Thailand Statistician
October 2026; 24(4): 783-802
http://statassoc.or.th
Contributed paper

Estimation Methods for the Modified Generalized Linear Exponential Distribution and Its Applications in Engineering Data

M.G.M. Ghazal*[a, b], M.A.W. Mahmoud [c] and H.M.M. Radwan [b]
[a] Department of Mathematics, Faculty of Science, Minia University, Minia, Egypt
[b] Department of Mathematics, University of Technology and Applied Sciences-Al Rustaq, Rustaq, Oman
[c] Mathematics Department, Faculty of Science, Al-Azhar University, Nasr city, Cairo, Egypt
*Corresponding author; e-mail: mohd.ibrahim1@mu.edu.eg

Received: 16 March 2023
Revised: 12 May 2026
Accepted: 16 June 2026

Abstract

This paper investigates different methods for estimating the unknown parameters of the modified generalized linear exponential distribution (MGLED), a flexible lifetime model that can accommodate bathtub-shaped, increasing, and decreasing hazard rate functions. The MGLED is of practical importance in reliability and engineering applications because it includes several well-known lifetime distributions as special cases, such as the exponential, Rayleigh, Weibull, modified Weibull, and generalized linear exponential distributions. Five classical estimation methods are considered, namely maximum likelihood estimation, least squares estimation, weighted least squares estimation, Cramér von-Mises estimation, and Anderson-Darling estimation. The performance of these estimation methods is assessed through two real engineering data sets and an extensive simulation study. For the real data applications, the fitted models are compared using non-parametric goodness-of-fit measures, including the Kolmogorov-Smirnov statistic with its corresponding P-value, the Anderson-Darling statistic, and the Cramér von-Mises statistic. In addition, parametric model selection criteria, including the Akaike information criterion, the corrected Akaike information criterion, and the Hannan-Quinn information criterion, are used to evaluate the adequacy of the MGLED relative to several competing lifetime models. The simulation study compares the proposed estimation methods in terms of mean squared error and average absolute bias. The numerical results show that the MGLED provides a better fit than the competing models for the considered engineering data sets, while the simulation results indicate that the maximum likelihood estimation and Anderson-Darling estimation methods generally perform better than the other estimation methods as the sample size increases.

Keywords: Modified generalized linear exponential distribution, hazard rate, simulation, Cramér von-Mises estimation, Anderson-Darling estimation.

1. Introduction

Statistical estimation is a fundamental concept in statistical inference that allows statisticians to estimate or predict population parameters based on sample information. This is critical since col-

lecting data from the entire population is often impractical or impossible. In statistical estimation, a statistician uses various methods, such as point estimation, interval estimation, and hypothesis testing, to estimate population parameters like the mean, variance, or proportion based on the available sample information. The accuracy of statistical estimation depends on several factors, including sample size, sampling method, and the level of variability in the population. Statistical estimation has numerous applications in various fields, including market research, epidemiology, finance, quality control, environmental science, education, and genetics. It is a powerful tool used in many different fields to make inferences about populations and to inform decision-making processes (see Nelson (1982) and Waller and Gotway (2004)).

Various classical estimation methods are utilized to estimate unknown parameters of known distributions. These methods include maximum likelihood estimators (MLEs), least-square estimators (LSEs), weighted least-square estimators (WLSEs), Cramér von-Mises estimators (CVMEs), and Anderson-Darling estimators (ADEs). Recently, many studies have investigated different methods of estimation for various parametric models. For instance, Afify et al. (2020) introduced a new heavy-tailed exponential distribution with diverse hazard rates and applied six estimation methods to estimate its parameters. They evaluated the performance of these estimators using Monte Carlo simulations. Similarly, Ibrahim et al. (2021) extended the Frchet model by introducing the Xgamma Frchet model and discussed the estimation of its parameters using various estimation methods. Kumar et al. (2021) examined the classical and Bayesian approaches for estimating the parameters and reliability characteristics of the inverse Pareto distribution, using different estimation methods. Ghazal et al. (2024) introduced an exponentiated extended extreme-value distribution characterized by various increasing or bathtub-shaped hazard rates. They examined four different methods for estimating the distribution's parameters. The performance of these estimators was assessed using average bias, mean squared errors, relative absolute biases, and root mean square error.

The modified generalized linear exponential distribution (MGLED) was introduced by Mahmoud et al. (2024) to offer a more adaptable lifetime distribution for analyzing real data sets in the most favorable circumstances. The MGLED's probability density function (PDF) and cumulative distribution function (CDF) are described by:

$$F(x;\underline{\Phi}) = 1-e^{-\left(\alpha\, x+\frac{\beta}{2}x^2\right)^{\xi}e^{\varphi\left(\alpha\, x+\frac{\beta}{2}x^2\right)}},\ \alpha\geq 0,\ \beta\geq 0,\ \xi>0,\ \varphi>0,\ x>0. \tag{1}$$

$$f(x;\underline{\Phi}) = (\alpha+\beta\, x)\left(\,\xi+\varphi\,(\alpha\, x+\frac{\beta}{2}x^2)\right)\left(\alpha\, x+\frac{\beta}{2}x^2\right)^{\xi-1} e^{\varphi\left(\alpha\, x+\frac{\beta}{2}x^2\right)} e^{-\left(\alpha\, x+\frac{\beta}{2}x^2\right)^{\xi}e^{\varphi\left(\alpha\, x+\frac{\beta}{2}x^2\right)}}, \tag{2}$$

where $\underline{\Phi}=(\alpha,\ \beta,\ \xi,\ \varphi)$.

Additionally, the survival and hazard rate functions of the MGLED are defined as follows:

$$S(x;\underline{\Phi}) = e^{-\left(\alpha\, x+\frac{\beta}{2}x^2\right)^{\xi}e^{\varphi\left(\alpha\, x+\frac{\beta}{2}x^2\right)}},\quad x>0. \tag{3}$$

$$h(x;\underline{\Phi}) = (\alpha+\beta\, x)\left(\,\xi+\varphi\,(\alpha\, x+\frac{\beta}{2}x^2)\right)\left(\alpha\, x+\frac{\beta}{2}x^2\right)^{\xi-1} e^{\varphi\left(\alpha\, x+\frac{\beta}{2}x^2\right)}. \tag{4}$$

The MGLED is a modification of the generalized linear exponential distribution, incorporating several key lifetime distributions used in survival analysis and reliability engineering, such as the Weibull, modified Weibull, linear exponential, and generalized linear exponential distributions. Mahmoud et al. (2024) examined the mathematical and statistical properties of the MGLED, including moments, moment generating function, quantiles, and modes. They also investigated the hazard rate, mean residual life, and variance residual life functions of the distribution. A notable feature of the MGLED is its ability to have a flexible bathtub-shaped curve (see Figure 1), which is useful for modeling reliability data.

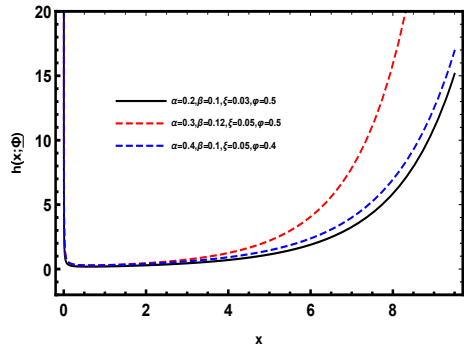


Figure 1 The hazard rate function of the MGLED with different values of parameters

This study considers five classical estimation methods for the unknown parameters of the MGLED, namely maximum likelihood estimation (MLE), least squares estimation (LSE), weighted least squares estimation (WLSE), Cramer von-Mises estimation (CVME), and Anderson-Darling estimation (ADE). The performance of these methods is assessed through two real engineering data applications and a simulation study. In the real data analysis, the fitted models are compared using the Kolmogorov-Smirnov (K-S) statistic and its corresponding P-value, together with the Anderson-Darling and Cramer von-Mises statistics. In addition, the Akaike information criterion (AIC), the corrected Akaike information criterion (AICc), and the Hannan-Quinn information criterion (HQIC) are used as model selection criteria. The simulation study evaluates the estimation methods in terms of mean squared errors (MSEs) and average absolute biases (AABs).

The main contribution of this paper is to provide a systematic comparison of several classical estimation methods for the MGLED under complete data. Unlike the original study of the MGLED, which focused mainly on its distributional properties and applications, the present work emphasizes the comparative performance of different estimation methods through both real engineering data and Monte Carlo simulation. This provides practical guidance on the choice of suitable estimation methods when applying the MGLED to reliability and lifetime data.

The remainder of this paper is organized as follows. Section 2 presents the different estimation methods considered for the MGLED parameters, including MLE, LSE, WLSE, CVME, and ADE. Section 3 provides the real data analysis and model comparison. Section 4 presents the simulation study. Finally, Section 5 gives the main conclusions and outlines possible directions for future research.

2. Different Methods of Estimation

In this section, five classical estimation methods are considered for estimating the unknown parameters of the modified generalized linear exponential distribution (MGLED): maximum likelihood estimation (MLE), least squares estimation (LSE), weighted least squares estimation (WLSE), Cramér von-Mises estimation (CVME), and Anderson-Darling estimation (ADE). The MLE is obtained by maximizing the likelihood function, whereas the LSE and WLSE are based on minimizing the squared distances between the theoretical distribution function and the empirical plotting positions. The CVME and ADE are minimum-distance estimation methods; the former measures the overall discrepancy between the fitted and empirical distribution functions, while the latter gives greater weight to discrepancies in the tails. Considering these methods provides a comprehensive comparison of the performance of different estimators for the MGLED parameters.

2.1. Maximum likelihood estimation

Assume that we have a random sample of size r, denoted by $x_1, ..., x_r$, drawn independently from the MGLED. The log-likelihood function can be derived from Eq. (2) as follows:

$$\begin{aligned}\ell(\underline{\Phi}) &= \sum_{i=1}^{r}\log\left(\alpha+\beta\, x_i\right)+\sum_{i=1}^{r}\log\Big(\,\xi+\varphi\left(\alpha\, x_i+\frac{\beta}{2}\, x_i^2\right)\Big)\\ &+(\xi-1)\sum_{i=1}^{r}\log\Big(\alpha\, x_i+\,\frac{\beta}{2}\, x_i^2\Big)\,-\sum_{i=1}^{r}\left(\alpha\, x_i+\frac{\beta}{2}\, x_i^2\right)^{\xi} e^{\varphi\,(\alpha\, x_i+\frac{\beta}{2}\, x_i^2)}.\\ &+\sum_{i=1}^{r}\varphi\left(\alpha\, x_i+\frac{\beta}{2}\, x_i^2\right). \qquad (5)\end{aligned}$$

By differentiating Eq. (5) with respect to α, β, ξ and φ, we get:

$$\begin{aligned}\ell_{\alpha}(\underline{\Phi}) &= \sum_{i=1}^{r}\frac{\varphi x_i}{\varphi\left(\alpha x_i+\frac{\beta x_i^2}{2}\right)+\xi}+(\xi-1)\sum_{i=1}^{r}\frac{x_i}{\alpha x_i+\frac{\beta x_i^2}{2}}+\sum_{i=1}^{r}\frac{1}{\alpha+\beta x_i}\\ &+\sum_{i=1}^{r}\varphi x_i-\sum_{i=1}^{r}\left(\xi x_i\left(\alpha x_i+\frac{\beta x_i^2}{2}\right)^{\xi-1} e^{\varphi\left(\alpha x_i+\frac{\beta x_i^2}{2}\right)}\right.\\ &\left.+\varphi x_i\left(\alpha x_i+\frac{\beta x_i^2}{2}\right)^{\xi} e^{\varphi\left(\alpha x_i+\frac{\beta x_i^2}{2}\right)}\right). \qquad (6)\\ \ell_{\beta}(\underline{\Phi}) &= \sum_{i=1}^{r}\frac{\varphi x_i^2}{2\left(\varphi\left(\alpha x_i+\frac{\beta x_i^2}{2}\right)+\xi\right)}+(\xi-1)\sum_{i=1}^{r}\frac{x_i^2}{2\left(\alpha x_i+\frac{\beta x_i^2}{2}\right)}+\sum_{i=1}^{r}\frac{x_i}{\alpha+\beta x_i}\\ &+\sum_{i=1}^{r}\frac{\varphi x_i^2}{2}-\sum_{i=1}^{r}\left(\frac{1}{2}\xi x_i^2\left(\alpha x_i+\frac{\beta x_i^2}{2}\right)^{\xi-1} e^{\varphi\left(\alpha x_i+\frac{\beta x_i^2}{2}\right)}\right.\\ &\left.+\frac{1}{2}\varphi x_i^2\left(\alpha x_i+\frac{\beta x_i^2}{2}\right)^{\xi} e^{\varphi\left(\alpha x_i+\frac{\beta x_i^2}{2}\right)}\right).\\ \ell_{\xi}(\underline{\Phi}) &= \sum_{i=1}^{r}\log\left(\alpha x_i+\frac{\beta x_i^2}{2}\right)+\sum_{i=1}^{r}\frac{1}{\varphi\left(\alpha x_i+\frac{\beta x_i^2}{2}\right)+\xi}\\ &-\sum_{i=1}^{r}\left(\alpha x_i+\frac{\beta x_i^2}{2}\right)^{\xi} e^{\varphi\left(\alpha x_i+\frac{\beta x_i^2}{2}\right)}\log\left(\alpha x_i+\frac{\beta x_i^2}{2}\right),\end{aligned}$$

and

$$\begin{aligned}\ell_{\varphi}(\underline{\Phi}) &= \sum_{i=1}^{r}\left(\alpha x_i+\frac{\beta x_i^2}{2}\right)+\sum_{i=1}^{r}\frac{\alpha x_i+\frac{\beta x_i^2}{2}}{\varphi\left(\alpha x_i+\frac{\beta x_i^2}{2}\right)+\xi}\\ &-\sum_{i=1}^{r}\left(\alpha x_i+\frac{\beta x_i^2}{2}\right)^{\xi+1} e^{\varphi\left(\alpha x_i+\frac{\beta x_i^2}{2}\right)}.\end{aligned}$$

The parameters $\alpha,\ \beta$, and φ are known:

The normal equation $\ell_{\xi}(\underline{\Phi})=0$ can be expressed as:

$$\frac{1}{\xi}=\frac{\sum_{i=1}^{r}\left(\alpha\, x_i+\frac{\beta}{2}\, x_i^2\right)^{\xi} e^{\varphi\left(\alpha\, x_i+\frac{\beta}{2}\, x_i^2\right)}\log\left(\alpha\, x_i+\frac{\beta}{2}\, x_i^2\right)-\sum_{i=1}^{r}\log\left(\alpha\, x_i+\frac{\beta}{2}\, x_i^2\right)}{\sum_{i=1}^{r}\frac{1}{\frac{\varphi\left(\alpha\, x_i+\frac{\beta}{2}\, x_i^2\right)}{\xi}+1}}. \qquad (7)$$

Equivalently, Eq. (7) can be written as

$$\frac{1}{\xi} = \Psi_1(\xi),$$

where

$$\Psi_1(\xi) = \frac{\sum_{i=1}^{r}\left(\alpha\, x_i + \frac{\beta}{2}\, x_i^2\right)^{\xi} e^{\varphi\left(\alpha\, x_i+\frac{\beta}{2}\, x_i^2\right)} \log\left(\alpha\, x_i + \frac{\beta}{2}\, x_i^2\right) - \sum_{i=1}^{r}\log\left(\alpha\, x_i + \frac{\beta}{2}\, x_i^2\right)}{\sum_{i=1}^{r}\frac{1}{\frac{\varphi\left(\alpha\, x_i+\frac{\beta}{2}\, x_i^2\right)}{\xi}+1}}. \quad (8)$$

It can be shown that $\Psi_1(\xi)$ is an increasing function of ξ. Since $1/\xi$ is a strictly decreasing function for $\xi > 0$, the graphical method can be used to identify the intersection point of the two functions. Hence, the MLE of ξ exists and is unique, as discussed by Balakrishnan and Kateri (2008).

The parameters $\alpha,\ \beta$, and ξ are known:

The equation $\ell_{\varphi}(\underline{\Phi}) = 0$ can be expressed as

$$\frac{1}{\varphi} = \frac{\sum_{i=1}^{r}\left(\alpha\, x_i + \frac{\beta}{2}\, x_i^2\right)^{\xi+1} e^{\varphi\left(\alpha\, x_i+\frac{\beta}{2}\, x_i^2\right)} - \sum_{i=1}^{r}\left(\alpha\, x_i + \frac{\beta}{2}\, x_i^2\right)}{\sum_{i=1}^{r}\frac{\alpha\, x_i+\frac{\beta}{2}\, x_i^2}{\left(\alpha\, x_i+\frac{\beta}{2}\, x_i^2\right)+\frac{\xi}{\varphi}}}. \quad (9)$$

Equivalently, Eq. (9) can be written as

$$\frac{1}{\varphi} = \Psi_2(\varphi),$$

where

$$\Psi_2(\varphi) = \frac{\sum_{i=1}^{r}\left(\alpha\, x_i + \frac{\beta}{2}\, x_i^2\right)^{\xi+1} e^{\varphi\left(\alpha\, x_i+\frac{\beta}{2}\, x_i^2\right)} - \sum_{i=1}^{r}\left(\alpha\, x_i + \frac{\beta}{2}\, x_i^2\right)}{\sum_{i=1}^{r}\frac{\alpha\, x_i+\frac{\beta}{2}\, x_i^2}{\left(\alpha\, x_i+\frac{\beta}{2}\, x_i^2\right)+\frac{\xi}{\varphi}}}. \quad (10)$$

It can be shown that $\Psi_2(\varphi)$ is an increasing function of φ. Since $1/\varphi$ is a strictly decreasing function for $\varphi > 0$, the graphical method can be used to identify the intersection point of the two functions. Therefore, the MLE of φ exists and is unique.

The parameters $\alpha,\ \beta,\ \xi$, and φ are known:

The MLE $\hat{\underline{\Phi}}$ of $\underline{\Phi} = (\alpha,\ \beta,\ \xi,\ \varphi)$ is determined by solving the four normal equations $\ell_{\alpha}(\underline{\Phi}) = 0$, $\ell_{\beta}(\underline{\Phi}) = 0$, $\ell_{\xi}(\underline{\Phi}) = 0$, and $\ell_{\varphi}(\underline{\Phi}) = 0$. These nonlinear equations cannot be solved analytically. Therefore, to obtain the solutions for the four nonlinear equations, numerical methods are used through employing `FindRoot` in Wolfram Mathematica 11 software. Initial parameter values can be obtained by fitting the sub-models of MGLED using `FindDistributionParameters` in the same software.

Since calculating the Fisher information matrix, which requires finding the expectation of the second derivative of Eq. (5), is quite challenging, it might be more practical to estimate these expected values using their MLEs. As a result, the asymptotic variance-covariance matrix can be represented as follows Cohen (1965):

$$\begin{pmatrix} Var(\hat{\alpha}) & Cov(\hat{\alpha},\hat{\beta}) & Cov(\hat{\alpha},\hat{\xi}) & Cov(\hat{\alpha},\hat{\varphi}) \\ Cov(\hat{\beta},\hat{\alpha}) & Var(\hat{\beta}) & Cov(\hat{\beta},\hat{\xi}) & Cov(\hat{\beta},\hat{\varphi}) \\ Cov(\hat{\xi},\hat{\alpha}) & Cov(\hat{\xi},\hat{\beta}) & Var(\hat{\xi}) & Cov(\hat{\varphi},\hat{\xi}) \\ Cov(\hat{\varphi},\hat{\alpha}) & Cov(\hat{\varphi},\hat{\beta}) & Cov(\hat{\varphi},\hat{\xi}) & Var(\hat{\varphi}) \end{pmatrix} =$$

$$\begin{pmatrix} -\ell_{\alpha\alpha}(\underline{\Phi}) & -\ell_{\alpha\beta}(\underline{\Phi}) & -\ell_{\alpha\xi}(\underline{\Phi}) & -\ell_{\alpha\varphi}(\underline{\Phi}) \\ -\ell_{\beta\alpha}(\underline{\Phi}) & -\ell_{\beta\beta}(\underline{\Phi}) & -\ell_{\beta\xi}(\underline{\Phi}) & -\ell_{\beta\varphi}(\underline{\Phi}) \\ -\ell_{\xi\alpha}(\underline{\Phi}) & -\ell_{\xi\beta}(\underline{\Phi}) & -\ell_{\xi\xi}(\underline{\Phi}) & -\ell_{\xi\varphi}(\underline{\Phi}) \\ -\ell_{\varphi\alpha}(\underline{\Phi}) & -\ell_{\varphi\beta}(\underline{\Phi}) & -\ell_{\varphi\xi}(\underline{\Phi}) & -\ell_{\varphi\varphi}(\underline{\Phi}) \end{pmatrix}^{-1}_{(\hat{\alpha},\hat{\beta},\hat{\xi},\hat{\varphi})}, \quad (11)$$

where $\ell_{\underline{\Phi}_i\underline{\Phi}_j}(\underline{\Phi}) = \frac{\partial^2 \ell}{\partial \underline{\Phi}_i \underline{\Phi}_j}$, $i, j = 1, 2, 3, 4$. The expressions for the second derivatives are available, but they are not shown here to save space. Thus, the approximate confidence interval for the parameters α, β, ξ, and φ based on the asymptotic variance-covariance matrix is given below:

$$\hat{\alpha} \pm z_{\frac{\tau}{2}} \sqrt{Var(\hat{\alpha})}, \hat{\beta} \pm z_{\frac{\tau}{2}} \sqrt{Var(\hat{\beta})}, \hat{\xi} \pm z_{\frac{\tau}{2}} \sqrt{Var(\hat{\xi})} \text{ and } \hat{\varphi} \pm z_{\frac{\tau}{2}} \sqrt{Var(\hat{\varphi})},$$

$z_{\frac{\tau}{2}}$ is the upper $\frac{\tau}{2}$ percentile of the standard normal distribution.

2.2. Least square and weighted least square estimations

The study by Swain et al. (1988) explored the application of LSEs and WLSEs for determining the parameters of the beta distribution. These techniques are also applicable for estimating the parameters of the MGLED. For this purpose, consider x_i, $i = 1, ..., r$ as the ordered values of a random sample of size r. The LSEs of the MGLED parameters can be found by minimizing the following function:

$$L1 = \sum_{i=1}^{r} \left(F(x_i; \underline{\Phi}) - \frac{i}{r+1} \right)^2. \tag{12}$$

The LSEs can be found by solving the non-linear equations related to the unknown parameters $\underline{\Phi} = (\alpha,\ \beta,\ \xi,\ \varphi)$. In other words, the LSEs are determined by minimizing the function with respect to $\underline{\Phi} = (\alpha,\ \beta,\ \xi,\ \varphi)$.

$$\frac{\partial L1}{\partial \alpha} = \sum_{i=1}^{r} 2 \left(1 - e^{-(\alpha\ x + \frac{\beta}{2} x_i^2)^{\xi}\ e^{\varphi(\alpha\ x_i + \frac{\beta}{2} x_i^2)}} - \frac{i}{r+1} \right) \delta_1(x_i; \underline{\Phi}) = 0,$$

$$\frac{\partial L1}{\partial \beta} = \sum_{i=1}^{r} 2 \left(1 - e^{-(\alpha\ x + \frac{\beta}{2} x_i^2)^{\xi}\ e^{\varphi(\alpha\ x_i + \frac{\beta}{2} x_i^2)}} - \frac{i}{r+1} \right) \delta_2(x_i; \underline{\Phi}) = 0,$$

$$\frac{\partial L1}{\partial \xi} = \sum_{i=1}^{r} 2 \left(1 - e^{-(\alpha\ x + \frac{\beta}{2} x_i^2)^{\xi}\ e^{\varphi(\alpha\ x_i + \frac{\beta}{2} x_i^2)}} - \frac{i}{r+1} \right) \delta_3(x_i; \underline{\Phi}) = 0,$$

and

$$\frac{\partial L1}{\partial \varphi} = \sum_{i=1}^{r} 2 \left(1 - e^{-(\alpha\ x + \frac{\beta}{2} x_i^2)^{\xi}\ e^{\varphi(\alpha\ x_i + \frac{\beta}{2} x_i^2)}} - \frac{i}{r+1} \right) \delta_4(x_i; \underline{\Phi}) = 0,$$

where

$$\begin{aligned}
\delta_1(x_i; \underline{\Phi}) &= \sum_{i=1}^{r} x_i\ e^{\varphi(\alpha\ x_i + \frac{\beta}{2} x_i^2) - (\alpha\ x + \frac{\beta}{2} x_i^2)^{\xi}\ e^{\varphi(\alpha\ x_i + \frac{\beta}{2} x_i^2)}} \\
&\quad \times (\alpha\ x_i + \frac{\beta}{2} x_i^2)^{\xi - 1} \Big(\xi + \varphi\ (\alpha\ x_i + \frac{\beta}{2} x_i^2) \Big), \qquad (13) \\
\delta_2(x_i; \underline{\Phi}) &= \sum_{i=1}^{r} \frac{1}{2}\ x_i^2 e^{\varphi(\alpha\ x_i + \frac{\beta}{2} x_i^2) - (\alpha\ x + \frac{\beta}{2} x_i^2)^{\xi}\ e^{\varphi(\alpha\ x_i + \frac{\beta}{2} x_i^2)}} \\
&\quad \times (\alpha\ x_i + \frac{\beta}{2} x_i^2)^{\xi - 1} \Big(\xi + \varphi\ (\alpha\ x_i + \frac{\beta}{2} x_i^2) \Big), \qquad (14) \\
\delta_3(x_i; \underline{\Phi}) &= \sum_{i=1}^{r} e^{\varphi(\alpha\ x_i + \frac{\beta}{2} x_i^2) - (\alpha\ x + \frac{\beta}{2} x_i^2)^{\xi}\ e^{\varphi(\alpha\ x_i + \frac{\beta}{2} x_i^2)}} \\
&\quad \times (\alpha\ x_i + \frac{\beta}{2} x_i^2)^{\xi}\ \log \Big[(\alpha\ x_i + \frac{\beta}{2} x_i^2) \Big], \qquad (15) \\
\text{and} \quad \delta_4(x_i; \underline{\Phi}) &= \sum_{i=1}^{r} e^{\varphi(\alpha\ x_i + \frac{\beta}{2} x_i^2) - (\alpha\ x + \frac{\beta}{2} x_i^2)^{\xi}\ e^{\varphi(\alpha\ x_i + \frac{\beta}{2} x_i^2)}}\ (\alpha\ x_i + \frac{\beta}{2} x_i^2)^{\xi + 1}. \qquad (16)
\end{aligned}$$

The MGLED parameters $\underline{\Phi} = (\alpha,\ \beta,\ \xi,\ \varphi)$ can be estimated using WLSEs by minimizing the following equation with respect to the parameters:

$$L2 = \sum_{i=1}^{r} \frac{(r+2)(r+1)^2}{(r-i+1)} \left(F(x_i) - \frac{i}{r+1} \right)^2.$$

The WLSEs can be computed by solving the subsequent nonlinear equations with regards to the unknown parameters $\underline{\Phi} = (\alpha,\ \beta,\ \xi,\ \varphi)$:

$$\frac{\partial L2}{\partial \alpha} = \sum_{i=1}^{r} 2\, \frac{(r+2)(r+1)^2}{(r-i+1)} \left(1 - e^{-(\alpha\, x + \frac{\beta}{2} x_i^2)^{\xi}\ e^{\varphi(\alpha\ x_i + \frac{\beta}{2} x_i^2)}} - \frac{i}{r+1} \right) \delta_1(x_i;\underline{\Phi}) = 0,$$

$$\frac{\partial L2}{\partial \beta} = \sum_{i=1}^{r} 2\, \frac{(r+2)(r+1)^2}{(r-i+1)} \left(1 - e^{-(\alpha\, x + \frac{\beta}{2} x_i^2)^{\xi}\ e^{\varphi(\alpha\ x_i + \frac{\beta}{2} x_i^2)}} - \frac{i}{r+1} \right) \delta_2(x_i;\underline{\Phi}) = 0,$$

$$\frac{\partial L2}{\partial \xi} = \sum_{i=1}^{r} 2\, \frac{(r+2)(r+1)^2}{(r-i+1)} \left(1 - e^{-(\alpha\, x + \frac{\beta}{2} x_i^2)^{\xi}\ e^{\varphi(\alpha\ x_i + \frac{\beta}{2} x_i^2)}} - \frac{i}{r+1} \right) \delta_3(x_i;\underline{\Phi}) = 0,$$

and

$$\frac{\partial L2}{\partial \varphi} = \sum_{i=1}^{r} 2\, \frac{(r+2)(r+1)^2}{(r-i+1)} \left(1 - e^{-(\alpha\, x + \frac{\beta}{2} x_i^2)^{\xi}\ e^{\varphi(\alpha\ x_i + \frac{\beta}{2} x_i^2)}} - \frac{i}{r+1} \right) \delta_4(x_i;\underline{\Phi}) = 0,$$

where $\delta_1(x_i;\underline{\Phi})$, $\delta_2(x_i;\underline{\Phi})$, $\delta_3(x_i;\underline{\Phi})$ and $\delta_4(x_i;\underline{\Phi})$ are given by (13), (14), (15), and (16) respectively.

2.3. Cramér von-Mises estimation

MacDonald (1971) demonstrated that the minimum distance estimator, CVMEs, has a smaller bias compared to other estimators in a previous study. To obtain the CVMEs of MGLED, the following function can be minimized:

$$W^2 = \frac{1}{12\, r} + \sum_{i=1}^{r} \left(F(x_i) - \frac{2\, i - 1}{2\, r} \right)^2.$$

The non-linear equations given below can be solved to obtain the CVMEs of the parameters:

$$\frac{\partial W^2}{\partial \alpha} = \sum_{i=1}^{r} 2 \left(1 - e^{-(\alpha\, x + \frac{\beta}{2} x_i^2)^{\xi}\ e^{\varphi(\alpha\ x_i + \frac{\beta}{2} x_i^2)}} - \frac{2\, i - 1}{2\, r} \right) \delta_1(x_i;\underline{\Phi}) = 0,$$

$$\frac{\partial W^2}{\partial \beta} = \sum_{i=1}^{r} 2 \left(1 - e^{-(\alpha\, x + \frac{\beta}{2} x_i^2)^{\xi}\ e^{\varphi(\alpha\ x_i + \frac{\beta}{2} x_i^2)}} - \frac{2\, i - 1}{2\, r} \right) \delta_2(x_i;\underline{\Phi}) = 0,$$

$$\frac{\partial W^2}{\partial \xi} = \sum_{i=1}^{r} 2 \left(1 - e^{-(\alpha\, x + \frac{\beta}{2} x_i^2)^{\xi}\ e^{\varphi(\alpha\ x_i + \frac{\beta}{2} x_i^2)}} - \frac{2\, i - 1}{2\, r} \right) \delta_3(x_i;\underline{\Phi}) = 0,$$

and

$$\frac{\partial W^2}{\partial \varphi} = \sum_{i=1}^{r} 2 \left(1 - e^{-(\alpha\, x + \frac{\beta}{2} x_i^2)^{\xi}\ e^{\varphi(\alpha\ x_i + \frac{\beta}{2} x_i^2)}} - \frac{2\, i - 1}{2\, r} \right) \delta_4(x_i;\underline{\Phi}) = 0,$$

where $\delta_1(x_i;\underline{\Phi})$, $\delta_2(x_i;\underline{\Phi})$, $\delta_3(x_i;\underline{\Phi})$ and $\delta_4(x_i;\underline{\Phi})$ are given by (13), (14), (15), and (16) respectively.

2.4. Anderson-Darling estimation

Boos (1981) examined the characteristics of ADEs in a separate investigation. Based on his findings, the ADEs of MGLED can be expressed by minimizing the subsequent function.

$$A^2 = -r - \sum_{i=1}^{r} \frac{(2\,i-1)}{r}\left(\log\Big(F(x_i)\Big) + \log\Big(1 - F(x_{r-i+1})\Big) \right).$$

The non-linear equations given below can be solved to obtain the ADEs of the parameters:

$$\frac{\partial A^2}{\partial \alpha} = -\sum_{i=1}^{r} \frac{(2\,i-1)}{r}\left(\frac{\delta_1(x_i;\underline{\Phi})}{1-e^{-(\alpha\,x+\frac{\beta}{2}x_i^2)^{\xi}\,e^{\varphi(\alpha\,x_i+\frac{\beta}{2}x_i^2)}}} - \frac{\delta_1(x_i;\underline{\Phi})}{e^{-(\alpha\,x+\frac{\beta}{2}x_i^2)^{\xi}\,e^{\varphi(\alpha\,x_i+\frac{\beta}{2}x_i^2)}}} \right) = 0,$$

$$\frac{\partial A^2}{\partial \beta} = -\sum_{i=1}^{r} \frac{(2\,i-1)}{r}\left(\frac{\delta_2(x_i;\underline{\Phi})}{1-e^{-(\alpha\,x+\frac{\beta}{2}x_i^2)^{\xi}\,e^{\varphi(\alpha\,x_i+\frac{\beta}{2}x_i^2)}}} - \frac{\delta_2(x_i;\underline{\Phi})}{e^{-(\alpha\,x+\frac{\beta}{2}x_i^2)^{\xi}\,e^{\varphi(\alpha\,x_i+\frac{\beta}{2}x_i^2)}}} \right) = 0,$$

$$\frac{\partial A^2}{\partial \xi} = -\sum_{i=1}^{r} \frac{(2\,i-1)}{r}\left(\frac{\delta_3(x_i;\underline{\Phi})}{1-e^{-(\alpha\,x+\frac{\beta}{2}x_i^2)^{\xi}\,e^{\varphi(\alpha\,x_i+\frac{\beta}{2}x_i^2)}}} - \frac{\delta_3(x_i;\underline{\Phi})}{e^{-(\alpha\,x+\frac{\beta}{2}x_i^2)^{\xi}\,e^{\varphi(\alpha\,x_i+\frac{\beta}{2}x_i^2)}}} \right) = 0,$$

and

$$\frac{\partial A^2}{\partial \varphi} = -\sum_{i=1}^{r} \frac{(2\,i-1)}{r}\left(\frac{\delta_4(x_i;\underline{\Phi})}{1-e^{-(\alpha\,x+\frac{\beta}{2}x_i^2)^{\xi}\,e^{\varphi(\alpha\,x_i+\frac{\beta}{2}x_i^2)}}} - \frac{\delta_4(x_i;\underline{\Phi})}{e^{-(\alpha\,x+\frac{\beta}{2}x_i^2)^{\xi}\,e^{\varphi(\alpha\,x_i+\frac{\beta}{2}x_i^2)}}} \right) = 0,$$

where $\delta_1(x_i;\underline{\Phi})$, $\delta_2(x_i;\underline{\Phi})$, $\delta_3(x_i;\underline{\Phi})$ and $\delta_4(x_i;\underline{\Phi})$ are given by (13), (14), (15), and (16) respectively.

3. Real Data Analysis

This section demonstrates the practical applicability and flexibility of the MGLED by analyzing two real engineering data sets. For each data set, the MGLED is compared with its sub-models, including the generalized linear exponential distribution (GLED) Mahmoud and Alam (2010), Weibull distribution (WD) Weibull (1951), linear failure rate distribution (LFRD) Bain (1974), Rayleigh distribution (RD) Bain (1974), and exponential distribution (ED) Bain (1974). In addition, several well-known competing distributions are considered to provide a broader assessment of the performance of the proposed model.

- The PDF of Marshall-Olkin GLED (M-O-GLED) which is given by Okasha and Kayid (2016)

$$f(x) = \frac{\xi\,\varphi\,e^{-(\alpha\,x+\frac{\beta}{2}x^2)^{\xi}}\,(\alpha\,x+\frac{\beta}{2}x^2)^{\xi-1}\,(\alpha+\beta\,x)}{\left(1-(1-\varphi)\,e^{-(\alpha\,x+\frac{\beta}{2}x^2)^{\xi}}\right)^2}.$$

- The PDF of Marshall-Olkin WD (M-O-WD) which is given by Ghitany et al. (2005)

$$f(x) = \frac{\alpha\,\xi\,\varphi\,e^{-(\alpha\,x)^{\xi}}\,(\alpha\,x)^{\xi-1}}{\left(1-(1-\varphi)\,e^{-(\alpha\,x)^{\xi}}\right)^2}.$$

- The PDF of the Transmuted linear exponential distribution (TLED), introduced by Tian et al. (2014), is written here using the reparameterization $\beta = \alpha$, $\theta = \beta$, and $\lambda = \varphi$ in order to maintain a consistent notation with the other competing models. Thus, its PDF is given by

$$f(x) = (\alpha+\beta x)e^{-\left(\alpha x+\frac{\beta}{2}x^2\right)}\left(1-\varphi+2\varphi e^{-\left(\alpha x+\frac{\beta}{2}x^2\right)}\right).$$

- The PDF of generalized exponential exponential distribution (GEED) which is given by Popović et al. (2016)

$$f(x) = \frac{\alpha\, e^{-\alpha\, x}}{\beta(1-e^{-\alpha\, x})\left(1-\frac{1}{\beta}\,\log(1-e^{-\alpha\, x})\right)^2}\,.$$

The scaled TTT transform plot is provided to determine the hazard rate shapes for the given data sets as

$$\underline{\Phi}_r(\tfrac{n}{r}) = \frac{\sum_{i=1}^{r} x_{i:r} + (r-n)\; x_{n:r}}{\sum_{i=1}^{r} x_i},$$

where $n = 1, ..., r$ and $x_{i:r}$ is the order statistics of the data Aarset (1987). Anderson-Darling test (A^*), Cramér von-Mises test (W^*) and K-S are used for non-parametric test statistic. These tests are defined as:

$$A^* = A^2\left(\frac{9}{4\,n^2} + \frac{3}{4\,n} + 1\right),$$
$$W^* = W^2\left(\frac{1}{2\,n} + 1\right),$$

where A^2 and W^2 are the same as in the previous section, and

$$K - S = \max\left(\tfrac{i}{r} - F(x_i), F(x_i) - \tfrac{i-1}{r}\right).$$

In addition, the Akaike information criterion (AIC), the corrected Akaike information criterion (AICc), and the Hannan-Quinn information criterion (HQIC) are used as parametric model selection criteria. These criteria are employed to compare the adequacy of the MGLED with the competing lifetime distributions. Smaller values of AIC, AICc, and HQIC indicate a better fit. AIC is defined as $AIC = -2\,\ell + 2\,k$, where ℓ is the log-likelihood function and k is the number of parameters. AICc is defined as $AICc = AIC + \frac{2\,k\,(k+1)}{r-k-1}$, and HQIC is defined as $HQIC = -2\,\ell + 2\,k\log\log r$. We use these criteria to evaluate and compare the performance of the MGLED with other well-known distributions, such as GLED, WD, LFRD, ED, RD, M-O-GLED, M-O-WD, TLED and GEED. We also use the scaled TTT transform plot to identify the shapes of hazard rate for given data sets. We perform all computations using the software Wolfram Mathematica 11.

3.1. The power transformer data

In this section, we consider a dataset of 52 transformers' lifetimes, which was previously analyzed by Zeng et al. (2016), obtained from the UK National Grid. The data are

1, 1, 2, 3, 4, 4, 6, 6, 7, 8, 8, 9, 10, 10, 13, 14, 14, 15, 15, 16, 19, 19, 20, 21, 22, 22, 22, 23,
24, 24, 25, 26, 26, 27, 27, 27, 28, 28, 29, 29, 30, 31, 34, 34, 35, 35, 37, 37, 39, 39, 48.

The TTT-plot of the transformer lifetimes shows that the hazard rate is increasing which is concave, as depicted in Figure 2. Table 1 summarizes the numerical values of non-parametric and parametric tests based on maximum likelihood estimates of all compared distributions for the transformer dataset. The results indicate that the MGLED provides a better fit for the dataset than other distributions. This is evident from its lower value on all non-parametric and parametric tests and higher P-value associated with the K-S test. Table 2 provides estimates of unknown parameters and the values of K-S with corresponding P-value based on different methods. According to Table 2, the LSE method provides the best performance for this data set based on the smallest K-S statistic and the largest corresponding P-value. Also, we present graphical representations of ξ and $\Psi_1(\xi)$ based on Eq. (7) in Figure 3 (a), and of φ and $\Psi_2(\varphi)$ based on Eq. (9) in Figure 3 (b) for the transformer dataset.

Table 1 The MLEs of the unknown parameters, along with their corresponding SE in parentheses, A^*, W^*, K-S with its corresponding P-value, AIC, $AICc$, and $HQIC$ values for various models, as applied to the power transformer data

Model	ML estimates	A^*	W^*	K-S	P-value	ℓ	AIC	AICc	HQIC
MGLED	α = 0.0143 (0.0088) β = 0.0004 (0.0012) ξ = 0.8983 (0.2638) φ = 1.2895 (2.5622)	0.2268	0.0362	0.0705	0.9583	-196.998	401.996	402.847	404.988
GLED	α = 0.0229 (0.0131) β = 0.0017 (0.0011) ξ = 1.2397 (0.2680)	0.6765	0.1139	0.1111	0.5423	-198.896	403.792	404.292	406.036
WD	α = 0.0441 (0.0038) ξ = 1.6916 (0.1979)	1.0276	0.1687	0.1319	0.3261	-200.74	405.479	405.724	406.975
LFRD	α = 0.0125 (0.0074) β = 0.0027 (0.0007)	0.7182	0.1343	0.1232	0.4089	-199.364	402.729	402.974	404.225
RD	α =0.0036 (0.0005)	1.5175	0.1664	0.1231	0.4099	-201.84	405.681	405.761	406.429
ED	α = 0.0489 (0.0068)	3.4747	0.6644	0.2014	0.0294	-208.916	419.831	419.911	420.579
M-O-GLED	α = 0.0460 (0.0379) β = 0.0026 (0.0028) ξ = 1.0126 (0.3243) φ = 3.2712 (3.4492)	0.4864	0.0746	0.0924	0.7663	-198.155	404.31	405.161	407.302
M-O-WD	α = 0.0793 (0.0367) ξ = 1.2399 (0.3223) φ =4.7157 (4.6229)	0.69	0.0983	0.097	0.7121	-199.231	404.462	404.962	406.706
TLED	α = 0.0200 (0.0143) β = 0.0027 (0.0008) φ =-0.3519 (0.3608)	0.6123	0.1076	0.1101	0.554	-198.956	403.912	404.412	406.156
GEED	α = 0.1303 (0.0172) β = 0.0754 (0.0339)	0.5014	0.0773	0.0964	0.7193	-200.238	404.476	404.72	405.972

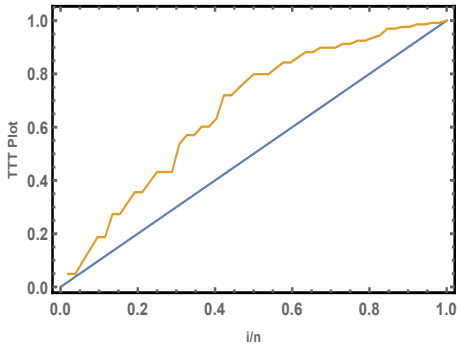
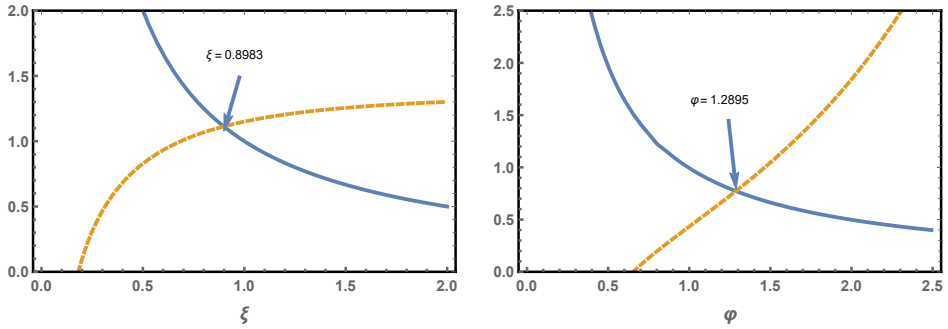


Figure 2 Scaled TTT-transform of the power transformer data

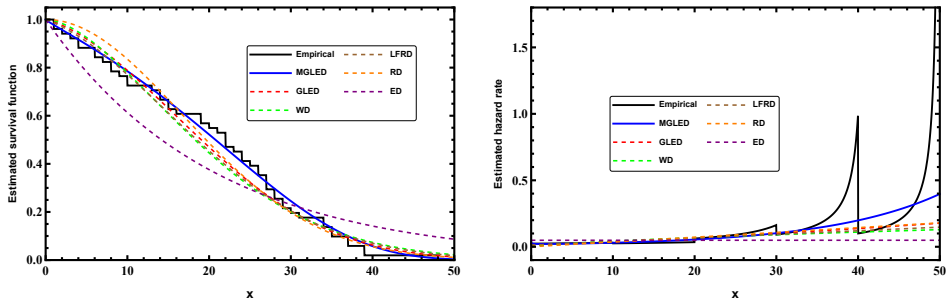
Table 2 The estimated values of unknown parameters using different methods along with the K-S test results and their corresponding P-values for the power transformer data

Methods	α	β	ξ	φ	K-S	P-value
MLEs	0.0143	0.0004	0.8983	1.2895	0.0705	0.9583
LSEs	0.0054	0.0005	0.5949	2.4244	0.0561	0.9967
WLSEs	0.0085	0.0005	0.7023	1.9255	0.0598	0.9924
ADEs	0.0098	0.0004	0.7503	1.8567	0.0644	0.9823



(a) **(b)**

Figure 3 (a) Plot of the $\frac{1}{\xi}$ and $\Psi_1(\xi)$ functions for the power transformer data (b) Plot of the $\frac{1}{\varphi}$ and $\Psi_2(\varphi)$ functions for the power transformer data.



(a) **(b)**

Figure 4 Estimated survival and hazard rate functions for the power transformer data based on the MGLED and its sub-models. Panel (a) displays the estimated survival functions, while panel (b) displays the corresponding estimated hazard rate functions

Figure 4 presents the estimated survival and hazard rate functions for the power transformer data using the MGLED and its sub-models, namely GLED, WD, LFRD, RD, and ED. As shown in panel (a), the fitted survival curve of the MGLED follows the empirical survival function closely over most of the observed data range. This indicates that the MGLED provides an adequate description of the survival behavior of the power transformer lifetimes. Panel (b) shows that the estimated hazard rate has an increasing pattern, which is consistent with the behavior suggested by the TTT plot for this data set. Compared with the simpler sub-models, the MGLED provides a more flexible fitted hazard rate curve and captures the increasing failure rate behavior more effectively.

3.2. The failure test data

The dataset from Lai et al. (2006) contains 50 observations of failure times for 50 items that were tested to failure. The failure times are

{0.12, 0.43, 0.92, 1.14, 1.24, 1.61, 1.93, 2.38, 4.51, 5.09, 6.79, 7.64, 8.45, 11.90, 11.94, 13.01, 13.25, 14.32, 17.47, 18.10, 18.66, 19.23, 24.39, 25.01, 26.41, 26.80, 27.75, 29.69, 29.84, 31.65, 32.64, 35.00, 40.70, 42.34, 43.05, 43.40, 44.36, 45.40, 48.14, 49.10, 49.44, 51.17, 58.62, 60.29, 72.13, 72.22, 72.25, 72.29, 85.20, 89.52}

The TTT-plot of the data shows that the hazard rate of the failure test data is bathtub-shaped, first convex and then concave, as depicted in Figure 5. Table 3 summarizes the numerical values of non-parametric and parametric tests based on maximum likelihood estimates of all comparison distributions for the failure test data. The results indicate that the MGLED distribution fits this data set better than other distributions, as it has the lowest value based on all non-parametric and parametric tests and the largest P-value associated with the K-S test. Additionally, Table 4 lists all estimates of the unknown parameters based on the different methods discussed in previous sections, along with the corresponding K-S values and P-values for the failure test data. According to the K-S test results, the ADE method is the optimal method for this data set.

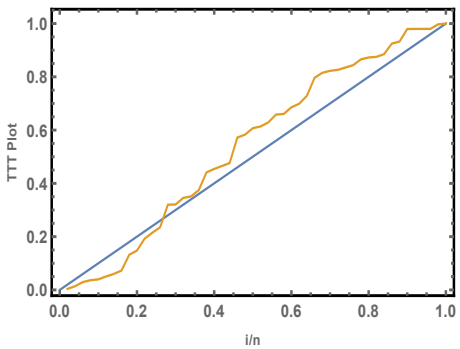


Figure 5 Scaled TTT-transform of the failure test data

Figure 7 (a) illustrates the graphical method for ξ and $\Psi_1(\xi)$ based on Eq. (7), while Figure 7 (b) presents the corresponding graphical method for φ and $\Psi_2(\varphi)$ based on Eq. (9) for the failure test data.

Figure 7 displays the estimated survival and hazard rate functions for the failure test data using the MGLED and its sub-models. In panel Figure 7 (a), the estimated survival function of the MGLED is close to the empirical survival function, particularly over the central part of the data range, which supports the suitability of the proposed model for these failure times. Panel Figure 7 (b) illustrates that the MGLED is able to capture a more flexible hazard rate pattern than the simpler sub-models. While some sub-models impose monotone or nearly constant hazard shapes, the MGLED provides a more adaptable hazard rate curve, which is useful for modeling engineering lifetime data with non-monotone failure behavior.

Table 3 The MLEs of the unknown parameters, along with their corresponding SE in parentheses, A^*, W^*, K-S with its corresponding P-value, AIC, $AICc$, and $HQIC$ values for various models, as applied to the failure test data

Model	ML estimates	A^*	W^*	K-S	P-value	ℓ	AIC	AICc	HQIC
MGLED	α = 0.0128 (0.0072) β = 0.0002 (0.0002) ξ = 0.6277 (0.1446) φ = 0.4940 (0.2396)	0.1775	0.0206	0.0562	0.9974	-216.184	440.368	441.257	443.281
GLED	α = 0.0159 (0.0093) b = 0.0009 (0.0006) ξ = 0.7569 (0.1464)	0.4438	0.0599	0.0839	0.8731	-217.918	441.836	442.357	444.02
WD	α = 0.0329 (0.0048) ξ = 1.0149 (0.1210)	0.9049	0.1281	0.1113	0.5654	-220.349	444.698	444.953	446.154
LFRD	α = 0.0246 (0.0064) β = 0.0003 (0.0002)	0.8568	0.0813	0.1023	0.6721	-219.05	442.1	442.355	443.556
RD	α =0.0013 (0.0002)	10.292	0.7594	0.2324	0.0090	-242.146	488.292	488.547	489.748
ED	α = 0.0331 (0.0047)	0.8992	0.1330	0.1143	0.5308	-220.357	444.713	444.969	446.17
M-O-GLED	α = 0.0374 (0.0455) β = 0.0016 (0.0019) ξ = 0.6484 (0.1950) φ = 2.5036 (2.8309)	0.3316	0.0400	0.0725	0.9553	-217.537	443.074	443.962	445.986
M-O-WD	α = 0.0839 (0.0656) ξ = 0.7525 (0.199) φ =4.4003 (4.4146)	0.5769	0.0649	0.0870	0.8436	-219.033	444.066	444.587	446.25
TLED	α = 0.0239 (0.0102) β = 0.0003 (0.0002) φ =0.0315 (0.4153)	0.8521	0.0815	0.1020	0.6756	-219.047	446.094	446.983	449.006
GEED	α = 0.0544 (0.0085) β = 0.2758 (0.1008)	0.3037	0.0773325	0.0649	0.9844	-219.735	443.469	443.724	444.925

Table 4 The estimated values of unknown parameters using different methods along with the K-S test results and their corresponding P-values for the failure test data

Methods	α	β	ξ	φ	K-S	P-value
MLEs	0.0128	0.00023	0.6277	0.4940	0.0562	0.9974
LSEs	0.0114	0.00025	0.5724	0.5247	0.0581	0.9959
WLSEs	0.0089	0.00066	0.4568	0.2017	0.0594	0.9945
CVMEs	0.0116	0.00026	0.5926	0.5216	0.0590	0.9949
ADEs	0.0129	0.00026	0.6125	0.4351	0.0550	0.9982

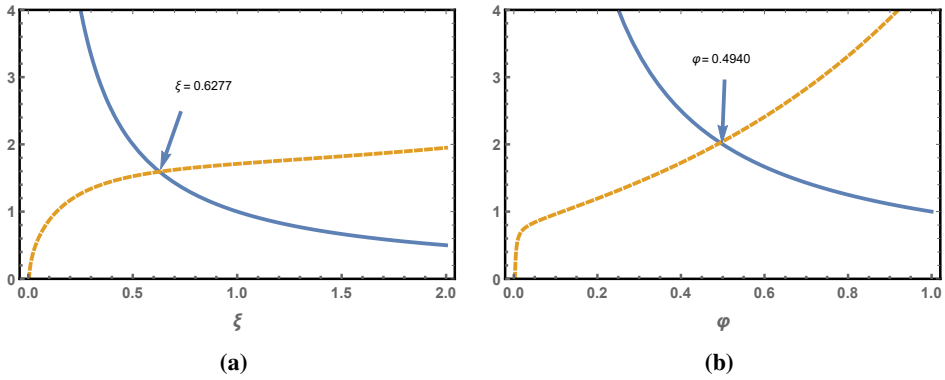


Figure 6 (a) Plot of the $\frac{1}{\xi}$ and $\Psi_1(\xi)$ functions for the failure test data (b) Plot of the $\frac{1}{\varphi}$ and $\Psi_2(\varphi)$ functions for the failure test data

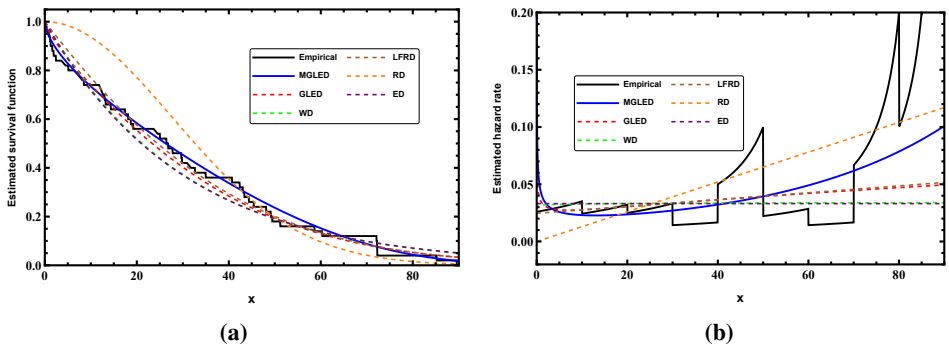


Figure 7 Estimated survival and hazard rate functions for the failure test data based on the MGLED and its sub-models. Panel (a) displays the estimated survival functions, while panel (b) displays the corresponding estimated hazard rate functions

4. Simulation Study

The purpose of the simulation study is to evaluate the finite-sample performance of the five estimation methods considered for the MGLED parameters. Two parameter settings are examined, namely $(\alpha, \beta, \xi, \varphi) = (1, 1, 1, 0.5)$ and $(\alpha, \beta, \xi, \varphi) = (1, 1, 2, 0.5)$. For each parameter setting, random samples are generated from the MGLED for different sample sizes. The unknown parameters are then estimated using the methods described in Section 2, namely MLE, LSE, WLSE, CVME, and ADE. This procedure is repeated 1000 times, and the performance of the estimators is assessed using mean squared errors (MSEs) and average absolute biases (AABs). The numerical results are reported in Tables 5–8, while Figures 8–11 provide graphical summaries of these results.

Table 5 MSEs and AAB for the parameter α under different proposed methods, as the sample size n varies

n	$(\alpha, \beta, \xi, \varphi)$		MLEs	LSEs	WLSEs	CVMEs	ADEs
20	$(1, 1, 1, 0.5)$	MSEs	0.3210	0.4092	0.4202	0.4079	0.3784
		AABs	0.4839	0.5667	0.5756	0.5653	0.5323
	$(1, 1, 2, 0.5)$	MSEs	0.1900	0.2541	0.2390	0.2297	0.2162
		AABs	0.3698	0.4379	0.4164	0.4133	0.3971
50	$(1, 1, 1, 0.5)$	MSEs	0.2021	0.2867	0.2995	0.2881	0.2351
		AABs	0.3692	0.4569	0.4654	0.4585	0.4023
	$(1, 1, 2, 0.5)$	MSEs	0.1312	0.1916	0.1740	0.1790	0.1574
		AABs	0.2913	0.3623	0.3389	0.3499	0.3231
80	$(1, 1, 1, 0.5)$	MSEs	0.1396	0.2372	0.2246	0.2353	0.1793
		AABs	0.3014	0.4044	0.3850	0.4018	0.3448
	$(1, 1, 2, 0.5)$	MSEs	0.1134	0.1669	0.1594	0.1591	0.1368
		AABs	0.2680	0.3338	0.3238	0.3255	0.2995
100	$(1, 1, 1, 0.5)$	MSEs	0.1154	0.2018	0.1896	0.2030	0.1453
		AABs	0.2755	0.3673	0.3514	0.3679	0.3096
	$(1, 1, 2, 0.5)$	MSEs	0.0960	0.1513	0.1508	0.1453	0.1194
		AABs	0.2396	0.3098	0.3049	0.3028	0.2704

Table 6 MSEs and AAB for the parameter β under different proposed methods, as the sample size n varies

n	$(\alpha, \beta, \xi, \varphi)$		MLEs	LSEs	WLSEs	CVMEs	ADEs
20	$(1, 1, 1, 0.5)$	MSEs	0.4812	0.6341	0.6511	0.6227	0.5892
		AABs	0.6015	0.7374	0.7509	0.7283	0.6940
	$(1, 1, 2, 0.5)$	MSEs	0.7045	0.7696	0.7477	0.7923	0.7512
		AABs	0.7735	0.8350	0.8161	0.8453	0.8111
50	$(1, 1, 1, 0.5)$	MSEs	0.4062	0.5496	0.6031	0.5297	0.5270
		AABs	0.5278	0.6556	0.6970	0.6365	0.6309
	$(1, 1, 2, 0.5)$	MSEs	0.6365	0.7596	0.7174	0.7627	0.7500
		AABs	0.7202	0.8238	0.7892	0.8213	0.8130
80	$(1, 1, 1, 0.5)$	MSEs	0.4049	0.5402	0.5613	0.5102	0.5109
		AABs	0.5173	0.6457	0.6592	0.6194	0.6144
	$(1, 1, 2, 0.5)$	MSEs	0.6074	0.7282	0.7068	0.7286	0.7070
		AABs	0.6995	0.7959	0.7766	0.7960	0.7808
100	$(1, 1, 1, 0.5)$	MSEs	0.4021	0.5097	0.5482	0.4820	0.4848
		AABs	0.5153	0.6159	0.6398	0.5918	0.5895
	$(1, 1, 2, 0.5)$	MSEs	0.6066	0.7102	0.7069	0.7128	0.7162
		AABs	0.7026	0.7782	0.7787	0.7809	0.7870

Table 7 MSEs and AAB for the parameter ξ under different proposed methods, as the sample size n varies

n	$(\alpha, \beta, \xi, \varphi)$		MLEs	LSEs	WLSEs	CVMEs	ADEs
20	$(1, 1, 1, 0.5)$	MSEs	0.1609	0.1590	0.1650	0.1824	0.1537
		AABs	0.3181	0.3294	0.3348	0.3476	0.3208
	$(1, 1, 2, 0.5)$	MSEs	0.1646	0.2989	0.2552	0.2083	0.2243
		AABs	0.2444	0.3618	0.3185	0.2734	0.3008
50	$(1, 1, 1, 0.5)$	MSEs	0.0770	0.1025	0.1212	0.1089	0.0826
		AABs	0.2180	0.2600	0.2843	0.2631	0.2288
	$(1, 1, 2, 0.5)$	MSEs	0.1302	0.2262	0.2091	0.1846	0.1725
		AABs	0.2163	0.3175	0.2956	0.2738	0.2655
80	$(1, 1, 1, 0.5)$	MSEs	0.0476	0.0789	0.0856	0.0821	0.0574
		AABs	0.1706	0.2262	0.2348	0.2286	0.1896
	$(1, 1, 2, 0.5)$	MSEs	0.1167	0.1947	0.1939	0.1702	0.1495
		AABs	0.2154	0.2904	0.2900	0.2619	0.2488
100	$(1, 1, 1, 0.5)$	MSEs	0.0385	0.0623	0.0721	0.0644	0.0454
		AABs	0.1556	0.2013	0.2117	0.2035	0.1714
	$(1, 1, 2, 0.5)$	MSEs	0.1013	0.1764	0.1840	0.1573	0.1326
		AABs	0.1893	0.2723	0.2780	0.2490	0.2274

Table 8 MSEs and AABs for the parameter φ under different proposed methods, as the sample size n varies

n	$(\alpha, \beta, \xi, \varphi)$		MLEs	LSEs	WLSEs	CVMEs	ADEs
20	$(1, 1, 1, 0.5)$	MSEs	0.1381	0.1559	0.1523	0.1565	0.1460
		AABs	0.3347	0.3736	0.3618	0.3676	0.3515
	$(1, 1, 2, 0.5)$	MSEs	0.1606	0.1682	0.1598	0.1653	0.1548
		AABs	0.3677	0.3862	0.3743	0.3772	0.3631
50	$(1, 1, 1, 0.5)$	MSEs	0.1075	0.1322	0.1266	0.1286	0.1134
		AABs	0.2822	0.3287	0.3198	0.3229	0.2979
	$(1, 1, 2, 0.5)$	MSEs	0.1417	0.1495	0.1491	0.1472	0.1371
		AABs	0.3429	0.3552	0.3580	0.3488	0.3374
80	$(1, 1, 1, 0.5)$	MSEs	0.0898	0.1178	0.1143	0.1180	0.1010
		AABs	0.2518	0.3043	0.3002	0.3047	0.2765
	$(1, 1, 2, 0.5)$	MSEs	0.1385	0.1393	0.1462	0.1387	0.1304
		AABs	0.3379	0.3391	0.3524	0.3376	0.3264
100	$(1, 1, 1, 0.5)$	MSEs	0.0845	0.1147	0.1068	0.1136	0.0944
		AABs	0.2423	0.2945	0.2852	0.2927	0.2622
	$(1, 1, 2, 0.5)$	MSEs	0.1327	0.1409	0.1440	0.1384	0.1311
		AABs	0.3314	0.3439	0.3522	0.3396	0.3294

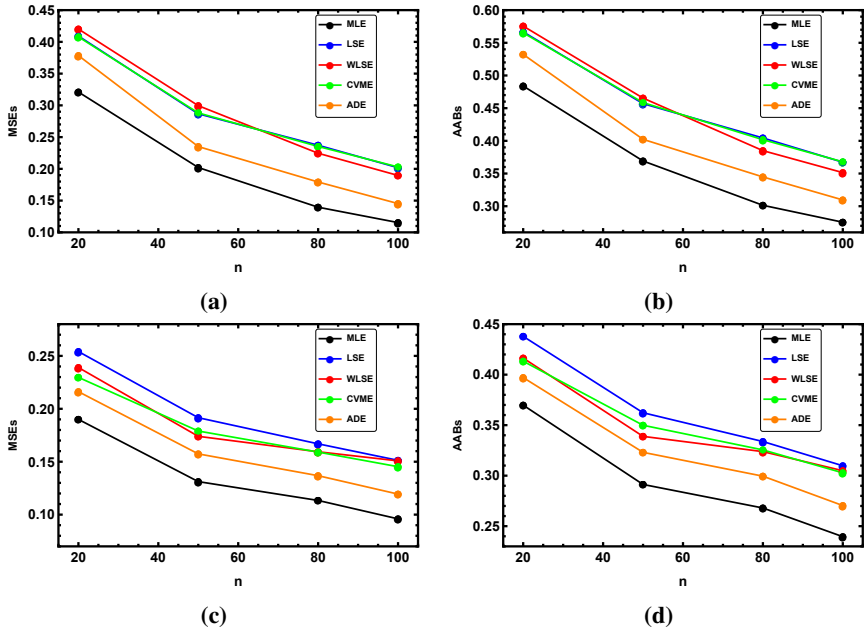


Figure 8 Graphical comparison of the MSEs and AABs for estimating the parameter α using different estimation methods under two parameter settings. Panels (a) and (b) correspond to MSEs and AABs for $(\alpha, \beta, \xi, \varphi) = (1, 1, 1, 0.5)$, respectively, while panels (c) and (d) correspond to MSEs and AABs for $(\alpha, \beta, \xi, \varphi) = (1, 1, 2, 0.5)$, respectively

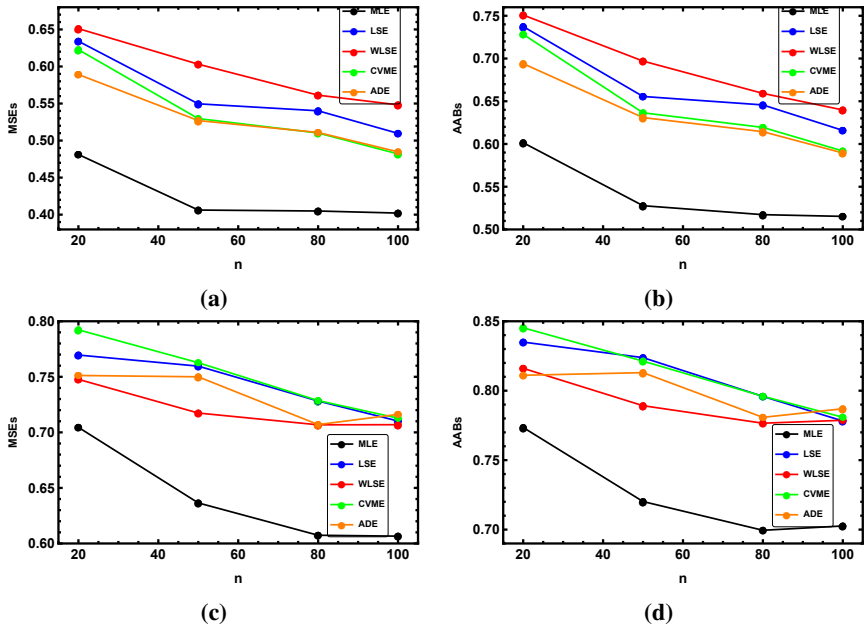


Figure 9 Graphical comparison of the MSEs and AABs for estimating the parameter β using different estimation methods under two parameter settings. Panels (a) and (b) correspond to MSEs and AABs for $(\alpha, \beta, \xi, \varphi) = (1, 1, 1, 0.5)$, respectively, while panels (c) and (d) correspond to MSEs and AABs for $(\alpha, \beta, \xi, \varphi) = (1, 1, 2, 0.5)$, respectively

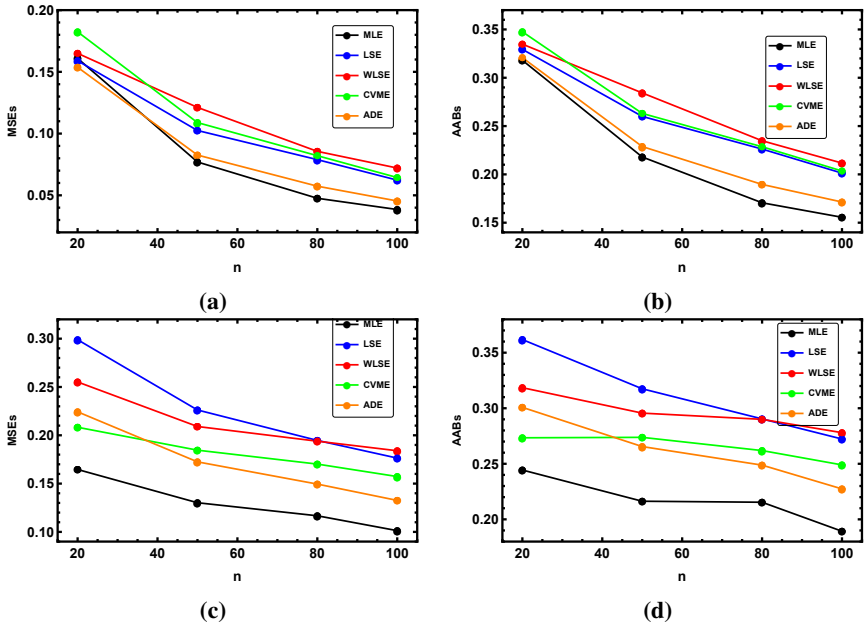


Figure 10 Graphical comparison of the MSEs and AABs for estimating the parameter ξ using different estimation methods under two parameter settings. Panels (a) and (b) correspond to MSEs and AABs for $(\alpha, \beta, \xi, \varphi) = (1, 1, 1, 0.5)$, respectively, while panels (c) and (d) correspond to MSEs and AABs for $(\alpha, \beta, \xi, \varphi) = (1, 1, 2, 0.5)$, respectively

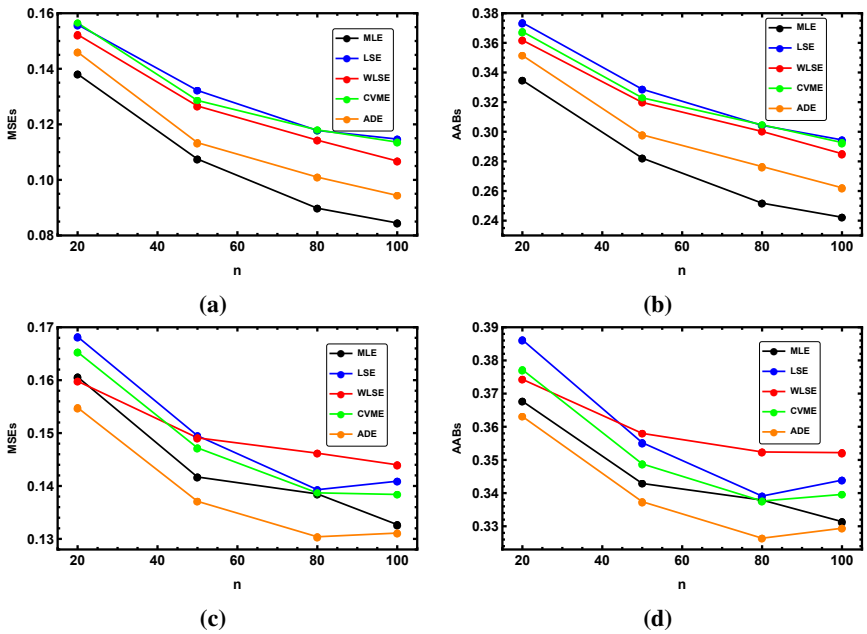


Figure 11 Graphical comparison of the MSEs and AABs for estimating the parameter φ using different estimation methods under two parameter settings. Panels (a) and (b) correspond to MSEs and AABs for $(\alpha, \beta, \xi, \varphi) = (1, 1, 1, 0.5)$, respectively, while panels (c) and (d) correspond to MSEs and AABs for $(\alpha, \beta, \xi, \varphi) = (1, 1, 2, 0.5)$, respectively

Figures 8–11 illustrate the behavior of the MSEs and AABs for the four parameters α, β, ξ, and φ under the two parameter settings. In each figure, panels (a) and (b) present the MSEs and AABs for the first setting, $(\alpha, \beta, \xi, \varphi) = (1, 1, 1, 0.5)$, whereas panels (c) and (d) present the corresponding results for the second setting, $(\alpha, \beta, \xi, \varphi) = (1, 1, 2, 0.5)$. Overall, the results show that both the MSEs and AABs generally decrease as the sample size increases, indicating improved estimation accuracy for larger samples. This decreasing pattern is particularly clear for the parameters α, ξ, and φ. For the parameter β, the reduction is less pronounced, suggesting that estimating β is relatively more challenging under the considered simulation settings.

The graphical results also indicate that the MLE and ADE methods generally provide competitive performance compared with the other estimation methods. In particular, MLE often yields smaller MSEs and AABs, especially for the parameters α, β, and ξ, whereas ADE performs well in several cases, particularly for estimating φ under the second parameter setting. These graphical findings are consistent with the numerical results in Tables 5–8 and support the conclusion that MLE and ADE are among the most effective estimation methods for the MGLED parameters.

5. Conclusions

In this paper, several classical methods were investigated for estimating the unknown parameters of the modified generalized linear exponential distribution (MGLED). These methods include maximum likelihood estimation (MLE), least square estimation (LSE), weighted least square estimation (WLSE), Cramér von-Mises estimation (CVME), and Anderson-Darling estimation (ADE). The main motivation for considering the MGLED is its flexibility in modeling different shapes of hazard rate functions, including increasing, decreasing, and bathtub-shaped hazard rates, which are commonly encountered in reliability and engineering lifetime data.

The usefulness of the MGLED was demonstrated through two real engineering data applications. The fitted MGLED was compared with several competing lifetime distributions using both non-parametric goodness-of-fit measures and parametric model selection criteria. The results showed that the MGLED provided a superior fit to the considered data sets, as indicated by smaller values of the Anderson-Darling statistic, Cramér von-Mises statistic, Kolmogorov-Smirnov statistic, Akaike information criterion (AIC), corrected Akaike information criterion (AICc), and Hannan-Quinn information criterion (HQIC), together with larger corresponding P-values. These findings confirm the practical applicability and competitiveness of the MGLED for modeling engineering lifetime data.

For the power transformer data, the LSE method produced the best performance among the considered estimation methods based on the Kolmogorov-Smirnov statistic and its corresponding P-value. In contrast, for the failure test data, the ADE method provided the best fit according to the same criterion. These results indicate that the relative performance of the estimation methods may depend on the nature and shape of the underlying data set. Therefore, considering several estimation methods is useful when applying flexible lifetime distributions such as the MGLED.

A simulation study was also conducted to compare the finite-sample performance of the different estimation methods in terms of mean squared error (MSE) and average absolute bias (AAB). The simulation results showed that both MSE and AAB generally decrease as the sample size increases, confirming the consistency and improved accuracy of the estimators for larger samples. Overall, the MLE and ADE methods showed better performance than the other methods in most of the simulation settings.

Future research may extend the present work in several directions. First, the estimation of the MGLED parameters can be studied under censored data schemes, which are frequently encountered in reliability and survival studies. Second, Bayesian estimation methods may be developed and compared with the classical methods considered in this paper. Third, the MGLED can be extended to regression modeling frameworks to incorporate covariate information. Finally, further applications may be considered in other fields such as survival analysis, risk modeling, medical data analysis, and industrial reliability to further assess the flexibility and practical value of the distribution.

References

Aarset MV. How to identify a bathtub hazard rate. IEEE Transactions on Reliability. IEEE Trans. Reliab. 1987; R-36(1):106–108.

Afify AZ, Gemeay AM, Ibrahim NK. The heavy-tailed exponential distribution: risk measures, estimation, and application to actuarial data. Mathematics. 2020; 8(8):1276.

Bain LJ. Analysis for the linear failure-rate life-testing distribution. Technometrics. 1974; 16(4):551–559.

Balakrishnan N, Kateri M. On the maximum likelihood estimation of parameters of Weibull distribution based on complete and censored data. Stat Probabil Lett. 2008; 78(17):2971–2975.

Boos DD. Minimum distance estimators for location and goodness of fit. J Am Stat Assoc. 1981; 76(375):663–670.

Cohen AC. Maximum likelihood estimation in the Weibull distribution based on complete and on censored samples. Technometrics. 1965; 7(4):579–588.

Ghitany ME, AL-Hussaini EK, Al-Jarallah RA. Marshall-Olkin extended Weibull distribution and its application to censored data. Journal of Applied Statistics. J Appl Stat. 2005; 32(10):1025–1034.

Ibrahim M, Handique L, Chakraborty S, Butt NS, Yousof HM. A new three-parameter xgamma frchet distribution with different methods of estimation and applications. Pak J Stat Oper.Res. 2021; 17(1): 291–308.

Kumar I, Jha S, Kumar K. On some estimation methods for the inverse pareto distribution. Ann Data Sci. 2021.

Lai CD, Murthy DN, Xie M. Weibull distributions and their applications. Springer London, London; 2006; 63–78.

MacDonald PDM. Comment on "An estimation procedure for mixtures of distributions" by Choi and Bulgren. J R Stat Soc B. 1971; 33(2):326–329.

Mahmoud MAW, Alam FMA. The generalized linear exponential distribution. Stat Probabil Lett. 2010; 80(11):1005–1014.

Mahmoud MAW, Ghazal MGM, and Radwan HMM. Modified generalized linear exponential distribution: Properties and applications. Stat Optim Inf Comput. 2024; 12:231–255.

Ghazal MGM, Tashkandy YA, Balogun OS and Bakr ME. Exponentiated extended extreme value distribution: Properties, estimation, and applications in applied fields. AIMS Math. 2024; 9(7):17634–17656.

Nelson W. Applied life data analysis. New York: John Wiley & Sons; 1982.

Okasha HM, Kayid M. A new family of MarshallOlkin extended generalized linear exponential distribution. J Comput Appl Math. 2016; 296:576–592.

Popović, BV., Ristić, M M., and Cordeiro GM. A two-parameter distribution obtained by compounding the generalized exponential and exponential distributions. Mediterr J Math. 2016; 13(5):2935–2954.

Swain JJ, Venkatraman S, Wilson JR. Least-squares estimation of distribution functions in johnsons translation system. J Stat Comput Simul. 1988; 29(4):271–297.

Tian Y, Tian M, and Zhu Q. Transmuted linear exponential distribution: a new generalization of the linear exponential distribution. Commun Stat Simulat. 2014; 43(10):2661–2677.

Weibull WA. Statistical distribution function of wide applicability. J Appl Mech. 1951; 18(3):293–297.

Waller A, Gotway C. Applied Spatial Statistics for Public Health Data. New York: John Wiley & Sons; 2004.

Zeng H, Lan T, Chen Q. Five and four-parameter lifetime distributions for bathtub-shaped failure rate using Perks mortality equation. Reliab Eng Syst Saf. 2016; 152:307–315.