Thailand Statistician
July 2026; 24(3): 765-782
http://statassoc.or.th
Contributed paper

Modeling of Cancer Patients Data Using Burr-X Log Logistic

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Received: 23 January 2024
Revised: 5 February 2026
Accepted: 9 April 2026

Abstract

The Burr-X Log Logistic probability distribution with three parameters is established with application. The projected distribution has a unimodal and bathtub shaped density function and hazard rate function with inverted bathtub shape. We considered its statistical properties to interpret about the nature of proposed distribution. Plots for its density function and survival function are drawn for different combination of parameters. Order statistics distribution and corresponding t^{th} moment for proposed distribution is also considered. Rényi entropy and Probability weighted moments are calculated. Maximum likelihood estimation technique is applied to compute parameter estimates. We executed a simulation study to compare performance of the estimators. We apply derived distribution on three real datasets and results show that it provides better fit than some existing distributions.

Keywords: Burr-X generator, order statistics, probability weighted moments, maximum likelihood estimation, Rényi entropy.

1. Introduction

The modern era on distribution theory has challenge faced by the researchers and thus results a number of models so that the data can be explored in different fields. Azzalini (1985), Marshall and Olkin (1997) and Gupta et al. (1998) were the pioneers in this area. Eugene et al. (2002) proposed beta-generated method. Cooray and Ananda (2005) proposed the composite method by combining two distributions. A new class of distribution by adding two more shape parameters was proposed by Cordeiro et al. (2013). Alzaatreh et al. (2013) introduced a general method named as transformed-transformer method. Modi and Gill (2015) derived Length-biased weighted Maxwell distribution., Tahir et al. (2015) establish a new family of continuous distributions namely odd generalized exponential family. Cordeiro, Silva and Ortega (2016) generated an extended-G geometric family. Cordeiro et al. (2017) proposed exponentiated Weibull-H distributions family. Power Lindley-G distributions were generated by Hassan and Nassr (2018). Bodhisuwan and Aryuyuen (2021) introduced Gompertz Weibull Frechet distribution. Family of power exponentiated distributions was proposed by Modi (2021). The Beta Topp-Leone generated family established by Watthanawisut et al. (2022).

In persistence of mentioned development, we have introduced Burr-X log logistic distribution with three parameters that can be handy to represent data in different fields. The developed model can be useful in various analysis techniques. The paper is structured as follows, Burr-X generator and log logistic distribution in Subsection 2.1 and Subsection 2.2 respectively. In Subsection 2.3, the Burr-X log logistic distribution is derived. In Subsection 2.4, we consider the hazard rate function for the new distribution. Expression to calculate moments of the Burr-X log logistic is given in Subsection 2.5. Probability weighted moments is evaluated in Subsection 2.6. The order statistics for the new distribution and Rényi entropy are discussed in Subsections 2.8 and 2.7, respectively. We use the method of maximum likelihood estimation in Subsection 2.9 to determine its parameters. A simulation study is given in Subsection 2.11. In Section 3, applicability of Burr-X log logistic distribution on a dataset is seen and better than some existing distributions. In our paper, we use following Lemmas.

Lemma 1 *We follow from Equation* (*1.110*) *on Page 25 of Gradshteyn and Ryzhik* (*2007*). *If* q *is a positive real non integer and* $|x| \le 1$, *then by binomial series expansion we have:*

$$(1-x)^{q-1} = \sum_{p=0}^{\infty} (-1)^p \binom{q-1}{p} x^p.$$

Lemma 2 *We follow from Equation* (*3.326.2*) *on Page 337 of Gradshteyn and Ryzhik* (*2007*). *For* $Re\ \beta > 0,\ Re\ a > 0,\ Re\ b > 0,$

$$\int_0^{\infty} x^a \exp\left(-\beta x^b\right) dx = \frac{\Gamma\left(\frac{a+1}{b}\right)}{b\beta^{\left(\frac{a+1}{b}\right)}}.$$

2. Model Definition and Estimation

2.1. Burr-X generator

Burr-X generator was proposed by Yousof et al. (2017) which can be applied to the data from various fields. The probability density function (pdf) $f(x)$ and cumulative distribution function (cdf) $F(x)$ of Burr-X generator are, respectively, given by

$$f(x) = \frac{2\theta g(x) G(x)}{\left(\bar{G}(x)\right)^3} \exp\left\{-\left(\frac{G(x)}{\bar{G}(x)}\right)^2\right\}\left[1-\exp\left\{-\left(\frac{G(x)}{\bar{G}(x)}\right)^2\right\}\right]^{\theta-1}, \ x>0, \theta>0, \tag{1}$$

and

$$F(x) = \left[1-\exp\left\{-\left(\frac{G(x)}{\bar{G}(x)}\right)^2\right\}\right]^{\theta}, \tag{2}$$

where θ is the shape parameter. The $g(x)$ and $G(x)$ denote the density and cumulative functions of considered baseline model and $\bar{G}(x) = 1 - G(x)$.

2.2. Log logistic distribution

The Log logistic distribution is used to study in different areas such as survival analysis, hydrology, networking and economy. For some uses of the log logistic distribution we can see to

Collet (2003), Shoukri et al. (1988), Bennett (1983) and Ashkar and Mahdi (2006). The distribution of a random variable X following log logistic distribution is given by:

$$g(x)=\frac{\beta}{\alpha}\frac{(x/\alpha)^{\beta-1}}{\left[1+(x/\alpha)^{\beta}\right]^{2}},\ x>0,\ \beta>0,\ \alpha>0. \tag{3}$$

$$G(x)=\frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}}, \tag{4}$$

where β is the shape parameter and α is the scale parameter.

2.3. Cdf and pdf of Burr-X log logistic distribution

For the Burr-X generator using the pdf and cdf defined in equations (1) and (2), respectively, we propose a new Burr-X log logistic distribution. Thus, the cdf of the Burr-X log logistic distribution can be defined as

$$F(x)=\left[1-\exp\left\{-\left(\frac{\frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}}}{1-\frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}}}\right)^{2}\right\}\right]^{\theta}=\left[1-\exp\left(-\frac{x^{2\beta}}{\alpha^{2\beta}}\right)\right]^{\theta}, \tag{5}$$

and its corresponding pdf is given by

$$f(x)=\frac{2\beta\theta}{\alpha^{2\beta}}x^{2\beta-1}\exp\left(-\frac{x^{2\beta}}{\alpha^{2\beta}}\right)\left\{1-\exp\left(-\frac{x^{2\beta}}{\alpha^{2\beta}}\right)\right\}^{\theta-1},\ x>0,\ \alpha>0,\ \theta>0,\ \beta>0. \tag{6}$$

Using Lemma 1,

$$f(x)=\frac{2\theta\beta}{\alpha^{2\beta}}\sum_{q=0}^{\infty}(-1)^{q}\binom{\theta-1}{q}x^{2\beta-1}\exp\left(-\frac{(q+1)x^{2\beta}}{\alpha^{2\beta}}\right). \tag{7}$$

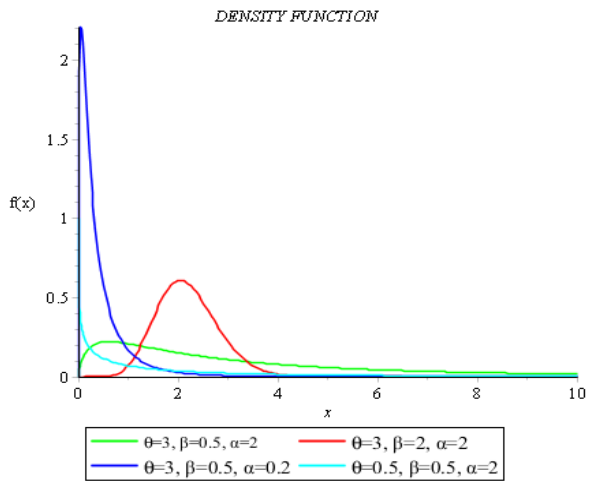


Figure 1 Probability density function of Burr-X log logistic distribution for various choices of its parameters θ, β and α.

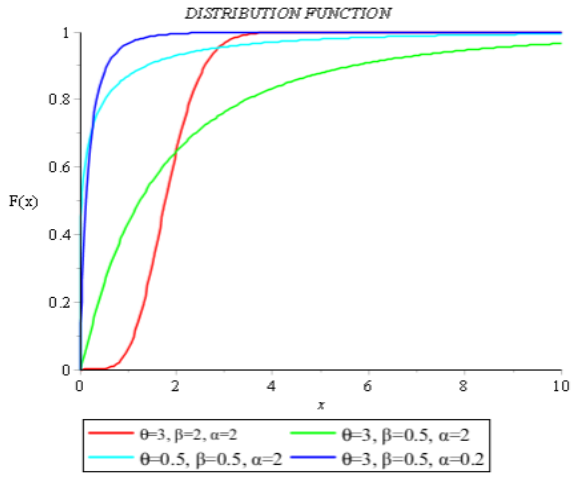


Figure 2. Distribution function of Burr-X log logistic distribution for various choices of its parameters θ, β and α.

2.4. Hazard rate function and survival function

Hazard rate function (hrf) defined by $h(x)=\frac{f(x)}{S(x)}$, is an important tool used to visualize life processes. According to the cdf and pdf given in equations (5) and (6), respectively, $h(x)$ and its survival function takes the form:

$$h(x)=\frac{\frac{2\beta\theta}{\alpha^{2\beta}}x^{2\beta-1}\exp\left(-\frac{x^{2\beta}}{\alpha^{2\beta}}\right).\left\{1-\exp\left(-\frac{x^{2\beta}}{\alpha^{2\beta}}\right)\right\}^{\theta-1}}{1-\left[1-\exp\left(-\frac{x^{2\beta}}{\alpha^{2\beta}}\right)\right]^{\theta}}, \tag{8}$$

and

$$S(x)=1-F(x)=1-\left[1-\exp\left(-\frac{x^{2\beta}}{\alpha^{2\beta}}\right)\right]^{\theta}. \tag{9}$$

2.5. Moments

The g^{th} moment (μ'_g) for a random variable (r.v.) X, having density function as defined in equation (7), can be calculated as:

$$\mu'_g=E\left(x^g\right)=\int_0^{\infty}x^g f(x)dx=\frac{2\beta\theta}{\alpha^{2\beta}}\sum_{l=0}^{\infty}(-1)^l\binom{\theta-1}{l}\int_0^{\infty}x^{g+2\beta-1}\exp\left(-\frac{(l+1)x^{2\beta}}{\alpha^{2\beta}}\right)dx.$$

Using Lemma 2,

$$\mu'_g=\frac{2\beta\theta}{\alpha^{2\beta}}\sum_{l=0}^{\infty}(-1)^l\binom{\theta-1}{l}\frac{\Gamma\left(\frac{g+2\beta}{2\beta}\right)}{2\beta\left(\frac{l+1}{\alpha^{2\beta}}\right)^{\left(\frac{g+2\beta}{2\beta}\right)}}, \tag{10}$$

where moments exists for $\left(\frac{g}{2\beta}+1\right)>0.$

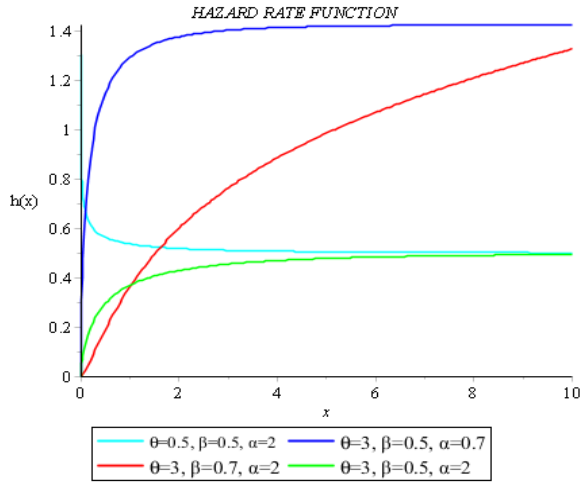


Figure 3. Hazard rate function of Burr-X log logistic distribution for various choices of its parameters θ, β and α.

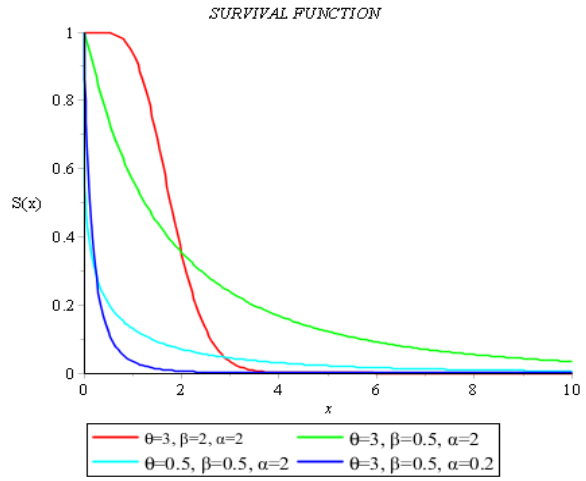


Figure 4. Survival function of Burr-X log logistic distribution for various choices of its parameters θ, β and α.

2.6. Probability weighted moment

With the density function defined in equation (6), then we can calculate its probability weighted moments (ρ) as:

$$\rho = E\left(x^k\left(F\left(x\right)\right)^m\right) = \int_0^\infty x^k\left(F\left(x\right)\right)^m f(x)dx,$$

$$= \frac{2\beta\theta}{\alpha^{2\beta}} \int_0^{\infty} x^{2\beta+k-1} \left[1-\exp\left(-\frac{x^{2\beta}}{\alpha^{2\beta}}\right)\right]^{\theta.m+\theta-1} \exp\left(-\frac{x^{2\beta}}{\alpha^{2\beta}}\right) dx$$

$$= \frac{2\beta\theta}{\alpha^{2\beta}} \sum_{q=0}^{\infty} (-1)^q \binom{\theta.m+\theta-1}{q} \int_0^{\infty} x^{2\beta+k-1} \exp\left(-\frac{(q+1)x^{2\beta}}{\alpha^{2\beta}}\right) dx.$$

Using Lemma 2,

$$\rho = \frac{2\beta\theta}{\alpha^{2\beta}} \sum_{q=0}^{\infty} (-1)^q \binom{\theta.m+\theta-1}{q} \frac{\Gamma\left(\frac{2\beta+k}{2\beta}\right)}{2\beta\left(\frac{q+1}{\alpha^{2\beta}}\right)^{\left(\frac{2\beta+k}{2\beta}\right)}}. \tag{11}$$

2.7. Rényi entropy

The deviation of the uncertainty (σ) is calculated by the entropy of a random variable. This entropy is defined by Rényi (1961), as:

$$E_X(\sigma) = \frac{1}{(1-\sigma)} \ln\left(\int_{-\infty}^{\infty} (f_X(x))^{\sigma} dx\right), \sigma > 0, \sigma \neq 1.$$

Using density function given in equation (6), we get,

$$E_X(\sigma) = \frac{1}{(1-\sigma)} \ln\left(\int_0^{\infty} \left(\frac{2\beta\theta}{\alpha^{2\beta}} x^{2\beta-1} \exp\left(-\frac{x^{2\beta}}{\alpha^{2\beta}}\right) \cdot \left\{1-\exp\left(-\frac{x^{2\beta}}{\alpha^{2\beta}}\right)\right\}^{\theta-1}\right)^{\sigma} dx\right),$$

$$= \frac{1}{(1-\sigma)} \ln\left(\left(\frac{2\beta\theta}{\alpha^{2\beta}}\right)^{\sigma} \int_0^{\infty} x^{(2\beta-1)\sigma} \exp\left(-\frac{x^{2\beta}}{\alpha^{2\beta}}\right)^{\sigma} \left\{1-\exp\left(-\frac{x^{2\beta}}{\alpha^{2\beta}}\right)\right\}^{\theta\sigma-\sigma} dx\right),$$

$$= \frac{1}{(1-\sigma)} \ln\left(\left(\frac{2\beta\theta}{\alpha^{2\beta}}\right)^{\sigma} \sum_{s=0}^{\infty} (-1)^s \binom{\theta\sigma-\sigma}{s} \int_0^{\infty} x^{(2\beta-1)\sigma} \exp\left(-\frac{(\sigma+s)x^{2\beta}}{\alpha^{2\beta}}\right) dx\right).$$

Using Lemma 2,

$$E_X(\sigma) = \frac{1}{(1-\sigma)} \ln\left(\left(\frac{2\beta\theta}{\alpha^{2\beta}}\right)^{\sigma} \sum_{s=0}^{\infty} (-1)^s \binom{\theta\sigma-\sigma}{s} \frac{\Gamma\left(\frac{2\beta\sigma-\sigma+1}{2\beta}\right)}{2\beta\left(\frac{\sigma+s}{\alpha^{2\beta}}\right)^{\left(\frac{2\beta\sigma-\sigma+1}{2\beta}\right)}}\right). \tag{12}$$

2.8. Order statistics

Let $X_1, X_2, ..., X_m$ be a sample data from Burr-X log logistic distribution with cdf defined in equation (5) and pdf in equation (6), respectively. Let $X_{(1:m)} \leq X_{(2:m)} \leq ... \leq X_{(m:m)}$ denote the order statistics for the considered sample. Here we find the probability density function $f_{r:m}(x)$, of $X_{r:m}$ and the t^{th} moments of $X_{r:m}$, $r = 1, 2, ..., m.$ The distribution, of the r^{th} order statistics of r.v. $X_{r:m}$, $r = 1, 2, ..., m$ as defined in, David (1981) is

$$f_{r:m}(x) = C_{r:m} \left[F(x;\alpha,\theta,\beta)\right]^{r-1} \left[1-F(x;\alpha,\theta,\beta)\right]^{m-r} f(x;\alpha,\theta,\beta), \ x > 0.$$

Using Lemma 1, we obtain

$$f_{r:m}(x) = C_{r:m}\sum_{s=0}^{\infty}(-1)^s\binom{m-r}{s}\left[F(x;\alpha,\theta,\beta)\right]^{r+s-1} f(x;\alpha,\theta,\beta)$$

$$= \frac{2\beta\theta}{\alpha^{2\beta}}C_{r:m}\sum_{s=0}^{\infty}(-1)^s\binom{m-r}{s}x^{2\beta-1}\exp\left(-\frac{x^{2\beta}}{\alpha^{2\beta}}\right)\left\{1-\exp\left(-\frac{x^{2\beta}}{\alpha^{2\beta}}\right)\right\}^{\theta r+\theta s-1}.$$

Using Lemma 1,

$$f_{r:m}(x) = \frac{2\beta\theta}{\alpha^{2\beta}}C_{r:m}\sum_{s=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{s+l}\binom{m-r}{s}\binom{\theta(r+s)-1}{l}x^{2\beta-1}\left\{\exp\left(-\frac{(l+1)x^{2\beta}}{\alpha^{2\beta}}\right)\right\}. \quad (13)$$

Expression for t^{th} moment above pdf, is

$$E\left(x^t\right) = \int_0^{\infty} x^t . f_{r:m}(x)dx$$

$$= \frac{2\beta\theta}{\alpha^{2\beta}}C_{r:m}\sum_{s=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{s+l}\binom{m-r}{s}\binom{\theta r+\theta s-1}{l}\int_0^{\infty} x^{2\beta+t-1}\left\{\exp\left(-\frac{(l+1)x^{2\beta}}{\alpha^{2\beta}}\right)\right\}dx$$

Using Lemma 2, we get

$$E\left(x^t\right) = \frac{2\beta\theta}{\alpha^{2\beta}}C_{r:m}\sum_{s=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{s+l}\binom{m-r}{s}\binom{\theta r+\theta s-1}{l}\frac{\Gamma\left(\frac{2\beta+t}{2\beta}\right)}{2\beta\left(\frac{l+1}{\alpha^{2\beta}}\right)^{\left(\frac{2\beta+t}{2\beta}\right)}}. \quad (14)$$

2.9. Maximum likelihood estimators

We may define the log-likelihood function of a random variable X that follows the Burr-X log logistic distribution as:

$$L(x;\alpha,\beta,\theta) = n\ln 2 + n\ln\beta + n\ln\theta - 2\beta n\ln\alpha + (2\beta-1)\sum_{i=1}^{n}\ln\left(x_i\right) - \sum_{i=1}^{n}\left(\frac{x_i^{2\beta}}{\alpha^{2\beta}}\right)$$

$$+(\theta-1)\sum_{i=1}^{n}\ln\left\{1-\exp\left(-\frac{x_i^{2\beta}}{\alpha^{2\beta}}\right)\right\}, \quad (15)$$

Thus, the normal equations are given as follows:

$$\frac{\partial L(x;\alpha,\beta,\theta)}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\ln\left\{1-\exp\left(-\frac{x_i^{2\beta}}{\alpha^{2\beta}}\right)\right\}, \quad (16)$$

$$\frac{\partial L(x;\alpha,\beta,\theta)}{\partial\beta} = \frac{n}{\beta} - 2n\ln\alpha + 2\sum_{i=1}^{n}\ln\left(x_i\right) + \frac{(\theta-1)}{\alpha^{2\beta}}\sum_{i=1}^{n}\frac{x_i^{2\beta}.\exp\left(-\frac{x_i^{2\beta}}{\alpha^{2\beta}}\right)\ln\left(\frac{x_i^2}{\alpha^2}\right)}{1-\exp\left(-\frac{x_i^{2\beta}}{\alpha^{2\beta}}\right)}$$

$$-\sum_{i=1}^{n}\left(\frac{x_i^{2\beta}}{\alpha^{2\beta}}\right)\ln\left(\frac{x_i^2}{\alpha^2}\right). \quad (17)$$

$$\frac{\partial L(x;\alpha,\beta,\theta)}{\partial\alpha} = \frac{2n\beta}{\alpha} + 2\beta\sum_{i=1}^{n}\left(\frac{x_i^{2\beta}}{\alpha^{2\beta+1}}\right) + \frac{2\beta(1-\theta)}{\alpha^{2\beta+1}}\sum_{i=1}^{n}\frac{x_i^{2\beta}.\exp\left(-\frac{x_i^{2\beta}}{\alpha^{2\beta}}\right)}{1-\exp\left(-\frac{x_i^{2\beta}}{\alpha^{2\beta}}\right)}. \quad (18)$$

We can find the approximate value of the parameters using method of maximum likelihood and solving above equations (16)-(18). To find the confidence interval, we use asymptotic normality results. Let the $\hat{\eta}=\left(\hat{\alpha},\hat{\beta},\hat{\theta}\right)'$, represents the MLE of $\eta=\left(\alpha,\beta,\theta\right)'$, then $\sqrt{n}\left(\hat{\eta}-\eta\right)\to N_3\left(0,I^{-1}\left(\eta\right)\right)$, where $I^{-1}\left(\eta\right)$ is the Fisher's information matrix given by

$$I\left(\eta\right)=-\frac{1}{n}\begin{bmatrix} E\left(\frac{\partial^2 L}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 L}{\partial\alpha\partial\beta}\right) & E\left(\frac{\partial^2 L}{\partial\alpha\partial\theta}\right) \\ E\left(\frac{\partial^2 L}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2 L}{\partial\beta^2}\right) & E\left(\frac{\partial^2 L}{\partial\beta\partial\theta}\right) \\ E\left(\frac{\partial^2 L}{\partial\theta\partial\alpha}\right) & E\left(\frac{\partial^2 L}{\partial\theta\partial\beta}\right) & E\left(\frac{\partial^2 L}{\partial\theta^2}\right) \end{bmatrix},$$

where

$$\frac{\partial^2 L(x;\alpha,\beta,\theta)}{\partial\theta^2}=-\frac{n}{\theta^2},\qquad \frac{\partial^2 L(x;\alpha,\beta,\theta)}{\partial\alpha\,\partial\theta}=-\sum_{i=1}^{n}\frac{2\beta {x_i}^{2\beta}\exp\left(-\frac{{x_i}^{2\beta}}{\alpha^{2\beta}}\right)}{\alpha^{2\beta+1}\left\{1-\exp\left(-\frac{{x_i}^{2\beta}}{\alpha^{2\beta}}\right)\right\}},$$

$$\frac{\partial^2 L(x;\alpha,\beta,\theta)}{\partial\beta\,\partial\theta}=-\sum_{i=1}^{n}\frac{{x_i}^{2\beta}\exp\left(-\frac{{x_i}^{2\beta}}{\alpha^{2\beta}}\right)}{\alpha^{2\beta}\left\{1-\exp\left(-\frac{{x_i}^{2\beta}}{\alpha^{2\beta}}\right)\right\}}\ln\left(\frac{{x_i}^2}{\alpha^2}\right),$$

$$\frac{\partial^2 L(x;\alpha,\beta,\theta)}{\partial\beta\,\partial\alpha}=-\frac{2n}{\alpha}+\sum_{i=1}^{n}\frac{2{x_i}^{2\beta}}{\alpha^{2\beta+1}}\left[1+\beta\ln\left(\frac{{x_i}^2}{\alpha^2}\right)\right]+(\theta-1)\sum_{i=1}^{n}\frac{2{x_i}^{2\beta}\exp\left(-\frac{{x_i}^{2\beta}}{\alpha^{2\beta}}\right)}{\alpha^{2\beta}\left\{1-\exp\left(-\frac{{x_i}^{2\beta}}{\alpha^{2\beta}}\right)\right\}^2}$$

$$\times\left[\left\{1-\exp\left(-\frac{{x_i}^{2\beta}}{\alpha^{2\beta}}\right)\right\}\left\{\frac{\beta {x_i}^{2\beta}}{\alpha^{2\beta}}\ln\left(\frac{{x_i}^2}{\alpha^2}\right)-1\right\}-\frac{\beta}{\alpha}\ln\left(\frac{{x_i}^2}{\alpha^2}\right)\left\{1-\exp\left(-\frac{{x_i}^{2\beta}}{\alpha^{2\beta}}\right)+\alpha^{2\beta}{x_i}^{2\beta}\exp\left(-\frac{{x_i}^{2\beta}}{\alpha^{2\beta}}\right)\right\}\right],$$

$$\frac{\partial^2 L(x;\alpha,\beta,\theta)}{\partial\alpha^2}=-\frac{2n\beta}{\alpha^2}-2\beta(2\beta+1)\sum_{i=1}^{n}\frac{{x_i}^{2\beta}}{\alpha^{2\beta+2}}+2\beta(1-\theta)\sum_{i=1}^{n}\frac{{x_i}^{2\beta}\exp\left(-\frac{{x_i}^{2\beta}}{\alpha^{2\beta}}\right)}{\alpha^{4\beta+2}\left\{1-\exp\left(-\frac{{x_i}^{2\beta}}{\alpha^{2\beta}}\right)\right\}^2}$$

$$\times\left[2\beta {x_i}^{2\beta}\left\{1-\exp\left(-\frac{{x_i}^{2\beta}}{\alpha^{2\beta}}\right)\right\}-\left\{(2\beta+1)\alpha^{2\beta}\left(1-\exp\left(-\frac{{x_i}^{2\beta}}{\alpha^{2\beta}}\right)\right)-2\beta {x_i}^{2\beta}\exp\left(-\frac{{x_i}^{2\beta}}{\alpha^{2\beta}}\right)\right\}\right],$$

$$\frac{\partial^2 L(x;\alpha,\beta,\theta)}{\partial\beta^2}=-\frac{n}{\beta^2}-\sum_{i=1}^{n}\frac{{x_i}^{2\beta}}{\alpha^{2\beta}}\ln^2\left(\frac{{x_i}^2}{\alpha^2}\right)+(\theta-1)\sum_{i=1}^{n}\ln\left(\frac{{x_i}^2}{\alpha^2}\right)\frac{{x_i}^{2\beta}\exp\left(-\frac{{x_i}^{2\beta}}{\alpha^{2\beta}}\right)}{\alpha^{2\beta}\left\{1-\exp\left(-\frac{{x_i}^{2\beta}}{\alpha^{2\beta}}\right)\right\}^2}$$

$$\times\left[\left\{1-\exp\left(-\frac{x_i^{2\beta}}{\alpha^{2\beta}}\right)\right\}\left\{2\ln x-\frac{x_i^{2\beta}}{\alpha^{2\beta}}\ln\left(\frac{x_i^2}{\alpha^2}\right)\right\}-\left\{2\ln\alpha.\left(1-\exp\left(-\frac{x_i^{2\beta}}{\alpha^{2\beta}}\right)\right)+\frac{x_i^{2\beta}}{\alpha^{2\beta}}\ln\left(\frac{x_i^2}{\alpha^2}\right)\exp\left(-\frac{x_i^{2\beta}}{\alpha^{2\beta}}\right)\right\}\right],$$

η being unknown, we can estimate $I^{-1}(\eta)$ by $I^{-1}(\hat{\eta})$ and applied it to find asymptotic confidence interval for parameters α, β and θ.

2.10. Quantile function

Using equation (5), the quantile function of proposed distribution is given by:

$$Q(x)=\left\{\ln\left\{\left(1-U^{1/\theta}\right)^{-\alpha^{2\beta}}\right\}\right\}^{1/2\beta}, \tag{19}$$

where U follows the uniform distribution in interval [0,1].

2.11. Simulation study

In this section, we assess the performance of the MLEs estimators for a finite sample of size n using a Monte Carlo simulation study. The simulation study based on the Burr-X log logistic distribution is carried out to examine the average biases (ABs), the mean square errors (MSEs), the root mean square errors (RMSEs), and maximum likelihood estimates (MLEs) for the model parameters β, θ and α. The simulation experiment was carried out using a variety of simulations with varying sample sizes and parameter values. To generate random samples for the Burr-X log logistic distribution, the quantile function is given in equation (19). The simulation study was repeated 1500 times, each with sample sizes n = 50, 100, ..., 1500, for the following parameter values β = 2.5, θ = 2.5 and α = 2.2. For each situation of simulated data, say, (estimates) for i = 1, 2, ..., 1500, the AB, RMSE, and MSE of the parameters were computed for β = 2.5, θ = 2.5 and α = 2.2. The MLE, AB, and RMSE values of the parameters β, θ and α are displayed from various sample sizes. Based on these findings, we conclude that the MLEs perform quite well in estimating the model parameters and that the estimates are fairly stable and are nearer to the true values for these sample sizes. Table 1 and Figure 5 show that as the sample size increases, the MSE, AB, and RMSE decrease as expected.

In addition, the MLEs of the parameters of the model are very close to the true value. As a result, the maximum likelihood estimates and their asymptotic results can be applied to construct confidence intervals for the model parameters even for a small sample size.

3. Results and Discussion

3.1. Application on the real life data

The projected Burr-X log logistic distribution is applied to fit an example. The results for the flexibility of distribution for the present data are analysed, using R software. We consider following distributions:

1) Burr-X log logistic distribution and log logistic distribution

$$f(x)=\frac{2\beta\theta}{\alpha^{2\beta}}x^{2\beta-1}\exp\left(-\frac{x^{2\beta}}{\alpha^{2\beta}}\right).\left\{1-\exp\left(-\frac{x^{2\beta}}{\alpha^{2\beta}}\right)\right\}^{\theta-1},\ x>0,\alpha>0,\theta>0,\beta>0$$

and

Table 1. Monte Carlo simulation results of MLE, AB and RMSE for the Burr-X log logistic distribution.

Parameters	n	MLE	AB	RMSE
β	50	2.855742	0.355741547	1.4449728
	100	2.664062	0.164062294	0.9075147
	300	2.543145	0.043144796	0.4161411
	600	2.513296	0.013296435	0.2816195
	900	2.504615	0.004614608	0.2308166
	1200	2.511855	0.011855128	0.2038188
	1500	2.506313	0.006312917	0.1805770
θ	50	19.296377	16.79637673	271.1875858
	100	3.743276	1.24327587	6.4203801
	300	2.707319	0.20731949	1.0790772
	600	2.606408	0.10640769	0.6681419
	900	2.579383	0.07938271	0.5331831
	1200	2.537264	0.03726407	0.4543218
	1500	2.532743	0.03274313	0.3825758
α	50	2.131726	-0.068274277	0.38370478
	100	2.164998	-0.035002164	0.25124685
	300	2.190265	-0.009735332	0.12663885
	600	2.193591	-0.006408579	0.08780186
	900	2.193567	-0.006432983	0.07282319
	1200	2.197924	-0.002075612	0.06432068
	1500	2.197496	-0.002504135	0.05596603

$$f(x)=\frac{b\left(\frac{x}{\alpha}\right)^{b-1}}{\alpha\left(1+\left(\frac{x}{\alpha}\right)^{b}\right)^{2}},\ x>0,\alpha>0,b>0.$$

2) Burr-X distribution and exponentiated exponential distribution

$$f(x)=2\theta p^{2}x.e^{-(p.x)^{2}}\left\{1-e^{-(p.x)^{2}}\right\}^{\theta-1},\ x>0,\ p>0,\theta>0.$$

and

$$f(x)=a\theta\left(1-e^{-\theta.x}\right)^{a-1}.e^{-\theta.x},\ x>0,a>0.$$

Let us assume the hypothesis at α = 5% level of significance (LOS),

H_0: The data fit to the Burr-X log logistic distribution

H_1: The data do not fit to the Burr-X log logistic distribution

Data Set 1: The data set taken from Efron (1988) represent the survival times of a collection of patients having Head and Neck cancer disease and radiation therapy (RT) treatment is used to cure them:

6.537, 10.42, 14.48, 16.10, 22.70, 34,41.55, 42,45.28, 49.40, 53.62, 63, 64, 83, 84, 91, 108, 112, 129, 133, 133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 146, 149, 154, 157, 160, 160, 165, 173, 176, 218, 225, 241, 248, 273, 277, 297, 405, 417, 420, 440, 523, 583, 594, 1101, 1146, 1417.

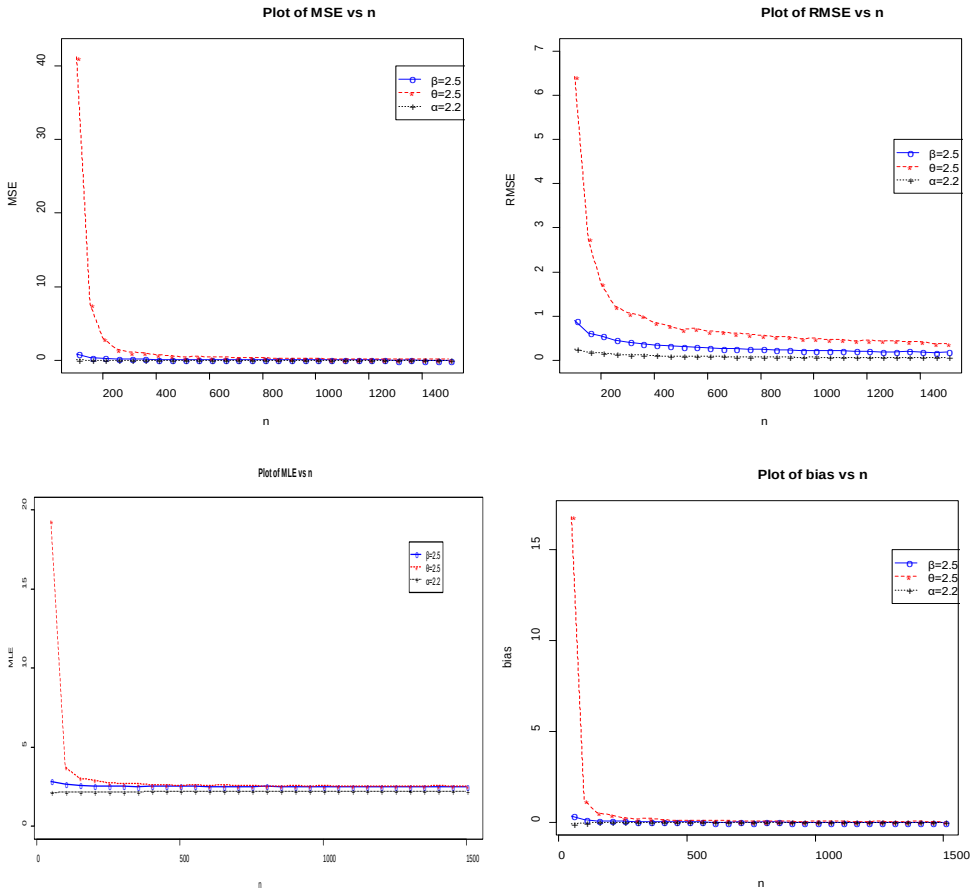


Figure 5. Plots for MLE, MSE, bias and RMSE for Burr-X log logistic distribution for parameter values $\beta = 2.5$, $\theta = 2.5$ and $\alpha = 2.2$.

Table 2. The ML estimate and AIC value of the Burr-X log logistic distribution parameters for data set 1.

Distributions	Estimates	Log-likelihood	D	p-value	AIC
Burr-X log logistic distribution	$\beta = 0.2010571$ $\theta = 17.3100263$ $\alpha = 6.2126303$	328.5121	0.12517	0.40132	663.02
Exponentiated exponential distribution	$\theta = 0.0039325$ $a = 0.9612724$	329.6353	0.15638	0.16502	663.26
Log logistic distribution	$\alpha = 41.3505392$ $b = 1.020789$	346.6794	0.39615	2.23e-07	697.34
Weibull distribution	$k = 0.09606526$ $\beta = 229.7656827$	366.9751	0.16649	0.08486	737.95
Burr-X distribution	$\theta = 1.84895516$ $p = 0.0051072$	370.9321	0.34427	1.12e-05	745.86

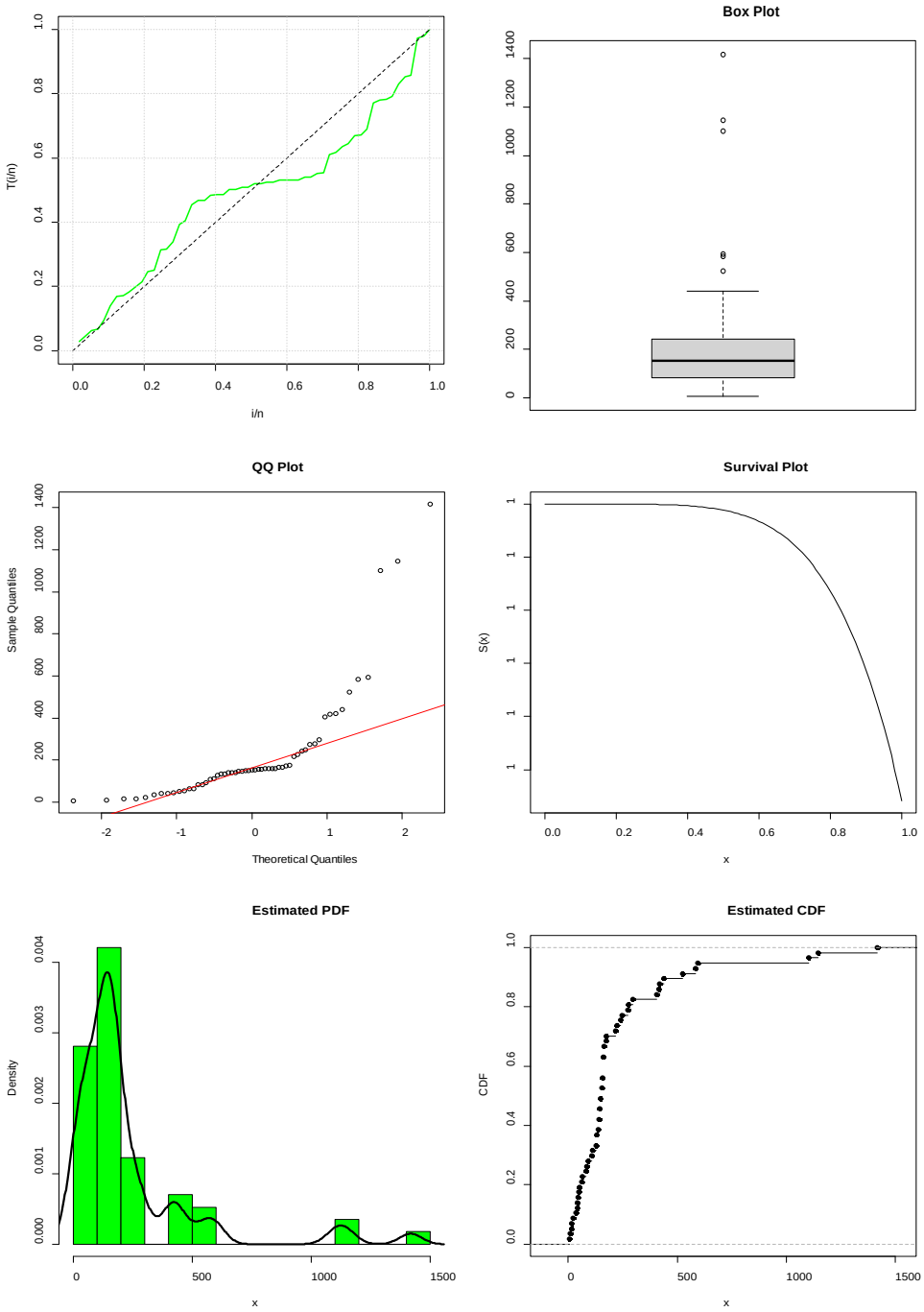


Figure 6. Plots of fitted Burr-X log logistic distribution for RT treatment data.

Plots of fitted Burr-X log logistic distribution, for RT treatment data are displayed in Figure 6. Box plot reveals that data is positively skewed. The Total Time on Test (TTT) plot in Figure 6 for data 1 has a convex shape which suggests a decreasing hazard shape. The empirical visualization suggests that the Burr-X log logistic distribution provides an improved fit for the data set 1.

Data Set 2: The second real data set represents the remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang (2003):

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Plots of fitted Burr-X log logistic distribution, for bladder cancer patient data are displayed in Figure 7. Box plot reveals that data is positively skewed. The TTT plot in Figure 7 for data set 2 has a concave shape then convex which suggests hrf is unimodal. The empirical visualization suggests that the Burr-X log logistic distribution provides an improved fit for the bladder cancer patient data.

Table 3 The ML estimate and AIC value of the Burr-X log logistic distribution parameters for data set 2.

Distributions	Estimates	Log-likelihood	D	p-value	AIC
Burr-X log logistic distribution	β = 0.3271428 θ = 3.3451843 α = 2.7953475	410.1775	0.034837	0.9977	826.3550
Log logistic distribution	α = 6.088111 b = 1.724475	411.4656	0.039983	0.9867	826.9312
Exponentiated exponential distribution	θ = 1.2171882 a = 0.1211278	413.0743	0.072451	0.5125	830.1486
Weibull distribution	k = 1.047734 β =9.5600007	414.0869	0.069985	0.5576	832.1738
Burr-X distribution	θ = 0.36401645 p = 0.04763012	429.2133	0.85684	< 2.2e-16	862.4266

Data Set 3: This data set has been taken from Nichols and Padgett (2006) on breaking stress of carbon fibres (in Gba):

3.56, 3.15, 3.70, 2.74, 2.73, 2.85, 2.56, 2.50, 3.60, 3.11, 3.27, 2.87, 2.17, 1.17, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.03, 1.80, 2.67, 2.93, 3.22, 3.39, 2.48, 1.18,2.81, 4.20, 3.33, 2.00, 1.22,2.55, 3.31, 3.31, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 1.61, 2.79, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 2.05, 3.65, 1.73, 1.59, 1.12, 1.71, 5.08, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 4.70, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82.

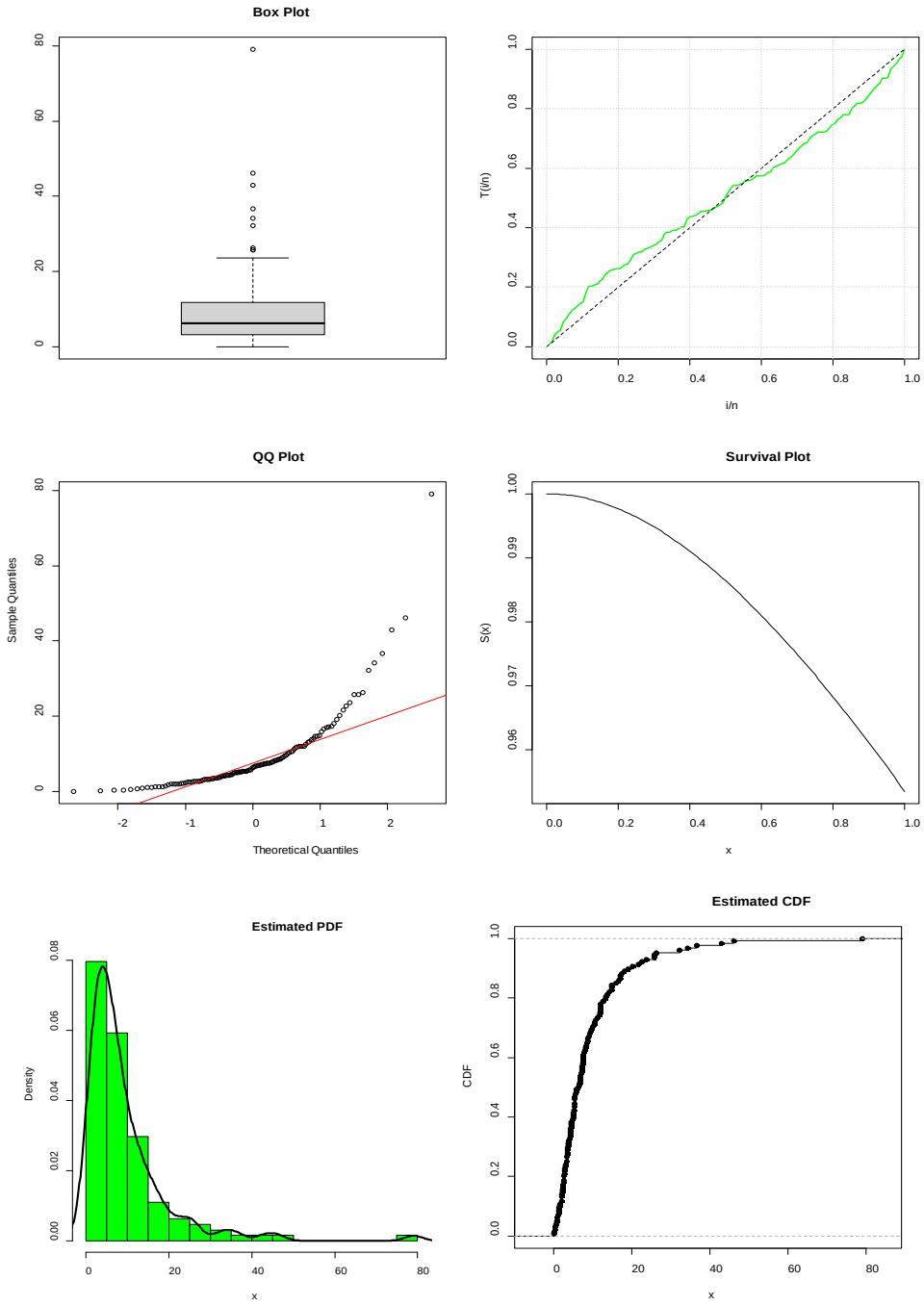


Figure 7. Plots of fitted Burr-X log logistic distribution for bladder cancer patients data.

Plots of fitted Burr-X log logistic distribution, for breaking stress of carbon fibres data are displayed in Figure 8. The box plot reveals that data is left-skewed. The TTT plot in Figure 8 for data 3 has a concave shape which suggests an increasing hazard shape. The empirical visualization suggests that the Burr-X log logistic distribution provides an improved fit for the data set 3.

Table 4. The ML estimate and AIC value of the Burr-X log logistic distribution parameters for data set 3.

Distributions	Estimates	Log-likelihood	D	p-value	AIC
Burr-X log logistic distribution	$\beta = 1.204560$ $\theta = 2.682411$ $\alpha = 1.316854$	141.3320	0.064427	0.8008	288.6640
Exponentiated Weibull distribution	$a = 1.316913$ $b = 2.409069$ $\lambda = 0.372806$	141.3350	0.064432	0.8008	288.6700
Burr-X distribution	$\theta = 1.8291496$ $p = 0.4253225$	142.5939	0.076934	0.5947	289.1878
Exponentiated exponential distribution	$\theta = 7.788574$ $a = 1.013191$	146.1823	0.10773	0.1961	296.3646
Log logistic distribution	$\alpha = 2.498361$ $b = 4.117942$	146.2795	0.09028	0.3889	296.5590

From Tables 2, 3 and 4, the Burr-X log logistic distribution has the maximum log-likelihood value and least AIC value as compared to other existing distributions. The AIC can be found using AIC = $-2\log_e L + 2k$, where $\log_e L$ denotes the log-likelihood function calculated at the maximum likelihood estimates, k is the number of parameters. Since p-value > α, we reject alternate hypothesis and accept that data fits Burr-X log logistic distribution for all three considered datasets.

3.2. Results obtained

The density function of order statistics of proposed distribution can be calculated by using equation (13) and its mean is given by equation (14). Expression for probability weighted moments is given in equation (11) and Rényi entropy in equation (12). Further we observe that the mean and probability weighted mean of the Burr-X log logistic distribution increases as θ increases or α decreases.

4. Conclusion

In above paper, the cdf and pdf of Burr-X log logistic distribution are introduced. The graphs depicts that the density function is unimodal and positively skewed with varying hazard rate function of derived distribution. The formula for its g^{th} moment of proposed distribution is provided. We have also calculated the expressions for Rényi entropy, Probability weighted moments and distribution for order statistics. To estimate model parameters, we applied the method of MLE. We obtain confidence interval of the proposed model using Fisher's information matrix. We executed simulation study to compare performance, of the estimators. The application of new distribution is justified using three data sets and observed that the Burr-X log logistic distribution exhibit consistently improved fit, than some existing distributions in literature. We hope that the proposed model may attract wider applications for modeling positive real data sets in many areas such as survival analysis, engineering, biology, hydrology, economics, and so on. As a future work, we can further extend this work to perform Bayesian estimation methods.

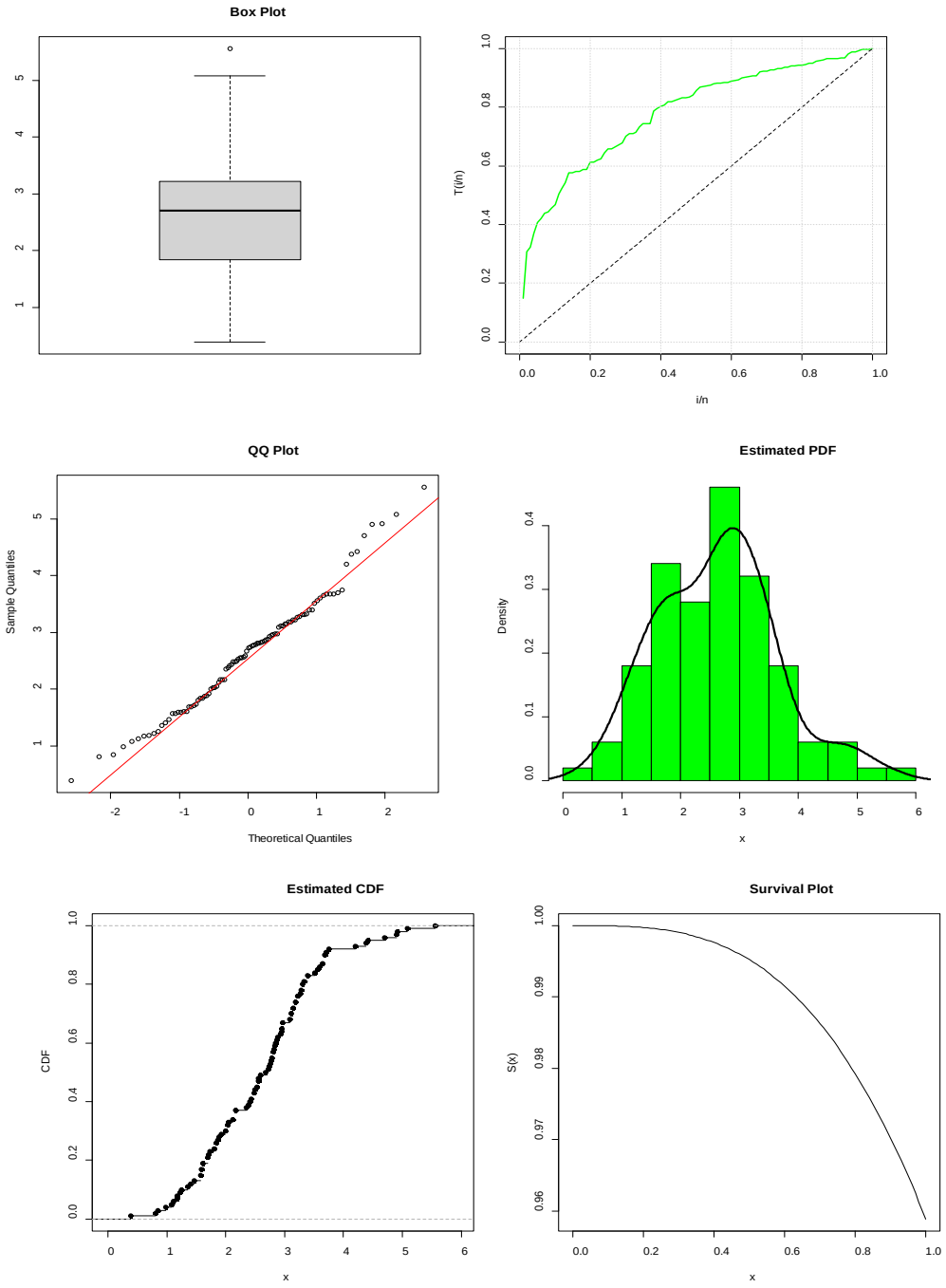


Figure 8. Plots of fitted Burr-X log logistic distribution for breaking stress of carbon fibres data.

Acknowledgments

We would like to thank the referees for their valuable comments and suggestions for the improvement of the manuscript.

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