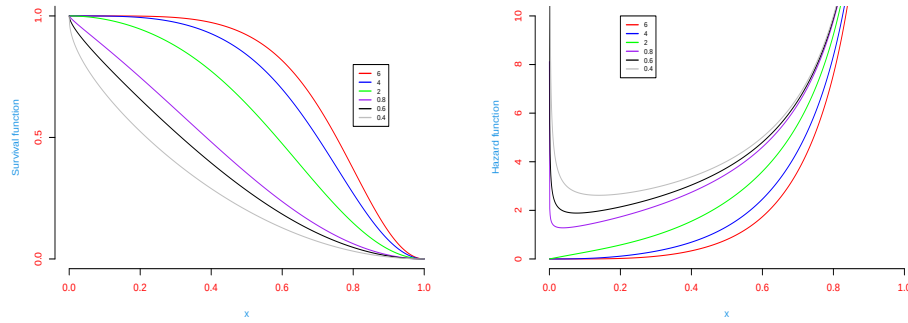


Figure 2 Survival and hazard function plots of the $SAS_{TL}(x;\xi)$ distribution at different choices of parameter.

4. Statistical Properties

In this section, some basic statistical properties of $SAS_{TL}(x;\xi)$ distribution such as moments, conditional moments, quantile function, order statistics, mean deviation and stochastic orderings etc. are discussed.

4.1. Moments

Since the key characteristics of a statistical distribution such as: central tendency, dispersion, skewness, and kurtosis can be analyzed through moments. We derive the r^{th} raw moment of the $SAS_{TL}(x;\xi)$ distribution. Let $X \sim SAS_{TL}(x;\xi)$, then the r-th raw moment is given as

$$\begin{aligned} E(X^r) = \frac{2\xi}{e}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{j!}\Bigg\{ & 2^{\xi j+\xi+k-1}\binom{\xi j+\xi-1}{k}\left(\frac{1}{r+\xi j+\xi+k}-\frac{1}{r+\xi j+\xi+k+1}\right) \\ & + 2^{\xi j+2\xi+k-1}\binom{\xi j+2\xi-1}{k}\left(\frac{1}{r+\xi j+2\xi+k}-\frac{1}{r+\xi j+2\xi+k+1}\right)\Bigg\}, \\ & r=1,2,3,\ldots. \end{aligned} \tag{9}$$

Proof: We know that

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x;\xi)\,dx; \qquad r=1,2,3,\ldots. \tag{10}$$

Using Equation (6) in Equation (10), we get

$$\mu'_r = \int_0^1 x^r \times 2\xi(1-x)\Big\{x(2-x)\Big\}^{\xi-1}\Big\{1+\{x(2-x)\}^{\xi}\Big\}\exp\Big\{\{x(2-x)\}^{\xi}-1\Big\}\,dx.$$

Using power series expansion of exponential function, i.e. $e^x = \sum_{j=0}^{\infty}\frac{x^j}{j!}$, we get

$$\begin{aligned} &= \frac{2\xi}{e}\sum_{j=0}^{\infty}\frac{1}{j!}\int_0^1\Big\{x^{r+\xi j+\xi-1}(1-x)(2-x)^{\xi j+\xi-1}+x^{r+\xi j+2\xi-1}(1-x)(2-x)^{\xi j+2\xi-1}\Big\}\,dx \\ &= \frac{2\xi}{e}\sum_{j=0}^{\infty}\frac{1}{j!}\int_0^1\Big\{(x^{r+\xi j+\xi-1}-x^{r+\xi j+\xi})(2-x)^{\xi j+\xi-1} \\ &\quad +(x^{r+\xi j+2\xi-1}-x^{r+\xi j+2\xi})(2-x)^{\xi j+2\xi-1}\Big\}\,dx. \end{aligned}$$

Now, here we use expansion of series, i.e. $(1-x)^n = \sum_{k=0}^{\infty} \binom{n}{k}(-1)^k x^k$ for all real values of n and $|x| < 1$, we get

$$= \frac{2\xi}{e}\sum_{j=0}^{\infty}\frac{1}{j!}\int_0^1 \left\{(x^{r+\xi j+\xi-1} - x^{r+\xi j+\xi})\sum_{k=0}^{\infty} 2^{\xi j+\xi-1}\left(\frac{-1}{2}\right)^k \binom{\xi j+\xi-1}{k} x^k\right\} dx$$
$$+ \frac{2\xi}{e}\sum_{j=0}^{\infty}\frac{1}{j!}\int_0^1 \left\{(x^{r+\xi j+2\xi-1} - x^{r+\xi j+2\xi})\sum_{k=0}^{\infty} 2^{\xi j+2\xi-1}\left(\frac{-1}{2}\right)^k \binom{\xi j+2\xi-1}{k} x^k\right\} dx.$$

By doing further simplification, we get

$$E(X^r) = \frac{2\xi}{e}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{j!}\left\{2^{\xi j+\xi+k-1}\binom{\xi j+\xi-1}{k}\left(\frac{1}{r+\xi j+\xi+k} - \frac{1}{r+\xi j+\xi+k+1}\right)\right.$$
$$\left. +\ 2^{\xi j+2\xi+k-1}\binom{\xi j+2\xi-1}{k}\left(\frac{1}{r+\xi j+2\xi+k} - \frac{1}{r+\xi j+2\xi+k+1}\right)\right\}.$$

■

Now, the mean, variance, skewness and kurtosis of proposed distribution can be easily obtained.

Remark 1 Complete derivation is done in A1.

4.2. Conditional moments

The r^{th} conditional moments of a random variable X having the $SAS_{TL}(x;\xi)$ distribution is obtained as below

$$E(X^r \mid X > t) = \frac{1}{S(t)}\int_t^1 x^r f(x)\,dx$$
$$= \frac{2\xi}{e\,S(t)}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{j!}$$
$$\times\left[2^{\xi j+\xi+k-1}\binom{\xi j+\xi-1}{k}\left(\frac{t^{r+\xi j+\xi+k}-1}{r+\xi j+\xi+k} - \frac{t^{r+\xi j+\xi+k+1}-1}{r+\xi j+\xi+k+1}\right)\right. \quad (11)$$
$$\left. +\ 2^{\xi j+2\xi+k-1}\binom{\xi j+2\xi-1}{k}\left(\frac{t^{r+\xi j+2\xi+k}-1}{r+\xi j+2\xi+k} - \frac{t^{r+\xi j+2\xi+k+1}-1}{r+\xi j+2\xi+k+1}\right)\right],$$
$$r = 1, 2, \ldots.$$

Remark 2 Derivation is very similar to A1.

4.3. Quantile function

Definition 1 Let $X \sim SAS_{TL}(x;\xi)$. Then, the p^{th} quantile function, say $Q(p)$, is given by

$$Q(p) = 1 - \sqrt{1 - (W_0(pe))^{1/\xi}}, \quad 0 < p < 1, \quad (12)$$

where $e = \exp(1)$ and $W_0(\cdot)$ denotes the principal branch of the Lambert-W function.

To derive Equation (12), equate the cumulative distribution function in Equation (5) to p, i.e.,

$$p = \{x(2-x)\}^{\xi}\exp\left(\{x(2-x)\}^{\xi} - 1\right).$$

Multiplying both sides by e gives

$$pe = \{x(2-x)\}^{\xi}\exp\left(\{x(2-x)\}^{\xi}\right).$$

Now, let

$$Z = (x(2-x))^{\xi}.$$

Then,

$$pe = Ze^{Z}.$$

Applying the principal branch of the Lambert-W function yields

$$W_0(pe) = Z = (x(2-x))^{\xi}.$$

Hence,

$$x(2-x) = (W_0(pe))^{1/\xi}.$$

Solving the quadratic equation in x, we obtain

$$x = 1 \pm \sqrt{1-(W_0(pe))^{1/\xi}}.$$

Since $0 < x < 1$, the admissible root is

$$Q(p) = 1 - \sqrt{1-(W_0(pe))^{1/\xi}}.$$

To obtain first, second (i.e. median) and third quartile of the $SAS_{TL}(x;\xi)$ distribution, we substitute $p = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, respectively, in Equation (12). Skewness and kurtosis are the statistical measures which are used to calculate the degree of long tail and degree of tail heaviness respectively.These can be assessed using quantile function. The Bowley measure of skewness, (see; Bowley (1926)), is defined as $\mathscr{B} = \frac{Q(\frac{3}{4})-2\cdot Q(\frac{1}{2})+Q(\frac{1}{4})}{Q(\frac{3}{4})-Q(\frac{1}{4})}$ and Moors kurtosis, (see; Moors (1998)), based on octiles, is given as $\mathscr{M} = \frac{Q(\frac{7}{8})-Q(\frac{5}{8})+Q(\frac{3}{8})-Q(\frac{1}{8})}{Q(\frac{6}{8})-Q(\frac{2}{8})}$.

4.4. Order statistics

In this sub-section, our aim to determine the PDF and CDF for the r-th order statistic, which refers to the r-th smallest value among a sample of n points when arranged in ascending order. Specifically, let $X_1, X_2, ..., X_n$ be a random sample of size n from $SAS_{TL}(x;\xi)$ distribution, and arrange these values in ascending order. We denote $X_{1:n} < X_{2:n} < ... < X_{n:n}$ as the corresponding order statistics. The PDF, $f_r(x)$, for the r-th order statistic $X_{r:n}$ is given by;

$$\begin{aligned} f_r(x) &= \frac{n!}{(r-1)!(n-r)!} F^{r-1}(x)\big[1-F(x)\big]^{n-r} f(x) \\ &= \frac{n!}{(r-1)!(n-r)!} \sum_{l=0}^{n-r} (-1)^l \binom{n-r}{l} F^{r+l-1}(x) f(x) \end{aligned}$$

and the corresponding CDF, $F_r(x)$ is given by

$$\begin{aligned} F_r(x) &= \sum_{p=r}^{n} \binom{n}{p} F^{p}(x)\big[1-F(x)\big]^{n-p} \\ &= \sum_{p=r}^{n} \sum_{q=0}^{n-p} (-1)^q \binom{n}{p} \binom{n-p}{q} F^{p+q}(x) \end{aligned}$$

where, $f(x)$ and $F(x)$ are the PDF and CDF of the random variable X. Now, using Equations (5) and (6), the PDF and the CDF of r-th order statistics based on random sample of size n for the proposed distribution are;

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!}\sum_{p=0}^{n-r}(-1)^p\binom{n-r}{p}2\xi(1-x)\big[x(2-x)\big]^{\xi r+\xi p-1} \big[x(2-x)^{\xi}+1\big]\big[\exp(\{x(2-x)\}^{\xi}-1)\big]^{r+p} \tag{13}$$

and

$$F_r(x) = \sum_{p=r}^{n}\sum_{q=0}^{n-p}(-1)^q\binom{n}{p}\binom{n-p}{q}\big[\{x(2-x)\}^{\xi}\exp(\{x(2-x)\}^{\xi}-1)\big]^{p+q} \tag{14}$$

respectively.

4.5. Mean deviation

The mean deviation from mean and median is defined by

$$\delta_1(x) = \int_0^1 |x-\mu| f(x)dx \tag{15}$$

and

$$\delta_2(x) = \int_0^1 |x-M| f(x)dx \tag{16}$$

respectively. Where μ is mean and M is median of X. Doing further simplification, mean deviation about mean and median will be

$$\delta_1(x) = 2\mu F(\mu) - 2\mu + 2\int_{\mu}^1 xf(x)dx$$

and

$$\delta_2(x) = -\mu + 2\int_{M}^1 xf(x)dx.$$

By Equation (11), we get

$$\int_{\mu}^1 xf_{SAS_{TL}}(x;\xi)\,dx = \frac{2\xi}{e}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{j!} \times\left[2^{\xi j+\xi+k-1}\binom{\xi j+\xi-1}{k}\left(\frac{\mu^{1+\xi j+\xi+k}-1}{1+\xi j+\xi+k}-\frac{\mu^{2+\xi j+\xi+k}-1}{2+\xi j+\xi+k}\right)\right. + \left. 2^{\xi j+2\xi+k-1}\binom{\xi j+2\xi-1}{k}\left(\frac{\mu^{1+\xi j+2\xi+k}-1}{1+\xi j+2\xi+k}-\frac{\mu^{2+\xi j+2\xi+k}-1}{2+\xi j+2\xi+k}\right)\right]$$

and

$$\int_{M}^1 xf_{SAS_{TL}}(x;\xi)\,dx = \frac{2\xi}{e}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{j!} \times\left[2^{\xi j+\xi+k-1}\binom{\xi j+\xi-1}{k}\left(\frac{M^{1+\xi j+\xi+k}-1}{1+\xi j+\xi+k}-\frac{M^{2+\xi j+\xi+k}-1}{2+\xi j+\xi+k}\right)\right. + \left. 2^{\xi j+2\xi+k-1}\binom{\xi j+2\xi-1}{k}\left(\frac{M^{1+\xi j+2\xi+k}-1}{1+\xi j+2\xi+k}-\frac{M^{2+\xi j+2\xi+k}-1}{2+\xi j+2\xi+k}\right)\right].$$

Thus, the mean deviation from mean and median are

$$\begin{aligned}\delta_1(x) &= 2\mu F(\mu) - 2\mu \\ &\quad + 2\Bigg\{\frac{2\xi}{e}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{j!} \\ &\quad \times \Bigg[2^{\xi j+\xi+k-1}\binom{\xi j+\xi-1}{k}\left(\frac{\mu^{1+\xi j+\xi+k}-1}{1+\xi j+\xi+k}-\frac{\mu^{2+\xi j+\xi+k}-1}{2+\xi j+\xi+k}\right) \\ &\quad + 2^{\xi j+2\xi+k-1}\binom{\xi j+2\xi-1}{k}\left(\frac{\mu^{1+\xi j+2\xi+k}-1}{1+\xi j+2\xi+k}-\frac{\mu^{2+\xi j+2\xi+k}-1}{2+\xi j+2\xi+k}\right)\Bigg]\Bigg\}\end{aligned}$$

and

$$\begin{aligned}\delta_2(x) &= 2\Bigg\{\frac{2\xi}{e}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{j!} \\ &\quad \times \Bigg[2^{\xi j+\xi+k-1}\binom{\xi j+\xi-1}{k}\left(\frac{M^{1+\xi j+\xi+k}-1}{1+\xi j+\xi+k}-\frac{M^{2+\xi j+\xi+k}-1}{2+\xi j+\xi+k}\right) \\ &\quad + 2^{\xi j+2\xi+k-1}\binom{\xi j+2\xi-1}{k}\left(\frac{M^{1+\xi j+2\xi+k}-1}{1+\xi j+2\xi+k}-\frac{M^{2+\xi j+2\xi+k}-1}{2+\xi j+2\xi+k}\right)\Bigg]\Bigg\} - \mu\end{aligned}$$

respectively.

4.6. Entropy

Entropy quantifies the average level of information or uncertainty associated with the possible outcomes of a random variable. Here, we discuss Rényi entropy (see; Rnyi (1961)) for the distribution under consideration. Let, X has PDF $f(x)$ then Rényi entropy is defined as:

$$\mathscr{R}_\beta(\xi) = \frac{1}{1-\beta}\log\Big[\int_0^1 f^\beta(x)\,dx\Big]$$

From Equation (6), we have

$$\int_0^1 f^\beta_{SAS_{TL}}(x)\,dx = \int_0^1 \big[2\xi(1-x)\{x(2-x)\}^{\xi-1}\big\{1+\{x(2-x)\}^\xi\big\}\exp(\{x(2-x)\}^\xi-1)\big]^\beta\,dx,$$

and after simplification we get,

$$\int_0^1 f^\beta_{SAS_{TL}}(x)\,dx = \left(\frac{2\xi}{e}\right)^\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{\beta^i}{i!}\binom{\beta}{j}\binom{A}{k}\binom{\beta}{\ell}(-1)^{k+\ell}\frac{2^{A-k}}{A+k+\ell+1}.$$

where,

$$A = \xi i + \xi j + \xi\beta - \beta$$

Therefore, the Rényi entropy is obtained as

$$\begin{aligned}\mathscr{R}_\beta(\xi) &= \frac{1}{1-\beta}\log\left[\left(\frac{2\xi}{e}\right)^\beta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{\beta^i}{i!}\binom{\beta}{j}\binom{A}{k}\binom{\beta}{\ell}(-1)^{k+\ell}\frac{2^{A-k}}{A+k+\ell+1}\right] \\ &= \frac{\beta}{1-\beta}\log\left(\frac{2\xi}{e}\right) + \frac{1}{1-\beta}\log\Bigg[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{\beta^i}{i!}\binom{\beta}{j}\binom{A}{k}\binom{\beta}{\ell} \\ &\quad \times (-1)^{k+\ell}\frac{2^{A-k}}{A+k+\ell+1}\Bigg].\end{aligned}$$

Remark 3 Complete derivation is done in A2.

4.7. Stochastic ordering

Definition 2 A random variable X is considered stochastically greater ($X \geq_{st} Y$) than Y if $F_X(x) \geq F_Y(x)$ for all x. Similarly, X is said to be greater than Y in the

- hazard rate order ($X \geq_{hz} Y$) if $h_Y(x) \geq h_X(x)$.
- likelihood ratio order ($X \geq_{lr} Y$) if $\frac{f_X(x)}{f_Y(x)}$ is increasing function of x.

Remark 4 The implication of this property illustrated by Shaked and Shanthikumar (1994), as $(X \geq_{lr} Y) \implies (X \geq_{hz} Y) \implies (X \geq_{st} Y)$.

Theorem 1 *Consider two independent random variables X and Y, where $X \sim SAS_{TL}(x;\xi_1)$ and $Y \sim SAS_{TL}(x;\xi_2)$. Then, $(X \geq_{lr} Y)$.*

Proof: Likelihood ratio of two independent random variables X and Y is given by

$$\frac{f_X(x;\xi_1)}{f_Y(x;\xi_2)} = \frac{\xi_1}{\xi_2} x^{\xi_1-\xi_2}(2-x)^{\xi_1-\xi_2} \frac{\{x(2-x)\}^{\xi_1}+1}{\{x(2-x)\}^{\xi_2}+1} \exp\left[\{x(2-x)\}^{\xi_1} - \{x(2-x)\}^{\xi_2}\right]$$

Since, for $\xi_1 \geq \xi_2$, $\frac{d}{dx}\left(\frac{f_X(x;\xi_1)}{f_Y(x;\xi_2)}\right) \geq 0$ for all $0 < x < 1$. Thus, the likelihood ratio is increasing function of x. Therefore, $X \geq_{lr} Y$ which further implies $X \geq_{hz} Y$ and in turn, $X \geq_{st} Y$.

■

4.8. Identifiability

Theorem 2 *The proposed $SAS_{TL}(x;\xi)$ distribution is identifiable.*

Proof: Suppose $f_{SAS_{TL}}(x;\xi_1) = f_{SAS_{TL}}(x;\xi_2)$ for all $x \in (0,1)$. Then

$$\begin{aligned} &2\xi_1(1-x)\{x(2-x)\}^{\xi_1-1}\left(1+\{x(2-x)\}^{\xi_1}\right)\exp\left(\{x(2-x)\}^{\xi_1}-1\right) \\ &= 2\xi_2(1-x)\{x(2-x)\}^{\xi_2-1}\left(1+\{x(2-x)\}^{\xi_2}\right)\exp\left(\{x(2-x)\}^{\xi_2}-1\right). \end{aligned}$$

After canceling common factors, we obtain

$$\begin{aligned} &\xi_1\{x(2-x)\}^{\xi_1-1}\left(1+\{x(2-x)\}^{\xi_1}\right)\exp\left(\{x(2-x)\}^{\xi_1}-1\right) \\ &= \xi_2\{x(2-x)\}^{\xi_2-1}\left(1+\{x(2-x)\}^{\xi_2}\right)\exp\left(\{x(2-x)\}^{\xi_2}-1\right). \end{aligned}$$

This implies

$$\frac{\xi_1}{\xi_2} = \{x(2-x)\}^{\xi_2-\xi_1}\frac{1+\{x(2-x)\}^{\xi_2}}{1+\{x(2-x)\}^{\xi_1}}\exp\left(\{x(2-x)\}^{\xi_2}-\{x(2-x)\}^{\xi_1}\right).$$

Left hand-side is constant, whereas right-hand side is a function of x. Since, $\{x(2-x)\} \in (0,1)$ for all $x \in (0,1)$, the right-hand side varies with x whenever $\xi_1 \neq \xi_2$. Hence, above equality holds for all x only if $\xi_1 = \xi_2$. Therefore, the parameter ξ is uniquely determined, and the proposed model is identifiable.

■

5. Methods of Estimation

In this section, we will examine several estimation techniques, such as maximum likelihood estimation (MLE), maximum product spacing (MPS) estimation, least square estimation and Cramèr-Von-Mises estimation etc., to estimate the unknown parameters of the model under consideration.

5.1. Maximum likelihood estimation (MLE)

Consider, $x_1, x_2, \ldots, x_n$ as a random samples of size n from the proposed distribution. The corresponding log-likelihood function for this distribution is defined as

$$\begin{aligned}\log L = n\log(2\xi) + \sum_{i=1}^{n}\log(1-x_i) + (\xi-1)\sum_{i=1}^{n}\log\left[x_i(2-x_i)\right] \\ + \sum_{i=1}^{n}\log\left[\left\{x_i(2-x_i)\right\}^{\xi}+1\right] + \sum_{i=1}^{n}\left[x_i(2-x_i)\right]^{\xi} - n. \end{aligned} \tag{17}$$

After differentiating Equation (17) with respect to ξ, we get

$$\begin{aligned}\frac{d\log L}{d\xi} = \frac{n}{\xi} + (\xi-1)\sum_{i=1}^{n}\left[\log(x_i)+\log(2-x_i)\right] \\ + \sum_{i=1}^{n}\left[\log(x_i)+\log(2-x_i)\right]\cdot\left[x_i(2-x_i)\right]^{\xi} \\ + \sum_{i=1}^{n}\frac{\left[\log(x_i)+\log(2-x_i)\right]\cdot\left[x_i(2-x_i)\right]^{\xi}}{\left[x_i(2-x_i)\right]^{\xi}+1}. \end{aligned} \tag{18}$$

By setting Equation (18) to zero, we obtain a non-linear likelihood equation. Solving this equation yields the MLE $\hat{\xi}$ of the parameter ξ. Since, this equation does not have closed form so they can't be solved analytically. In this situation, a numerical technique is employed to obtain the MLE of the parameter.

5.2. Maximum product spacing estimation (MPS)

Let, $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ represent the ordered sample of size n . We define the spacing as follow

$$\begin{aligned}D_i(\xi) = \int_{x_{(i-1)}}^{x_{(i)}} f(x;\xi)dx; \qquad i = 1,2,\ldots,n \\ = F(x_{(i)};\xi) - F(x_{(i-1)};\xi) \end{aligned} \tag{19}$$

where $F(x_{(0)};\xi) = 0$ and $F(x_{(n+1)};\xi) = 1$. Clearly, $\sum_{i=1}^{n+1} D_i(\xi) = 1$. For more details, one can see: Cheng and Amin (1983). Now, using Equations (5) and (19) we get

$$\begin{aligned}D_i = \{x_{(i)}(2-x_{(i)})\}^{\xi}\exp\left(\{x_{(i)}(2-x_{(i)})\}^{\xi}-1\right) \\ - \{x_{(i-1)}(2-x_{(i-1)})\}^{\xi}\exp\left(\{x_{(i-1)}(2-x_{(i-1)})\}^{\xi}-1\right); i = 1,2,\ldots,n+1. \end{aligned} \tag{20}$$

The MPS estimator, $\hat{\xi}$, for the parameter ξ is obtained by maximizing geometric mean of the observed difference D_i, i.e.

$$G = \left(\prod_{i=1}^{n+1} D_i\right)^{\frac{1}{n+1}}$$

and taking logarithm of G we get,

$$\log G = \frac{1}{n+1}\left(\sum_{i=1}^{n+1}\log D_i\right). \tag{21}$$

Now substituting the value of D_i from Equation (20) in Equation (21) , we get

$$\log G = \frac{1}{n+1}\sum_{i=1}^{n+1}\log\Bigg[\left\{x_{(i)}(2-x_{(i)})\right\}^{\xi}\exp\left(\left\{x_{(i)}(2-x_{(i)})\right\}^{\xi}-1\right) - \left\{x_{(i-1)}(2-x_{(i-1)})\right\}^{\xi}\exp\left(\left\{x_{(i-1)}(2-x_{(i-1)})\right\}^{\xi}-1\right)\Bigg]. \tag{22}$$

By differentiating Equation (22) with respect to ξ and equating the derivative to zero, we obtain a nonlinear equation that must be solved numerically for parameter ξ.

5.3. Least square estimation (LSE)

Let, $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ represent the ordered sample of size n from $SAS_{TL}(x;\xi)$ distribution. Then, LSE of the parameter, i.e. $\hat{\xi}$ of the parameter ξ is obtaied by minimizing the

$$Z(\xi) = \sum_{i=1}^{n}\{F(x_{(i)},\xi) - E(F(x_{(i)})\}^2$$

where, $E(F(x_{(i)}) = \frac{i}{n+1}$ $; i = 1, 2, \ldots, n$. Then,

$$Z(\xi) = \sum_{i=1}^{n}\left[\{x_{(i)}(2-x_{(i)})\}^{\xi}\exp(\{x_{(i)}(2-x_{(i)})\}^{\xi}-1) - \frac{i}{n+1}\right]^2. \tag{23}$$

To minimize $Z(\xi)$ as defined in Equation (23), we differentiate it with respect to ξ and set the derivative equal to zero. This yields a nonlinear equation in parameter ξ, which must be solved numerically.

5.4. Cramèr-Von-Mises Estimation (CVME)

The CVME is a type of minimum distance estimator, also popularly known as maximum goodness of-fit estimator. It is based on the difference between the estimate of the CDF and the emirical distribution function.

Let, $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$ represent the ordered sample of size n from $SAS_{TL}(x;\xi)$ distribution. Therefore, the CVME, $\hat{\xi}_{CVM}$, is obtained by minimizing the function $C(\xi)$ with respect to ξ where

$$C(\xi) = \frac{1}{12n} + \sum_{i=1}^{n}\left[\{x_{(i)}(2-x_{(i)})\}^{\xi}\exp(\{x_{(i)}(2-x_{(i)})\}^{\xi}-1) - \frac{2i-1}{2n}\right]^2. \tag{24}$$

Since, the equation resulting from differentiating Equation (24) with respect to ξ does not admit a closed-from solution, the corresponding optimization problem leads to a nonlinear equation in parameter. Consequently, numerical technique must be employed to obtain the estimate of the parameter ξ.

6. Simulation

In this section, the Monte Carlo simulation study is conducted to evaluate the performance of the different estimators derived from various estimation methods, including ML, MPS and LS estimation methods. m samples were generated for different variation in sample sizes and parameter values, specifically $n = 20, 40, \ldots, 180, 200$ and $\xi = (0.8, 2, 4)$, respectively. The average estimates and mean squared error (MSE) for the parameters, calculated from $N = 1000$ replications, which is summarized in Tables 1 - 3 respectively.

The parameter estimate i.e, $\hat{\xi}$ of ξ is obtained by

$$\hat{\xi} = \frac{1}{N}\sum_{i=1}^{N}\hat{\xi}_i.$$

Also, Bias and MSE are obtained by

$$\mathit{Bias}(\hat{\xi}) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\xi}_i - \xi\right)$$

and

$$\mathit{MSE}(\hat{\xi}) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\xi}_i - \xi\right)^2,$$

respectively.

Table 1 Simulation results: MLE, Bias, and MSE

n	$\xi = 0.8$			$\xi = 2.0$			$\xi = 4.0$		
	MLE	Bias	MSE	MLE	Bias	MSE	MLE	Bias	MSE
20	0.85767	0.05767	0.05766	2.14417	0.14417	0.36035	4.28835	0.28835	1.44139
40	0.82368	0.02368	0.02595	2.05921	0.05921	0.16220	4.11841	0.11841	0.64882
60	0.81584	0.01584	0.01460	2.03960	0.03960	0.09123	4.07920	0.07920	0.36492
80	0.81319	0.01319	0.01199	2.03298	0.03298	0.07491	4.06597	0.06597	0.29966
100	0.80929	0.00929	0.00937	2.02324	0.02324	0.05857	4.04647	0.04647	0.23426
120	0.80372	0.00372	0.00734	2.00930	0.00930	0.04590	4.01859	0.01859	0.18362
140	0.80770	0.00770	0.00655	2.01925	0.01925	0.04096	4.03850	0.03850	0.16383
160	0.80646	0.00646	0.00563	2.01616	0.01616	0.03518	4.03232	0.03232	0.14074
180	0.80794	0.00794	0.00461	2.01986	0.01986	0.02880	4.03971	0.03971	0.11519
200	0.80435	0.00435	0.00454	2.01088	0.01088	0.02838	4.02176	0.02176	0.11352

Table 2 Simulation results: MPSE, Bias, and MSE

n	$\xi = 0.8$			$\xi = 2.0$			$\xi = 4.0$		
	MPSE	Bias	MSE	MPSE	Bias	MSE	MPSE	Bias	MSE
20	0.79502	-0.00498	0.04678	1.98754	-0.01246	0.29235	3.97509	-0.02492	1.16942
40	0.78457	-0.01543	0.02320	1.96142	-0.03858	0.14499	3.92283	-0.07717	0.57995
60	0.78663	-0.01337	0.01360	1.96658	-0.03342	0.08498	3.93316	-0.06684	0.33991
80	0.78944	-0.01056	0.01124	1.97361	-0.02639	0.07022	3.94721	-0.05279	0.28089
100	0.78918	-0.01082	0.00895	1.97296	-0.02704	0.05596	3.94592	-0.05408	0.22383
120	0.78600	-0.01400	0.00722	1.96500	-0.03500	0.04512	3.93000	-0.07000	0.18049
140	0.79202	-0.00798	0.00631	1.98006	-0.01994	0.03945	3.96011	-0.03989	0.15780
160	0.79214	-0.00786	0.00546	1.98036	-0.01964	0.03412	3.96072	-0.03928	0.13646
180	0.79495	-0.00505	0.00443	1.98738	-0.01262	0.02770	3.97475	-0.02525	0.11079
200	0.79250	-0.00750	0.00445	1.98126	-0.01874	0.02784	3.96251	-0.03749	0.11137

Table 3 Simulation results: LSE, Bias, and MSE

n	$\xi = 0.8$			$\xi = 2.0$			$\xi = 4.0$		
	LSE	Bias	MSE	LSE	Bias	MSE	LSE	Bias	MSE
20	0.84223	0.04223	0.06715	2.10557	0.10557	0.41971	4.21114	0.21114	1.67883
40	0.81855	0.01855	0.03223	2.04639	0.04639	0.20142	4.09277	0.09277	0.80567
60	0.80957	0.00957	0.01697	2.02393	0.02393	0.10606	4.04786	0.04786	0.42422
80	0.80829	0.00829	0.01405	2.02071	0.02071	0.08782	4.04143	0.04143	0.35129
100	0.80636	0.00636	0.01177	2.01590	0.01590	0.07355	4.03179	0.03179	0.29419
120	0.80297	0.00297	0.00894	2.00743	0.00743	0.05587	4.01485	0.01485	0.22350
140	0.80637	0.00637	0.00791	2.01593	0.01593	0.04946	4.03186	0.03186	0.19783
160	0.80541	0.00541	0.00703	2.01352	0.01352	0.04396	4.02703	0.02703	0.17583
180	0.80628	0.00628	0.00565	2.01569	0.01569	0.03533	4.03138	0.03138	0.14131
200	0.80308	0.00308	0.00535	2.00771	0.00771	0.03346	4.01542	0.01542	0.13385

Table 4 Simulation results: CVME, Bias, and MSE

n	$\xi = 0.8$			$\xi = 2.0$			$\xi = 4.0$		
	CVME	Bias	MSE	CVME	Bias	MSE	CVME	Bias	MSE
20	0.84824	0.04824	0.06810	2.12061	0.12061	0.42564	4.24122	0.24122	1.70257
40	0.82167	0.02167	0.03250	2.05416	0.05416	0.20310	4.10833	0.10833	0.81242
60	0.81168	0.01168	0.01706	2.02921	0.02921	0.10664	4.05842	0.05842	0.42657
80	0.80987	0.00987	0.01412	2.02467	0.02467	0.08825	4.04934	0.04934	0.35301
100	0.80762	0.00762	0.01181	2.01906	0.01906	0.07380	4.03811	0.03811	0.29522
120	0.80402	0.00402	0.00896	2.01005	0.01005	0.05601	4.02011	0.02011	0.22405
140	0.80728	0.00728	0.00794	2.01819	0.01819	0.04961	4.03639	0.03639	0.19845
160	0.80620	0.00620	0.00705	2.01550	0.01550	0.04407	4.03099	0.03099	0.17626
180	0.80698	0.00698	0.00567	2.01744	0.01744	0.03543	4.03488	0.03488	0.14171
200	0.80372	0.00372	0.00536	2.00929	0.00929	0.03353	4.01858	0.01858	0.13411

Figure 3 illustrates the MSEs obtained from four different methods across varying sample sizes.

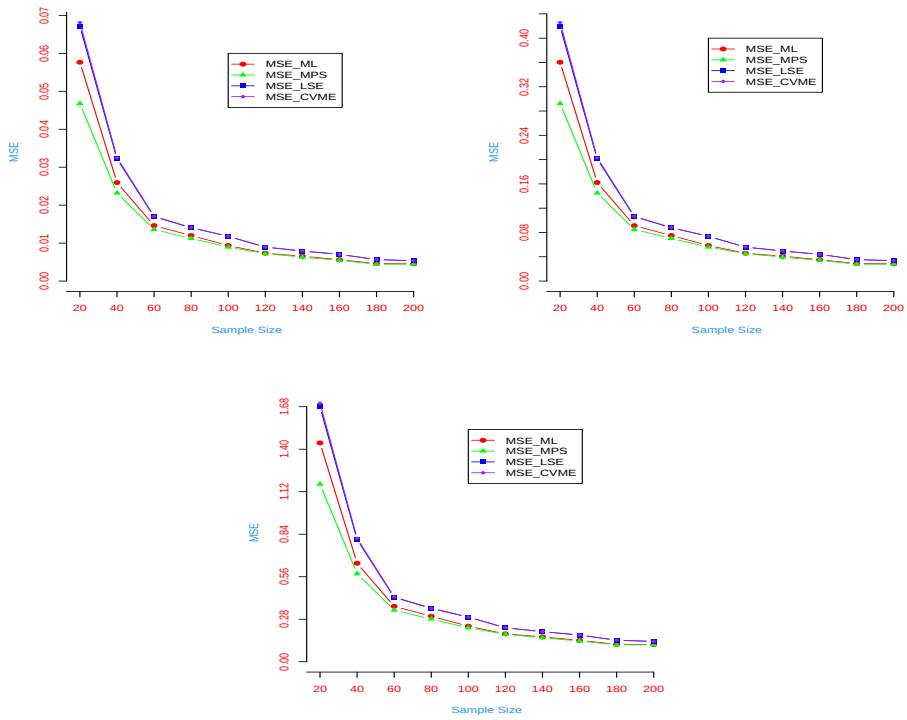


Figure 3 Graphical comparison of the MSEs for the different parameter values $\xi = \{0.8, 2, 4\}$.

Empirical results, as illustrated through tableau and MSE plots, demonstrate that increasing the sample size consistently reduces the MSE.

7. Real Data Application

In order to discuss, how the proposed distribution can be implemented in real life, we analyze the three datasets. The data is scaled to the unit interval $[0, 1]$ using min-max normalization, i.e.,

$$\frac{\text{data} - \min(\text{data})}{\max(\text{data}) - \min(\text{data})}.$$

Define a small positive constant ε to handle edge cases during normalization.To avoid computational issues associated with boundaries values, any data point equal to 0 or 1 after normalization are replaced with ε and $1 - \varepsilon$, respectively, ensuring all values lies strictly within open interval $(0, 1)$.

The capacity factor, defined as the ratio of actual electricity output to the maximum possible output. This quantity is the key measure of reliability, lies in the unit interval. It is commonly used to assess the operational efficacy of power unit and holds particular significance for nuclear power plants.First dataset is taken from Caramanis et al. (1983) providing capacity factors by SC16 algorithms. The second data-set contains the survival times, measured in weeks, of 33 patients diagnosed with acute Myelogenous Leukemia. It was recently analyzed by Jamal et al. (2017), who fitted it using the odd Burr-III Lomax distribution. The third dataset, consists of 58 observations related to the monthly metrics on the unemployment claim from July 2008 to April 2013. One can find out this

dataset from Alanzi et al. (2022). These data sets are reported in Tables 5 - 7 respectively. Summary of all the datasets are given in Table 8.

Table 5 Computation time of SC16-algorithms

0.974	0.867	0.989	0.924	0.819	0.623	0.564	0.462
0.395	0.247	0.136	0.136	0.109	0.084	0.072	0.059
0.045	0.038	0.028	0.020	0.017	0.014	0.011	

Table 6 Acute Myelogenous Leukemia data

0.4129	0.9999	0.6387	0.8580	0.0969	0.6903	0.7741	0.0195
0.2452	0.9160	0.3549	0.1614	0.1356	0.0001	0.0001	0.0259
0.4129	0.3549	0.4129	0.1033	0.0388	0.0969	0.1356	0.0130
0.0195	0.0066	0.0453	0.0195	0.0130	0.1872	0.0195	0.2710

Table 7 Unemployment Claim data

0.823	0.864	0.816	0.841	0.831	0.833	0.894	0.869	0.866
0.860	0.837	0.826	0.804	0.809	0.758	0.770	0.778	0.707
0.814	0.825	0.906	0.924	0.927	0.920	0.770	0.544	0.550
0.608	0.630	0.650	0.820	0.873	0.900	0.916	0.899	0.862
0.695	0.650	0.751	0.862	0.702	0.530	0.764	0.898	0.897
0.908	0.902	0.879	0.645	0.739	0.765	0.803	0.708	0.669
0.561	0.579	0.701	0.839					

Table 8 Summary of datasets under consideration

Data	Min.	Q1	Median	Mean	Q3	Max.
I	0.0060	0.0325	0.1160	0.2881	0.5180	0.8660
II	0.0060	0.0190	0.1350	0.2652	0.4130	0.9900
III	0.5300	0.7073	0.8180	0.7857	0.8682	0.9270

Furthermore, the total time on test (TTT-plot) plot of first two datasets indicates a bathtub hazard rate while third dataset has increasing hazard rate which is shown in Figures (4) and (5). For detailed study related to TTT-plot one can read Aarset (1987).

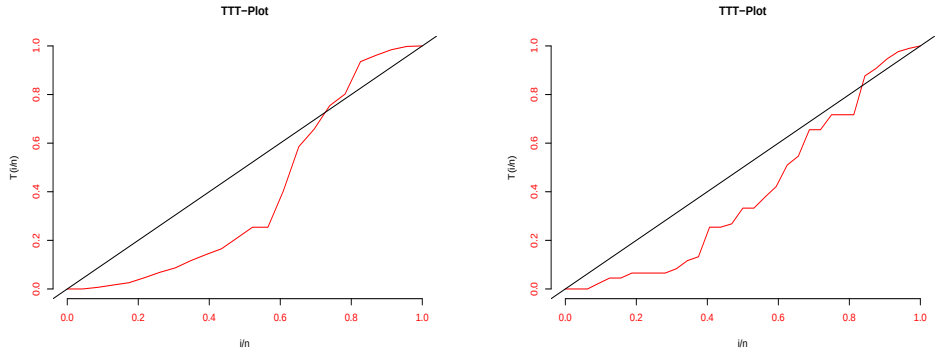


Figure 4 TTT plots for Data Sets I and II.

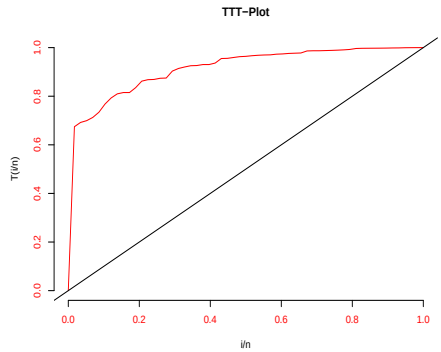


Figure 5 TTT plots for dataset- III .

The probability distributions considered for comparison purpose, includes: Topp-Leone distribution(TLD) suggested by Genc (2013), unit new X-Lindley distribution (UNXLD) proposed by Irshad et al. (2024), Unit Teissier Distribution (UTD) suggested by Krishna et al. (2022), Power function distribution (PFD) introduced by Meniconi and Barry (1996),Unit Zeghdoudi distribution (UZD) introduced by Bashiru et al. (2025), Unit Rayleigh distribution (URD) introduced by Bantan et al. (2020), Unit Lindley distribution (ULD) introduced by Mazucheli et al. (2019). The PDF's of these distributions, all defined on the unit interval (0, 1), were used to compare against the proposed density function:

1. **Topp-Leone Distribution** with PDF:

$$f(x,\xi) = 2 \cdot \xi x^{\xi-1} \cdot (1-x) \cdot (2-x)^{\xi-1}; \quad 0 < x < 1,\ \xi > 0$$

2. **Unit new X-Lindley Distribution** with PDF:

$$f(x,\xi) = \frac{\xi}{2} \cdot x^{\xi-1} \cdot \left[1 - \xi \cdot \log(x)\right]; \quad 0 < x < 1,\ \xi > 0$$

3. **Unit Teissier Distribution** with PDF:

$$f(x,\xi) = \xi \cdot (x^{-\xi} - 1) \cdot x^{-(\xi+1)} \cdot \exp\left\{-x^{-\xi} + 1\right\}; \quad 0 < x < 1,\ \xi > 0$$

4. **Power Function Distribution** with PDF:

$$f(x,\xi) = \xi \cdot x^{\xi-1}; \quad 0 < x < 1,\ \xi > 0$$

5. **Unit Rayleigh distribution** with PDF:

$$f(x,\xi) = -\frac{2\xi}{x}\log(x)\exp\Big(-\xi(\log(x))^2\Big); \quad 0<x<1,\ \xi>0$$

6. **Unit Lindley distribution** with PDF:

$$f(x,\xi) = \frac{\xi^2}{1+\xi}(1-x)^{-3}\exp\Big(-\frac{\xi x}{1-x}\Big); \quad 0<x<1,\ \xi>0$$

7. **One-parameter Beta distribution** with PDF:

$$f(x,\xi) = \frac{\xi^2}{1+\xi}(1-x)^{-3}\exp\Big(-\frac{\xi x}{1-x}\Big); \quad 0<x<1,\ \xi>0$$

8. **Unit Zeghdoudi distribution** with PDF:

$$f(x,\xi) = \frac{\xi^3 x}{(1-x)^4(\xi+2)}\exp\Big(-\frac{\xi x}{1-x}\Big); \quad 0<x<1,\ \xi>0$$

Furthermore, Figures 6 and 7 displays the empirical cumulative distribution function (ECDF) alongside the fitted CDF for the above mentioned datasets. Also, we compare the fitting of the $SAS_{TL}(x;\xi)$ distribution with aforementioned well known comparative models on the basis of P-P and Q-Q plots which is shown in Figures 8, 9, and 10. The figure demonstrates that the proposed model provides better fit compared to the competing models. Consequently, the proposed model can be regarded as a strong alternative to several existing lifetime models.

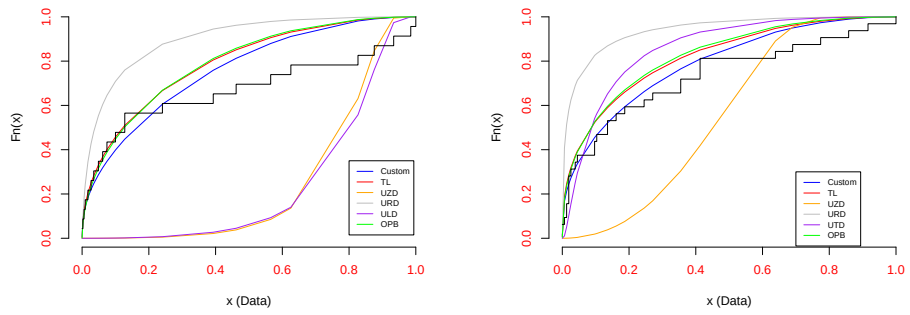


Figure 6 The Empirical vs fitted CDF plots for datasets I, II.

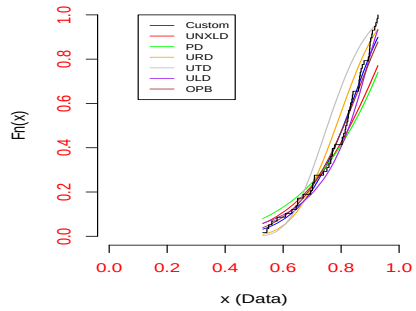


Figure 7 The Empirical vs fitted CDF plots for data III.

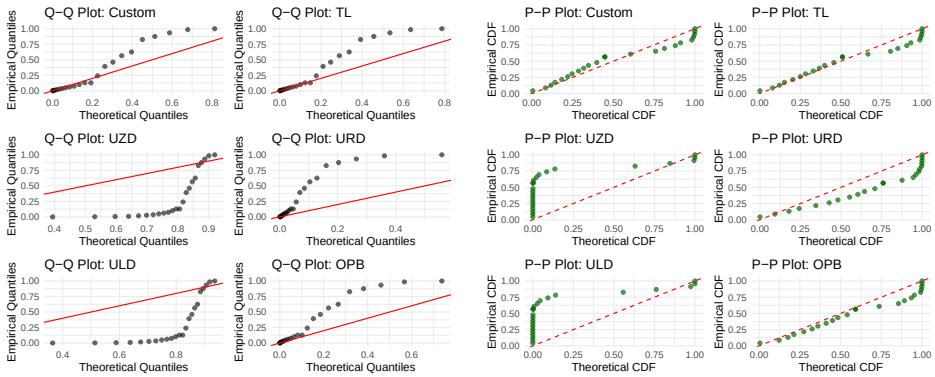


Figure 8 Q-Q and P-P plots for data I.

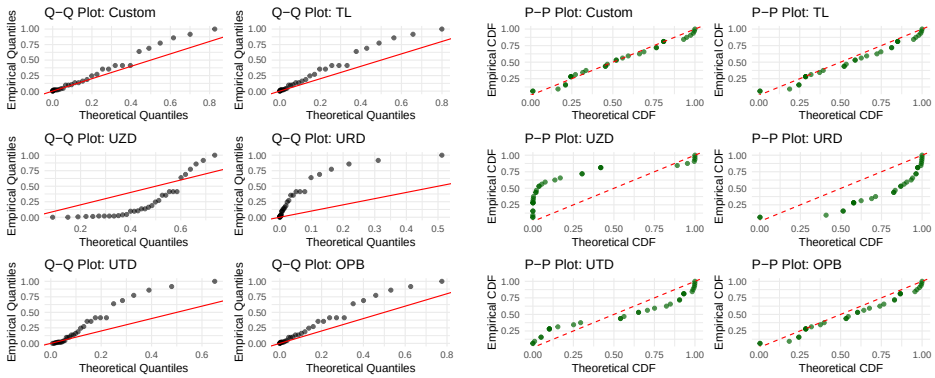


Figure 9 Q-Q and P-P plots for data II.

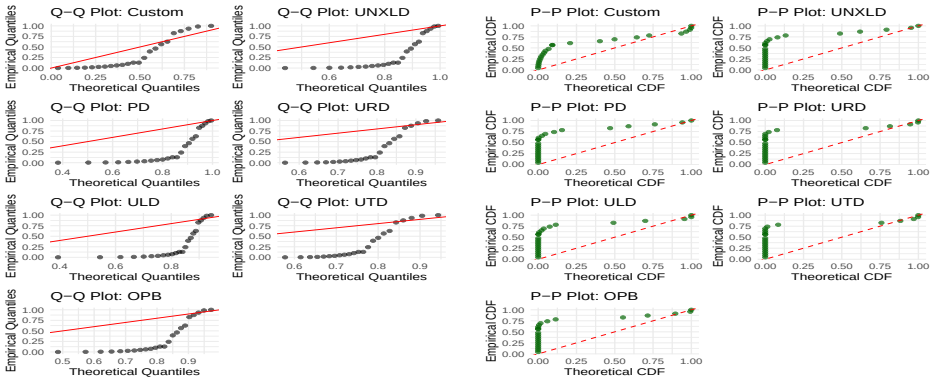


Figure 10 Q-Q and P-P plots for data III.

The Kolmogorov-Smirnov (KS) test statistics and associated p-value are employed to quantify the degree of fit between a candidate model's CDF and the empirical distribution of the observed data Seal et al. (2023). Complementing this approach, we also consider model selection criteria such as Akaike Information criterion (AIC), Corrected Akaike Information criterion (AICc), Bayesian Information criterion (BIC), and log-likelihood (LL) values to ensure more accurate and comprehensive comparison of competing models. In general, the smallest value of AIC, AICc and BIC alongside highest value of LL indicates the better fit to the data.

Table 9 log(L), AIC, AICc, BIC, KS-statistics and p-values for fitted models based on Real Data I

Model	MLE	log(L)	AIC	AICc	BIC	KS-stat.	p-value
SAS$_{TL}\{\xi\}$	0.3111	-2.8378	7.6757	7.8662	8.8112	0.1987	0.3242
TL$\{\xi\}$	0.4715	-3.8510	9.7019	9.8924	10.8374	0.2102	0.2612
UZD$\{\xi\}$	0.6291	-150.1284	302.2568	302.4473	303.3923	0.5578	1e-8
URD$\{\xi\}$	0.0649	-18.6090	39.2180	39.4085	40.3535	0.3362	0.0110
ULD$\{\xi\}$	0.3916	-78.5101	159.0202	159.2107	160.1557	0.4704	0.0001
OPB$\{\xi\}$	0.5140	-4.9039	11.8078	11.9983	12.9433	0.2169	0.2294

It is observed from the Table 9 that, $SAS_{TL}(x;\xi)$ has the smallest value for AIC, AICc and BIC while for LL it has maximum value among all other competitive model. As a result, the suggested $SAS_{TL}(x;\xi)$ model is considered as the best model for the data-I.

Table 10 log(L), AIC, AICc, BIC, KS-statistics and p-values for fitted models based on Real Data II

Model	MLE	log(L)	AIC	AICc	BIC	KS-stat.	p-value
SAS$_{TL}\{\xi\}$	0.2573	14.3832	-26.7663	-26.6330	-25.3006	0.1186	0.7586
TL$\{\xi\}$	0.3838	11.6598	-21.3196	-21.1863	-19.8539	0.1571	0.4086
UZD$\{\xi\}$	2.5552	-122.2946	246.5892	246.7225	248.0549	0.4062	0.0001
URD$\{\xi\}$	0.0352	-17.9475	37.8950	38.0284	39.3608	0.4503	1e-6
UTD$\{\xi\}$	0.4012	-85.8466	173.6931	173.8265	175.1589	0.2489	0.0380
OPB$\{\xi\}$	0.4110	10.4749	-18.9499	-18.8165	-17.4841	0.1700	0.3135

Tables 10 and 11 also indicates that the $SAS_{TL}(x;\xi)$ model achieves superior performance, as evidenced by its minimum AIC, AICc and BIC values, alongside the highest LL-value. Consequently, it is deemed the best fitting model for datasets II and III.

Table 11 log(L), AIC, AICc, BIC, KS-statistics and p-values for fitted models based on Real Data III

Model	MLE	log(L)	AIC	AICc	BIC	KS-stat.	p-value
SAS$_{TL}\{\xi\}$	10.1095	50.5711	-99.1423	-99.0708	-97.0818	0.1011	0.5935
UNXLD$\{\xi\}$	6.2090	40.7958	-79.5916	-79.5202	-77.5312	0.2284	0.0047
PD$\{\xi\}$	3.9707	36.5862	-71.1724	-71.1010	-69.1119	0.2599	0.0008
URD$\{\xi\}$	11.6441	49.4570	-96.9141	-96.8427	-94.8537	0.1571	0.1141
ULD$\{\xi\}$	0.3462	49.1132	-96.2265	-96.1551	-94.1660	0.1088	0.4987
UTD$\{\xi\}$	3.4740	43.0292	-84.0585	-83.9870	-81.9980	0.2695	0.0004
OPB$\{\xi\}$	7.4730	49.2177	-96.4354	-96.3640	-94.3750	0.1229	0.3453

8. Conclusions

The ongoing trend in statistical studies involves proposing new lifetime distribution or modifying existing ones. This article presents a single parameter $SAS_{TL}(x;\xi)$ distribution constructed through the SAS-transformation (1). Some of the important statistical properties such as survival and hazard rate function, moments, conditional moments, quantile function, mean deviation, order statistics, entropy and stochastic ordering are obtained. To estimate the parameter, four different estimation method such as maximum likelihood estimation, maximum product spacings, least square and Cramèr-Von-Mises estimation are employed. These estimators are compared using Monte Carlo simulation for different sample sizes. The simulation results demonstrate that the estimators are consistent and unbiased with increasing sample size. Furthermore, empirical analysis with three real datasets suggest that the proposed $SAS_{TL}(x;\xi)$ offers superior fitting compared to the existing models, validating its utility for lifetime data modeling.

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Appendix

A1. Derivation of the r-th order Moment

Let X be a random variable with PDF

$$f_{SAS_{TL}}(x;\xi) = 2\xi(1-x)\{x(2-x)\}^{\xi-1}\left[1+\{x(2-x)\}^{\xi}\right] \times \exp\left[\{x(2-x)\}^{\xi}-1\right], \quad 0<x<1,\ \xi>0.$$

The r-th moment about the origin is defined as

$$\mu'_r = E(x^r) = \int_{-\infty}^{\infty} x^r f(x;\xi)\,dx; \qquad r = 1,2,3,.....$$

Substituting the pdf, we obtain

$$\mu'_r = \int_0^1 x^r\, 2\xi(1-x)\{x(2-x)\}^{\xi-1}\left[1+\{x(2-x)\}^{\xi}\right] \times \exp\left[\{x(2-x)\}^{\xi}-1\right]dx.$$

Using the series expansion

$$\exp(z) = \sum_{i=0}^{\infty}\frac{z^i}{i!},$$

we write

$$\exp\left([x(2-x)]^{\xi}-1\right) = \frac{1}{e}\sum_{i=0}^{\infty}\frac{[x(2-x)]^{\xi i}}{i!}.$$

Thus,

$$\mu'_r = \frac{2\xi}{e}\sum_{i=0}^{\infty}\frac{1}{i!}\int_0^1 x^r(1-x)\left\{x(2-x)\right\}^{\xi i+\xi-1}\left[1+\{x(2-x)\}^{\xi}\right]dx.$$

Note that,

$$(1-x)x^r = x^r - x^{r+1}.$$

Hence,

$$\mu'_r = \frac{2\xi}{e}\sum_{i=0}^{\infty}\frac{1}{i!}\int_0^1 (x^r-x^{r+1})x^{\xi i+\xi-1}(2-x)^{\xi i+\xi-1}\left[1+\{x(2-x)\}^{\xi}\right]dx.$$

Expanding the term $\left[1+\{x(2-x)\}^{\xi}\right]$, we obtain

$$\mu'_r = \frac{2\xi}{e}\sum_{i=0}^{\infty}\frac{1}{i!}\left(I_1+I_2\right),$$

where

$$I_1 = \int_0^1 (x^r-x^{r+1})x^{\xi i+\xi-1}(2-x)^{\xi i+\xi-1}dx,$$
$$I_2 = \int_0^1 (x^r-x^{r+1})x^{\xi i+2\xi-1}(2-x)^{\xi i+2\xi-1}dx.$$

Using the generalized binomial expansion

$$(2-x)^A = 2^A\sum_{j=0}^{\infty}\binom{A}{j}\left(-\frac{x}{2}\right)^j, \quad |x|<2,$$

where

$$\binom{A}{j} = \frac{A(A-1)\cdots(A-j+1)}{j!},$$

we expand each term.

After simplification and evaluation of integrals of the form

$$\int_0^1 x^m dx = \frac{1}{m+1},$$

the r-th moment reduces to

$$\mu'_r = \frac{2\xi}{e}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j}{i!}\left[2^{\xi i+\xi-j-1}\binom{\xi i+\xi-1}{j}\left(\frac{1}{\xi i+\xi+r+j}-\frac{1}{\xi i+\xi+r+j+1}\right)\right.$$
$$\left.+\,2^{\xi i+2\xi-j-1}\binom{\xi i+2\xi-1}{j}\left(\frac{1}{\xi i+2\xi+r+j}-\frac{1}{\xi i+2\xi+r+j+1}\right)\right]. \tag{25}$$

Remark 5 The above series converges since the binomial expansion is valid for $|x| < 2$, which holds for $x \in (0, 1)$.

A2. Derivation: Rényi Entropy of the Proposed Model

The Rényi entropy of order $\beta > 0$, $\beta \neq 1$, for a continuous random variable with PDF $f(x;\xi)$ is defined as

$$\mathscr{R}_\beta(\xi) = \frac{1}{1-\beta}\log\left[\int_0^1 f^\beta(x;\xi)dx\right].$$

The PDF of proposed model is,

$$f_{SAS_{TL}}(x;\xi) = 2\xi(1-x)\{x(2-x)\}^{\xi-1}\left[1+\{x(2-x)\}^{\xi}\right]$$
$$\times\exp\left[\{x(2-x)\}^{\xi}-1\right], \quad 0<x<1,\ \xi>0.$$

Raising $f_{SAS_{TL}}(x;\xi)$ to the power β, we obtain

$$f^\beta_{SAS_{TL}}(x;\xi) = \left(\frac{2\xi}{e}\right)^\beta(1-x)^\beta\{x(2-x)\}^{\beta(\xi-1)}$$
$$\times\left[1+\{x(2-x)\}^{\xi}\right]^\beta\exp\left[\beta\{x(2-x)\}^{\xi}\right].$$

Using the series expansion of the exponential function,

$$\exp(\beta z) = \sum_{i=0}^{\infty}\frac{\beta^i}{i!}z^i,$$

we write

$$\exp\left[\beta\{x(2-x)\}^{\xi}\right] = \sum_{i=0}^{\infty}\frac{\beta^i}{i!}\{x(2-x)\}^{\xi i}.$$

Thus,

$$f^\beta_{SAS_{TL}}(x;\xi) = \left(\frac{2\xi}{e}\right)^\beta\sum_{i=0}^{\infty}\frac{\beta^i}{i!}(1-x)^\beta\{x(2-x)\}^{\xi i+\beta\xi-\beta}\left[1+\{x(2-x)\}^{\xi}\right]^\beta$$

Next, using the binomial expansions

$$(1-x)^{\beta} = \sum_{\ell=0}^{\infty} \binom{\beta}{\ell} (-1)^{\ell} x^{\ell},$$
$$(2-x)^{A} = \sum_{k=0}^{\infty} \binom{A}{k} (-1)^{k} 2^{A-k} x^{k}.$$

where

$$A = \xi i + \xi j + \xi\beta - \beta$$

we obtain

$$f^{\beta}_{SAS_{TL}}(x;\xi) = \left(\frac{2\xi}{e}\right)^{\beta} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{\beta^i}{i!}\binom{\beta}{j}\binom{A}{k}\binom{\beta}{\ell}(-1)^{k+\ell}2^{A-k}x^{A+k+\ell}.$$

Integrating over $(0,1)$,

$$\int_0^1 x^{A+k+\ell}dx = \frac{1}{A+k+\ell+1},$$

we get

$$\int_0^1 f^{\beta}_{SAS_{TL}}(x;\xi)dx = \left(\frac{2\xi}{e}\right)^{\beta} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{\beta^i}{i!}\binom{\beta}{j}\binom{A}{k}\binom{\beta}{\ell}(-1)^{k+\ell}2^{A-k}\frac{1}{A+k+\ell+1}.$$

Therefore, the Rényi entropy is

$$\mathscr{R}_{\beta}(\xi) = \frac{1}{1-\beta}\log\left[\left(\frac{2\xi}{e}\right)^{\beta} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{\beta^i}{i!}\binom{\beta}{j}\binom{A}{k}\binom{\beta}{\ell}(-1)^{k+\ell}2^{A-k}\frac{1}{A+k+\ell+1}\right]$$

Or,

$$\mathscr{R}_{\beta}(\xi) = \frac{\beta}{1-\beta}\log\left(\frac{2\xi}{e}\right) + \frac{1}{1-\beta}\log\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{\beta^i}{i!}\binom{\beta}{j}\binom{A}{k}\binom{\beta}{\ell} \times (-1)^{k+\ell}\frac{2^{A-k}}{A+k+\ell+1}\right]. \tag{26}$$

A3. R Code for Fitting and Simulation

The complete R code used for parameter estimation, goodness-of-fit analysis, and simulation studies is given below.

Listing 1 Complete R code for fitting the proposed and competing models and simulation study

```
# Clear workspace
rm(list = ls(all = TRUE))
# Libraries
library(MASS)
library(goftest)
library(DataSetsUni)
library(ADGofTest)
```

```
\# ------------------------------------
\# Data
\# ------------------------------------
data <- data
x <- (data - min(data)) / (max(data) - min(data))
eps <- 1e-6
x[x == 0] <- eps
x[x == 1] <- 1 - eps
n <- length(x)
# ------------------------------------
# 1. SAS-TL (Proposed Model)
# ------------------------------------
pdf_SAS_TL <- function(x, theta) {
val <- 2 * theta * (1 - x) * ((x * (2 - x))^(theta - 1)) *
(1 + (x * (2 - x))^theta) * exp(((x * (2 - x))^theta - 1))
val[!is.finite(val) | val <= 0] <- .Machine$double.eps
return(val)
}

cdf_SAS_TL <- function(x, theta) {
val <- ((x * (2 - x))^theta) * exp(((x * (2 - x))^theta - 1))
return(pmin(1, pmax(0, val)))
}
# ------------------------------------
# 2. Topp-Leone (TL)
# ------------------------------------
pdf_TL <- function(x, theta) {
val <- 2 * theta * (1 - x) * (x^(theta - 1)) * (2 - x)^(theta - 1)
val[!is.finite(val) | val <= 0] <- .Machine$double.eps
return(val)
}

cdf_TL <- function(x, theta) {
val <- (x * (2 - x))^theta
return(pmin(1, pmax(0, val)))
}
# ------------------------------------
# 3. Unit Zeghdoudi (UZD)
# ------------------------------------
pdf_UZD <- function(x, theta) {
val <- ((theta^3 * x) / ((1 - x)^4 * (theta + 2))) *
exp(-theta * x / (1 - x))
val[!is.finite(val) | val <= 0] <- .Machine$double.eps
return(val)
}

cdf_UZD <- function(x, theta) {
val <- (1 - exp(-theta * x / (1 - x)) *
(1 + (theta * x * (theta + 2 - 2 * x)) /
((theta + 2) * (1 - x)^2)))
return(pmin(1, pmax(0, val)))
```

```
}
# ------------------------------------
# 4. Unit Lindley (ULD)
# ------------------------------------
pdf_ULD <- function(x, theta) {
val <- (theta^2 / (1 + theta)) *
exp((-theta * x) / (1 - x)) / (1 - x)^3
val[!is.finite(val) | val <= 0] <- .Machine$double.eps
return(val)
}

cdf_ULD <- function(x, theta) {
val <- 1 - (1 - (theta * x) / ((1 + theta)*(1-x))) *
exp((-theta * x) / (1 - x))
return(pmin(1, pmax(0, val)))
}
# ------------------------------------
# 5. Unit Rayleigh (URD)
# ------------------------------------
pdf_URD <- function(x, theta) {
val <- (-2 * theta / x) * log(x) * exp(-theta*(log(x))^2)
val[!is.finite(val) | val <= 0] <- .Machine$double.eps
return(val)
}

cdf_URD <- function(x, theta) {
val <- exp(-theta*(log(x))^2)
return(pmin(1, pmax(0, val)))
}
# ------------------------------------
# 6. One-Parameter Beta (OPB)
# ------------------------------------
pdf_OPB <- function(x, mu) {
const <- mu * (mu + 1)
val <- const * x^(mu - 1) * (1 - x)
val[!is.finite(val) | val <= 0] <- .Machine$double.eps
return(val)
}

cdf_OPB <- function(x, mu) {
const <- mu * (mu + 1)
val <- const * ((x^mu)/mu - (x^(mu+1))/(mu+1))
return(pmin(1, pmax(0, val)))
}
# ------------------------------------
# Model Fitting
# ------------------------------------
dist_list <- list(
SAS_TL = list(pdf = pdf_SAS_TL, cdf = cdf_SAS_TL),
TL     = list(pdf = pdf_TL,     cdf = cdf_TL),
UZD    = list(pdf = pdf_UZD,    cdf = cdf_UZD),
```

```
ULD     = list(pdf = pdf_ULD,     cdf = cdf_ULD),
URD     = list(pdf = pdf_URD,     cdf = cdf_URD)
)

results <- lapply(names(dist_list), function(name) {
dist <- dist_list[[name]]
fit <- tryCatch({fitdistr(x, dist$pdf, start = list(theta = 2), lower =
if (is.null(fit)) {
return(data.frame(Distribution = name, Theta = NA))
}

theta_hat <- fit$estimate
LL <- fit$loglik
k <- length(theta_hat)

AIC  <- -2 * LL + 2 * k
AICc <- AIC + (2*k^2 + 2*k)/(n-k-1)
BIC  <- -2 * LL + log(n)*k

cdf_fn <- function(q) dist$cdf(q, theta_hat)
ks <- ks.test(x, cdf_fn)

data.frame(Distribution = name, Theta = round(theta_hat, 4),
LL = round(LL, 4), AIC = round(AIC, 4),
AICc = round(AICc, 4), BIC = round(BIC, 4),
KS_stat = round(ks$statistic , 4), KS_p    = round(ks$p.value , 4))
})

final_results <- do.call(rbind, results)
print(final_results)
###################################
# Simulation Study
###################################
library(lamW)
# Simulation settings
t <- 1000
n_values <- c(20,40,60,80,100,120,140,160,180,200)

# Storage matrices
MLE_n  <- matrix(nrow=t, ncol=length(n_values))
MPS_n  <- matrix(nrow=t, ncol=length(n_values))
LS_n   <- matrix(nrow=t, ncol=length(n_values))
CVME_n <- matrix(nrow=t, ncol=length(n_values))

colnames(MLE_n)  <- paste0("n_", n_values)
colnames(MPS_n)  <- paste0("n_", n_values)
colnames(LS_n)   <- paste0("n_", n_values)
colnames(CVME_n) <- paste0("n_", n_values)

# Random number generation
set.seed()
```

```
generate_random_numbers <- function(theta, n) {
u <- runif(n)
w_value <- lambertW0(u * exp(1))
sqrt_term <- sqrt(1 - (w_value)^(1 / theta))
result <- 1 - sqrt_term
return(result)
}

# PDF and CDF
f <- function(x, theta) {
val <- 2 * theta * (1 - x) * ((x * (2 - x))^(theta - 1)) *
(1 + (x * (2 - x))^theta) * exp(((x * (2 - x))^theta - 1))
val[!is.finite(val) | val <= 0] <- .Machine$double.eps
return(val)
}

F <- function(x, theta) {
val <- ((x * (2 - x))^theta) * exp(((x * (2 - x))^theta - 1))
return(pmin(1, pmax(0, val)))
}

# Estimation methods
LL <- function(par, x) {
theta <- par
return(-sum(log(f(x, theta))))
}

MPS <- function(par, x) {
theta <- par
n <- length(x)
x <- sort(x)
D <- numeric(n + 1)
D[1] <- F(x[1], theta)
for (j in 2:n) {
 D[j] <- F(x[j], theta) - F(x[j-1], theta)
}
D[n+1] <- 1 - F(x[n], theta)

return(-(1/(n+1)) * sum(log(D)))
}

LS <- function(par, x) {
theta <- par
n <- length(x)
x <- sort(x)

Z <- numeric(n)
for (j in 1:n) {
Z[j] <- F(x[j], theta) - (j/(n+1))
}
```

```
return(sum(Z^2))
}

CVME <- function(par, x) {
theta <- par
n <- length(x)
x <- sort(x)

C <- numeric(n)
for (j in 1:n) {
C[j] <- F(x[j], theta) - ((2*j-1)/(2*n))
}

return((1/(12*n)) + sum(C^2))
}

# Simulation loop
theta_true <- 0.8

for (i in 1:length(n_values)) {
        n <- n_values[i]

for (k in 1:t) {
x <- generate_random_numbers(theta_true, n)

        MLE_n[k,i]  <- nlm(LL, p=3, x=x)$estimate
        MPS_n[k,i]  <- nlm(MPS, p=3, x=x)$estimate
        LS_n[k,i]   <- nlm(LS, p=3, x=x)$estimate
        CVME_n[k,i] <- nlm(CVME, p=3, x=x)$estimate
        }
}

# Performance measures
results_df <- data.frame(n = n_values,
Avg_MLE = colMeans(MLE_n), Bias_MLE = colMeans(MLE_n - theta_true),
MSE_MLE = colMeans((MLE_n - theta_true)^2))
print(results_df)
```