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Logarithmic Transformed Topp-Leone Family of Distributions: Properties, and Applications

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Abstract

This study introduces a novel family of continuous distributions called the logarithmic transformed Topp-Leone family. Key statistical properties of the proposed family are investigated, including the hazard rate, reverse hazard rate, odds function, moments, moment-generating function, residual and reverse residual life moments, entropy measures, order statistics, probability-weighted moments, and quantile function. Parameter estimation is performed using both maximum likelihood and maximum product spacing methods. To demonstrate the flexibility of the proposed family, the Lindley distribution is considered the baseline distribution, and the logarithmic transformed Topp-Leone Lindley distribution is derived, for which comprehensive statistical properties are also derived. The performance of the proposed estimators is evaluated through comprehensive Monte Carlo simulations.

Keywords: Logarithmic transformation, logarithmic transformed Topp-Leone family of distributions, probability weighted moments, maximum likelihood estimation.

1. Introduction

In the modern era, the modelling of lifetime processes received immense attention. In this regard, several continuous lifetime distributions are available in the literature and are widely used to best handle them in many areas such as engineering, economics, finance, medical science, environmental science, etc. In many practical situations, due to advancements in electric equipment and lifestyle, existing distributions do not provide a satisfactory statistical model to fit practical lifetime data. Therefore, during the last few years, various types of generalized classes of distributions have been proposed by adding an extra parameter or mixing several well-known families of distributions. These distributions gives more flexibility to the model, but complexity is involved in the estimation of the parameters. The statistical literature has witnessed a substantial proliferation of generated families of lifetime distributions, each seeking to overcome the limitations of classical models. Mudholkar and Srivastava (1993) pioneered the exponentiated Weibull family, accommodating unimodal, bathtub-shaped, and a broad spectrum of monotone failure rates within a unified framework. Building on this foundation, Gupta et al. (1998) developed the generalised exponential distribution, demonstrating its capacity to capture both monotonic and non-monotonic failure rates despite arising from a monotonic baseline. Eugene et al. (2002) constructed the beta-normal distribution by embedding the normal distribution within a beta generator, yielding flexibility in skewness and kurtosis for symmetric and asymmetric data. Shaw and Buckley (2009) proposed the rank transmutation map (RTM)

method to create skew-kurtotic distributions by composing CDFs and quantile functions, generating flexible skew-normal and lifetime models beyond Gram-Charlier limitations. Alzaatreh et al. (2013) developed the T-X family generator, offering heavy tails and monotone and non-monotone hazard shapes. Bourguignon et al. (2014) introduced the Weibull-G family, using a Weibull generator to capture diverse hazard rate profiles in lifetime modelling. Alizadeh et al. (2015) extended this line of development through the Kumaraswamy Marshall–Olkin family, which utilises the Kumaraswamy generator with the Marshall–Olkin family to produce highly adaptable lifetime models. Tahir et al. (2016) further enriched the literature with the Logistic-X family, leveraging the logistic generator to construct lifetime distributions.

Recently, various generalised families have been developed to increase flexibility in modelling complex lifetime data. For instance, the transmuted Topp-Leone G (TTL-G) family extends the transmuted class via the Topp-Leone generator for flexible hazard shapes (Yousof et al., 2017). Afify et al. (2017) proposed the beta transmuted-H family for lifetime data, extending the transmuted family by combining beta and transmuted generators to achieve greater flexibility in hazard shapes. Afify et al. (2017) introduced the odd exponentiated half-logistic-G family by merging odd transformations with the exponentiated half-logistic generator for complex hazard shapes. Granzotto et al. (2017) presented the cubic rank transmuted (CRT) family through a cubic rank transmutation map, extending the quadratic transmutation approach by incorporating two additional parameters to generate highly flexible distributions from baseline models. In the same year, Brito et al. (2017) proposed the Topp–Leone odd log–logistic family, which combines the Topp–Leone and odd log–logistic generators to model a wide variety of hazard rate shapes in lifetime data. Cordeiro et al. (2017) developed the exponentiated Weibull–H family, introducing an exponentiation parameter to increase modelling flexibility. Later, Cordeiro et al. (2018) introduced the Burr XII system of densities, a general class obtained by adding two extra shape parameters to a baseline distribution.

Further developments include the bivariate Gumbel–G family proposed by Eliwa and El Morshedy (2019), which extends the univariate Gumbel generator to model dependent lifetime data with flexible marginal and joint hazard behaviours. Aslam et al. (2018) introduced the cubic transmuted-G family, extending the quadratic rank transmutation map by incorporating two additional parameters through a cubic transformation. Rahman et al. (2018) proposed the general transmuted family, a flexible extension of quadratic and cubic transmuted distributions based on higher-order transmutation maps involving multiple parameters. Ahmad et al. (2018) developed the Weibull-X family, which is generated from a Weibull random variable and provides a versatile framework for constructing new lifetime models. In addition, Alizadeh et al. (2018) introduced the extended odd Weibull-G family, augmenting the odd Weibull generator with an additional shape parameter to broaden its capacity for capturing a wider spectrum of hazard rates in lifetime data.

Other notable contributions include the Marshall–Olkin generalised-G family proposed by Yousof et al. (2018), which extends the Marshall–Olkin model through a generalised construction to produce flexible lifetime distributions. Hamedani et al. (2018) introduced the Type I general exponential (TIGE) class, designed to generate distributions capable of modelling both monotone and non-monotone hazard rate functions. Bantan et al. (2019) developed the truncated inverted Kumaraswamy-G (TIK-G) family, obtained by truncating the inverted Kumaraswamy distribution. Similarly, Nassar et al. (2019) proposed the Marshall–Olkin alpha power family, which combines the Marshall–Olkin family with the alpha power transformation to generate flexible lifetime distributions. Nofal et al. (2019) introduced the generalised Kumaraswamy-G (GKw-G) family of distributions as an extension of the Kumaraswamy-G class.

More recently, Badr et al. (2020) proposed the transmuted odd Fréchet-G family by extending the odd Fréchet-G class via the Shaw–Buckley transmutation map, enabling flexible hazard rate modelling. Awodutire (2020) developed the Chen–Pareto distribution, a generalised Pareto model enhanced by two additional shape parameters for improved flexibility in modelling lifetime and heavy-tailed data. Furthermore, Reyad et al. (2021) introduced the Fréchet Topp–Leone-G family, combining the Fréchet generator with the Topp–Leone model. In the same year, Eghwerido et al. (2021)

proposed the transmuted alpha power-G (TAPO-G) family, incorporating alpha power transformation and a baseline-G generator into a single flexible lifetime model, and subsequently, Eghwerido et al. (2021) proposed the alpha power Marshall–Olkin-G family, which couples alpha power transformation with the Marshall–Olkin extension families for enhanced distribution. Finally, Awodutire (2022) studied the exponentiated Chen-G family, generalising the Chen-G class through an additional shape parameter to broaden its capacity for capturing varied hazard rate profiles. Collectively, these contributions reflect the sustained research impetus toward constructing flexible distributional families capable of adequately modelling lifetime and reliability data.

Keeping in view the aforementioned literature, the motivation of the present study is to develop more flexible probability distributions that can serve as alternatives to existing models, with the aim of providing a better fit for real-life datasets. In many practical applications, traditional distributions often fail to adequately capture complex data behaviour such as skewness, heavy tails, or varying hazard rate patterns. Therefore, the development of flexible distributional families has become an important area of research in statistical modelling. In this context, Topp and Leone (1955) introduced the Topp–Leone (TL) distribution, which was initially proposed to model empirical data exhibiting a J-shaped behaviour. Such patterns commonly arise in reliability studies, including data related to powered band tool failures and automatic calculating machine failures. Due to its simple structure and useful properties, the TL distribution has attracted considerable attention in reliability and lifetime data analysis. Subsequently, Nadarajah and Kotz (2003) further investigated the properties of the TL distribution and demonstrated that it can display different shapes of the hazard rate function depending on the value of its shape parameter. Specifically, when $0 < \alpha < 1$, the distribution is capable of modelling a bathtub-shaped hazard rate, which is particularly important in reliability engineering as it reflects the three stages of a product life cycle: early failures, constant failure rate, and wear-out failures. On the other hand, when $\alpha \geq 1$, the TL distribution exhibits a monotonically decreasing hazard rate, making it suitable for modelling systems where the failure risk decreases over time.

The construction of new probability distributions has become a common approach to improving model flexibility and adaptability to diverse data structures. Typically, this process involves two fundamental components: a generator function and a baseline distribution. The generator modifies the baseline distribution to create a new family with enhanced shape characteristics and greater modelling capability. Several researchers, including Jones (2004) and Alzaatreh et al. (2013), have proposed different generator mechanisms for constructing such families of distributions. Using the generator approach proposed by Alzaatreh et al. (2013), Sangsanit and Bodhisuwan (2016) introduced the Topp–Leone generated (TL-G) family of distributions. This family extends the classical TL distribution by using it as a generator applied to an arbitrary baseline distribution. As a result, the TL-G family inherits the flexibility of both the Topp–Leone generator and the parent distribution, allowing it to accommodate a wide variety of distributional shapes and hazard rate behaviours.

The cumulative distribution function (CDF) and probability density function (PDF) of the Topp Leone generated family of distributions are given as follows:

$$F(x) = [G(x)]^{\alpha} [2 - G(x)]^{\alpha} \qquad x \in \mathbb{R},\ \alpha > 0, \tag{1}$$

$$f(x) = 2\alpha g(x) [1 - G(x)] [G(x)]^{\alpha-1} [2 - G(x)]^{\alpha-1} \qquad x \in \mathbb{R}, \tag{2}$$

respectively. Where $G(x)$ and $g(x)$ are the CDF and PDF of the baseline distribution, respectively.

Recently, Maurya et al. (2018) proposed a new logarithmic transformation with CDF and PDF as:

$$W(x) = \frac{\log [1 + F(x)]]}{\log(2)} \qquad x \in \mathbb{R}, \tag{3}$$

$$w(x) = \frac{f(x)}{(1 + F(x)) \log(2)} \qquad x \in \mathbb{R}, \tag{4}$$

respectively. Motivated by the aforementioned developments in the statistical literature, this study aims to construct a new class of distributions, namely the logarithmic transformed Topp-Leone (LTTL)

family, by systematically integrating the Topp-Leone generated family with logarithmic transformation. The proposed family is derived through the composition of the logarithmic transformation with the Topp-Leone generated family, yielding a more flexible class of lifetime distributions. A comprehensive investigation of its fundamental statistical properties is carried out, encompassing key structural and inferential characteristics.

The rest of the paper is organised as follows: Section 2 presents the derivation of the proposed LTTL family of distributions. Section 3 establishes the linear representation of this family. Section 4 examines a comprehensive set of statistical properties, including the odds function, cumulative hazard function, moments, moment-generating function, residual and reversed residual life moments, Rényi entropy, order statistics, probability-weighted moments, and quantile function. Finally, Section 5 adopts the Lindley distribution as a baseline and derives the corresponding statistical characteristics of the resulting sub-model to illustrate the practical applicability of the LTTL family. Section 6 addresses parameter estimation for the logarithmically transformed Topp-Leone Lindley (LTTLL) distribution under a complete sample, employing both the maximum likelihood estimation (MLE) and maximum product spacing estimation (MPSE) methods. Section 7 evaluates the finite-sample performance of the proposed estimators through a Monte Carlo simulation study. The empirical relevance and superiority of the proposed model are demonstrated in Section 8 via comparative analysis of two real-world datasets, followed by conclusions in Section 9.

2. Logarithmic Transformed Topp-Leone Family of Distributions

Let $F(x)$ denote the CDF of a Topp-Leone family of distribution with random variable X. The CDF of the logarithmic transformed Topp-Leone (LTTL) family is obtained by substituting (1) into (3), and is given by

$$W_{LTTL}(x) = \frac{\log[1 + \{G(x)(2 - G(x))\}^{\alpha}]}{\log(2)} \qquad x \in \mathbb{R}, \tag{5}$$

where $G(x)$ is the baseline cumulative distribution function. The corresponding PDF of proposed model is given by

$$w_{LTTL}(x) = \frac{2\alpha[1 - G(x)][G(x)(2 - G(x))]^{\alpha-1} g(x)}{[1 + \{G(x)(2 - G(x))\}^{\alpha}] \log(2)} \tag{6}$$

where $g(x)$ baseline PDF.

The survival function $S_{LTTL}(x)$ and the hazard function $h_{LTTL}(x)$ of the proposed model is given as:

$$S_{LTTL}(x) = 1 - \frac{\log[1 + \{G(x)(2 - G(x))\}^{\alpha}]}{\log(2)} \tag{7}$$

$$h_{LTTL}(x) = \frac{2\alpha[1 - G(x)][G(x)(2 - G(x))]^{\alpha-1} g(x)}{[1 + \{G(x)(2 - G(x))\}^{\alpha}] \log[\frac{2}{1+\{G(x)(2-G(x))\}^{\alpha}}]}. \tag{8}$$

3. Linear Representation of LTTL Family of Distributions

In this section, a useful linear representation for the PDF and CDF of the LTTL family of distributions is established. Such a representation proves instrumental in the systematic derivation of the family's statistical properties, as it expresses the PDF and CDF as weighted sums of baseline density and distribution functions, thereby facilitating analytical tractability. Expanding the PDF of the LTTL family, we get

$$w_{LTTL}(x) = \frac{2\alpha[1 - G(x)][G(x)(2 - G(x))]^{\alpha-1} g(x)}{[1 + \{G(x)(2 - G(x))\}^{\alpha}] \log(2)}. \tag{9}$$

Consider the Binomial expression

$$(1+x)^{-1} = \sum_{i=0}^{\infty} (-1)^i x^i$$

where $|x| < 1$. Then the expression

$$[1 + \{G(x)(2 - G(x))\}^{\alpha}]^{-1} = \sum_{i=0}^{\infty} (-1)^i [G(x)(2 - G(x))]^{\alpha n} \tag{10}$$

from Equations (9) and (10), we have

$$w_{LTTL}(x) = \frac{2\alpha}{\log(2)} g(x)[1 - G(x)] \sum_{i=0}^{\infty} (-1)^i [G(x)(2 - G(x))]^{\alpha(i+1)-1} \tag{11}$$

after simplification of (11), we have

$$\begin{aligned} w_{LTTL}(x) = \frac{2\alpha}{\log(2)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{2j+1} \frac{(-1)^{i+j+k}}{(k+1)} \binom{(\alpha(i+1)-1)}{j} \binom{2j+1}{k} \\ \times (k+1) g(x)[G(x)]^k \end{aligned} \tag{12}$$

after rearranging Equation (12), we get

$$w_{LTTL}(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{2j+1} t_h r_{k+1} \tag{13}$$

where

$$t_h = \frac{(-1)^{i+j+k}}{(k+1)} \binom{(\alpha(i+1)-1)}{j} \binom{2j+1}{k} \frac{2\alpha}{\log(2)}$$

and

$$r_{k+1} = (k+1) g(x) G(x)^k.$$

Equation (13) shows that the PDF of the LTTL family is an infinite linear combination of baseline PDF. Integrating both sides of (13) with respect to x gives the corresponding linear representation of the CDF of the LTTL family, given by

$$W_{LTTL}(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{2j+1} t_h R_{k+1} \tag{14}$$

where

$$R_{k+1} = G(x)^{k+1}$$

Therefore, the LTTL density function is a infinite linear combination of density of baseline. Which is very useful to study statistical properties of LTTL family of distribution.

4. Statistical Properties of LTTL Family of Distributions

In this section, different statistical properties of the LTTL family of distributions, including the odds function, cumulative hazard function, moments, moment generating function, moments of the residual and reversed residual life, Rényi entropy, order statistics, probability weighted moments, and quantile function, are discussed comprehensively.

4.1. Odds function

The odds function is defined as the ratio of distribution function and the survival function. The odds function of proposed LTTL family of distributions is given by

$$O_{LTTL}(x) = \frac{\frac{\log\left[1+\left\{G(x)(2-G(x))\right\}^{\alpha}\right]}{\log(2)}}{1-\frac{\log\left[1+\left\{G(x)(2-G(x))\right\}^{\alpha}\right]}{\log(2)}}. \tag{15}$$

The above expression, given in (15) can be simplified by considering the Binomial expression as:

$$(1-x)^{-1} = \sum_{l=0}^{\infty} x^l$$

where $|x| < 1$. Then the expression given in denominator is simplified as

$$\left[1-\frac{\log[1+\{G(x)(2-G(x))\}^{\alpha}]}{\log(2)}\right]^{-1} = \sum_{l=0}^{\infty}\left[\frac{\log[1+\{G(x)(2-G(x))\}^{\alpha}]}{\log(2)}\right]^{l} \tag{16}$$

substituting Equation (16) into (15), we get

$$O_{LTTL}(x) = \sum_{l=0}^{\infty}\left[\frac{\log[1+\{G(x)(2-G(x))\}^{\alpha}]}{\log(2)}\right]^{l+1}. \tag{17}$$

Using Equation (14) in (17), we get

$$O_{LTTL}(x) = \sum_{l=0}^{\infty}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} t_h R_{k+1}\right]^{l+1}. \tag{18}$$

Hence, using the result of power series explored in the work of Gradshteyn and Ryzhik (2014)

$$\left(\sum_{k=0}^{\infty} f_k z^k\right)^n = \sum_{k=0}^{\infty} C_{n,k} z^k \tag{19}$$

where, n is natural number and coefficient $C_{n,k}$ can be derived from recurrence relation as

$$C_{n,k} = (k(C_{n,k})^{\frac{1}{n}})^{-1}\sum_{l=1}^{k}(l(n+1)-k)((C_{n,l})^{\frac{1}{n}})(C_{n,k-l}).$$

Therefore, using Equations (18) and (19) the final explicit solution for odds function is given by

$$O_{LTTL}(x) = \sum_{l=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} C_{h,l+1} R_{k+1}. \tag{20}$$

4.2. Cumulative hazard function(CHF)

The CHF is an important function to study the survival characteristic of any phenomenon. The cumulative hazard function of LTTL family of distribution is obtained by the following relation between survival function, $S(x)$ and CHF.

$$H(x) = -\log(S(x))$$

$$H(x) = -\log(1 - W_{LTTL}(x)) = -\log\left[1 - \frac{\log[1 + \{G(x)(2 - G(x))\}^{\alpha}]}{\log(2)}\right]. \tag{21}$$

By using the result based on logarithmic series expansion and Equation (14). The expression for CHF is obtained as

$$H(x) = -\log(1 - W_{LTTL}(x)) = \sum_{l=0}^{\infty} \frac{1}{(l+1)} \left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} t_h R_{k+1}\right]^{l+1}. \tag{22}$$

Now, using Equation (19), the final expression for CHF is given by

$$H(x) = -\log(1 - W_{LTTL}(x)) = \sum_{l=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} \frac{1}{(l+1)} C_{h,l+1} R_{k+1}. \tag{23}$$

4.3. Moments

Moments are very useful to describe several characteristics of the distributions such as skewness, kurtosis, dispersion, and measure of central tendency. In this subsection, moments of LTTL family of distributions is considered.

The r^{th} moment of the LTTL family of distributions is defined as

$$\mu_r^{'} = \int_0^{\infty} x^r \frac{2\alpha[1 - G(x)][G(x)(2 - G(x))]^{\alpha-1} g(x)}{[1 + \{G(x)(2 - G(x))\}^{\alpha}]\log(2)} dx. \tag{24}$$

Hence, using (12) and (24)

$$\begin{aligned} \mu_r^{'} = \frac{2\alpha}{\log(2)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} \frac{(-1)^{i+j+k}}{(k+1)} \binom{(\alpha(i+1)-1)}{j}\binom{2j+1}{k} \\ \times \int_0^{\infty} x^r (k+1) g(x)[G(x)]^k dx \end{aligned} \tag{25}$$

after rearranging the above expression, $\mu_r^{'}$ is given by

$$\mu_r^{'} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} t_h N(x)$$

where
$$N(x) = \int_0^{\infty} x^r (k+1) g(x)[G(x)]^k dx. \tag{26}$$

The r^{th} cumulants k_r and central moments μ_r of the distribution are computed by the following equations:

$$k_r = \mu_r^{'} - \sum_{l=1}^{r-1} \binom{r-1}{l-1} k_l \mu_{r-l}^{'} \tag{27}$$

and
$$\mu_r = \sum_{l=1}^{r} (-1)^l \binom{r}{l} \left(\mu_1^{'}\right)^l \mu_{r-l}^{'} \tag{28}$$

respectively. The variance (V), skewness (S), and kurtosis (K) measure can be obtained from the usual moments relationship;

$$V = \mu_2^{'} - (\mu_1^{'})^2; \quad S = \frac{\mu_3}{\mu_2^{3/2}}; \quad K = \frac{\mu_4}{\mu_2^2}.$$

4.4. Moment generating function

The moment generating function $M_X(t)$ of LLTL family of distributions is computed by

$$M_X(t) = \int_0^\infty e^{tx} \frac{2\alpha[1-G(x)][G(x)(2-G(x))]^{\alpha-1} g(x)}{[1+\{G(x)(2-G(x))\}^\alpha]\log(2)} dx \tag{29}$$

using Equations (12) and (29)

$$\begin{aligned} M_X(t) = \frac{2\alpha}{\log(2)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} \frac{(-1)^{i+j+k}}{(k+1)} \binom{(\alpha(i+1)-1)}{j}\binom{2j+1}{k} \\ \times \int_0^\infty e^{tx}(k+1) g(x)[G(x)]^k dx. \end{aligned} \tag{30}$$

Equation (30) can be written as

$$M_X(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} t_h R(x) \tag{31}$$

where, t_h is as previously defined, and

$$R(x) = \int_0^\infty e^{tx}(k+1) g(x)[G(x)]^k dx.$$

4.5. Moments of the residual life and reversed residual life

The m^{th} moment of the residual life of LTTL family of distributions, say $h_m(t)$ for $m = 1, 2..$ is defined as

$$h_m(t) = E[(X-t)^m | X \geq t] = \frac{1}{1-W_{LTTL}(x)} \int_t^\infty (x-t)^m dW_{LTTL}(x). \tag{32}$$

Therefore, from (12) and (32), we get

$$\begin{aligned} h_m(t) = \frac{1}{1-W_{LTTL}(x)} \frac{2\alpha}{\log(2)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1} \frac{(-1)^{i+j+k}}{(k+1)} \binom{\alpha(i+1)-1}{j}\binom{2j+1}{k} \\ \times \int_t^\infty (k+1)(x-t)^m g(x)[G(x)]^k dx. \end{aligned} \tag{33}$$

Hence, using (13) and (33), we have

$$h_m(t) = \frac{1}{1-W_{LTTL}(x)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{2j+1}\sum_{l=0}^{m} (-1)^{m-l}(k+1)\binom{m}{l} t^{m-l} t_h E_k \tag{34}$$

where

$$E_k = \int_t^\infty x^l g(x)[G(x)]^k dx.$$

Similarly, the m^{th} moment of the reverse residual life of LTTL family of distributions, say $B_m(t)$ is given as

$$B_m(t) = E[(t-X)|X \leq t] = \frac{1}{W_{LTTL}(x)} \int_0^t (t-x)^m dW_{LTTL}(x) \tag{35}$$

after substituting (12) in (35),we get

$$B_m(t) = \frac{1}{W_{LTTL}(x)} \frac{2\alpha}{\log(2)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{2j+1} \frac{(-1)^{i+j+k}}{(k+1)} \binom{\alpha(i+1)-1}{j} \binom{2j+1}{k} \times \int_0^t (k+1)(t-x)^m g(x)[G(x)]^k dx \tag{36}$$

using (13) and (36), we have

$$B_m(t) = \frac{1}{W_{LTTL}(x)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{2j+1} \sum_{l=0}^{m} (-1)^l (k+1) \binom{m}{l} t^{m-l} t_h E_k. \tag{37}$$

4.6. Rényi entropy

Rényi entropy plays a pivotal role in quantifying uncertainty and complexity across a broad range of disciplines, including electronics, engineering, and the characterisation of chaotic dynamical systems (Kurths et al., 1995). As a generalisation of Shannon entropy, Hartley entropy, and collision entropy within the framework of information theory, Rényi entropy offers a unified and versatile measure of distributional uncertainty. In the discrete setting, Rényi entropy of order two is most widely employed as a measure of economic diversity (Hart, 1975), while in quantum information theory it serves as a well-established measure of entanglement. The Rényi entropy of the LTTL family of distributions is derived as follows:

$$J_R(\gamma) = \frac{1}{1-\gamma} \log \left[\int_0^{\infty} w_{LTTL}^{\gamma}(x) dx \right] \quad ; \gamma \neq 1, \gamma > 0. \tag{38}$$

Therefore, from (6) and (38)

$$J_R(\gamma) = \frac{1}{1-\gamma} \log \left[\left(\frac{2\alpha}{\log(2)} \right)^{\gamma} \int_0^{\infty} \frac{g(x)^{\gamma} [1-G(x)]^{\gamma} [G(x)(2-G(x))]^{\gamma(\alpha-1)}}{[1+\{G(x)(2-G(x))\}^{\alpha}]^{\gamma}} dx \right] \tag{39}$$

again considering the series approximation

$$(1+x)^{-n} = \sum_{i=0}^{\infty} (-1)^i \frac{n(n+1)...(n+i-1)}{i\,!} x^i,$$

now, in our considered case

$$\Big[1+\{G(x)(2-G(x))\}^{\alpha}\Big]^{-\gamma} = \sum_{i=0}^{\infty} (-1)^i \frac{\gamma(\gamma+1)...(\gamma+i-1)}{i\,!} \Big[G(x)(2-G(x))\Big]^{\gamma\alpha}. \tag{40}$$

Hence, from (39) and (40), we have

$$J_R(\gamma) = \frac{1}{1-\gamma} \log \left[\left(\frac{2\alpha}{\log(2)} \right)^{\gamma} \sum_{i=0}^{\infty} (-1)^i \frac{\gamma(\gamma+1)...(\gamma+i-1)}{i\,!} \times \int_0^{\infty} g(x)^{\gamma} \Big[1-G(x)\Big]^{\gamma} \Big[G(x)(2-G(x))\Big]^{\gamma(2\alpha-1)} dx \right], \tag{41}$$

after simplification of the above equation, we get

$$J_R(\gamma) = \frac{1}{1-\gamma} \log \left[\left(\frac{2\alpha}{\log(2)} \right)^{\gamma} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{i+j+k} \frac{\gamma(\gamma+1)...(\gamma+i-1)}{i\,!} \times \binom{\gamma(2\alpha-1)}{j} \binom{2j+\gamma}{k} R_{\gamma k} \right] \tag{42}$$

where $R_{\gamma k} = \int_0^{\infty} g(x)^{\gamma} G(x)^k dx.$

4.7. Order statistics

Suppose $X_1, X_2, ...X_n$ be the random sample of size n from the LTTL family of distributions and $X_{(1)}, X_{(2)}, ...X_{(n)}$ be the corresponding order statistics. Then the PDF of r^{th} order statistic $X_{(r)}$, say $w_{r:n}$ is obtained by

$$w_{r:n}(x_r) = \frac{1}{B(r, n-r+1)} w_{LTTL}(x_r) \sum_{l=0}^{n-r} (-1)^l \binom{n-r}{l} W^{r+l-1}(x_r). \tag{43}$$

Therefore,

$$W^{r+l-1}(x_r) = \left[\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{2j+1} t_h G(x)^{k+1} \right]^{r+l-1} \tag{44}$$

where t_h is as previously defined. Substituting (20) into (44), we get

$$W^{r+l-1}(x_r) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{2j+1} C_{h,r+l-1} G(x)^{k+1} \tag{45}$$

substituting (13) and (45) into (43), the PDF of $X_{(r)}$ of LTTL family of distributions is derived as;

$$\begin{aligned} w_{r:n}(x_r) = \frac{1}{B(r, n-r+1)} \sum_{l=0}^{n-r} (-1)^l \binom{n-r}{l} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{2j+1} t_h C_{h,r+l-1} \\ \times (k+1) g(x) G(x)^{2k+1}. \end{aligned} \tag{46}$$

4.8. Probability weighted moments

Probability weighted moments (PWMs) were introduced by Greenwood et al. (1979) as a tool for estimating the parameters of probability distribution. The probability-weighted moments can be defined for a random variable whose ordinary moments exist and used for estimating the parameters of the distribution whose inverse form cannot be expressed explicitly. Estimates based on probability-weighted moments are often considered to be superior to standard moment-based estimates. The $(s, r)^{th}$ probability weighted moment of the LTTL family of distributions, say $\rho_{s,r}$ is defined as

$$\rho_{s,r} = E[X^s W_{LTTL}^r(x)] = \int_{-\infty}^{\infty} x^s W_{LTTL}^r(x) w_{LTTL}(x) dx. \tag{47}$$

Therefore, using Equations (13) and (19) into Equation (47), the $(s, r)th$ probability weighted moment of LTTL family of distribution is given by

$$\rho_{s,r} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{2j+1} (k+1) t_h C_{h,r} \int_0^{\infty} x^s g(x) G(x)^{2k+1}. \tag{48}$$

4.9. Quantile function

Let X be the real valued random variable with distribution function F. Then, for $p \in (0, 1)$, a value of x such that

$$F(x^-) = P(X < x) \leq p \quad and \quad F(x) = P(X \leq x) \geq p$$

Table 1 Some special cases of the Logarithmic Transformed Topp-Leone family of distributions

Sr. no.	Baseline Model	CDF of LTTL Family	Generated Model	Support
1	Lindley	$\frac{\log\left[1+\left\{1-\left(\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2\right\}^{\alpha}\right]}{\log(2)}$	LTTL-Lindley	$x \in \mathbb{R}^+$
2	Chen	$\frac{\log\left[1+\left\{1-e^{2\lambda(1-e^{x^{\beta}})}\right\}^{\alpha}\right]}{\log(2)}$	LTTL-Chen	$x \in \mathbb{R}^+$
3	Pareto	$\frac{\log\left[1+\left\{1-(\frac{m}{x})^{2\lambda}\right\}^{\alpha}\right]}{\log(2)}$	LTTL-Pareto	$x \in [m, \infty)$
4	Lomax	$\frac{\log\left[1+\left(1-(1+\lambda x)^{-2k}\right)^{\alpha}\right]}{\log(2)}$	LTTL-Lomax	$x \in \mathbb{R}^+$
5	Gamma	$\frac{\log\left[1+\left\{\left(\frac{1}{\Gamma(\lambda)}\gamma(\lambda,\beta x)\right)\left(2-\frac{1}{\Gamma(\lambda)}\gamma(\lambda,\beta x)\right)\right\}^{\alpha}\right]}{\log(2)}$	LTTL-Gamma	$x \in \mathbb{R}^+$
6	Weibull	$\frac{\log\left[1+\left\{1-e^{-2\beta x^{\lambda}}\right\}^{\alpha}\right]}{\log(2)}$	LTTL-Weibull	$x \in \mathbb{R}^+$
7	Uniform	$\frac{\log\left[1+\left\{\frac{x}{b}\left(2-\frac{x}{b}\right)\right\}^{\alpha}\right]}{\log(2)}$	LTTL-Uniform	$x \in (0, b)$
8	Rayleigh	$\frac{\log\left[1+\left\{1-e^{-2\beta x^2}\right\}^{\alpha}\right]}{\log(2)}$	LTTL-Rayleigh	$x \in \mathbb{R}^+$
9	XGamma	$\frac{\log\left[1+\left\{1-\frac{\left(1+\theta+\theta x+\frac{\theta^2 x^2}{2}\right)^2}{(1+\theta)^2}e^{-2\theta x}\right\}^{\alpha}\right]}{\log(2)}$	LTTL-XGamma	$x \in \mathbb{R}^+$
10	Gompertz	$\frac{\log\left[1+\left\{1-e^{\frac{-2\lambda}{\mu}\left(e^{\mu x}-1\right)}\right\}^{\alpha}\right]}{\log(2)}$	LTTL-Gompertz	$x \in \mathbb{R}^+$
11	Exponential	$\frac{\log\left[1+\left\{1-e^{-2\lambda x}\right\}^{\alpha}\right]}{\log(2)}$	LTTL-Exponential	$x \in \mathbb{R}^+$
12	Burr-X	$\frac{\log\left[1+\left\{\left(1-e^{-x^2}\right)^{\theta}\left(2-\left(1-e^{-x^2}\right)^{\theta}\right)\right\}^{\alpha}\right]}{\log(2)}$	LTTL-Burr-X	$x \in \mathbb{R}^+$
13	Burr-XII	$\frac{\log\left[1+\left\{1-(1+x^c)^{-2k}\right\}^{\alpha}\right]}{\log(2)}$	LTTL-Burr-XII	$x \in \mathbb{R}^+$
14	Kumaraswamy	$\frac{\log\left[1+\left\{1-\left(1-x^{\lambda}\right)^{-2\beta}\right\}^{\alpha}\right]}{\log(2)}$	LTTL-Kumaraswamy	$x \in [0, 1]$
15	Standard Gumbel	$\frac{\log\left[1+\left\{\left(e^{-e^{-x}}\right)\left(2-e^{-e^{-x}}\right)\right\}^{\alpha}\right]}{\log(2)}$	LTTL-Standard Gumbel	$x \in \mathbb{R}$

is called a quantile of order p for the distribution. The quantile function x_q of LTTL family of distributions can be derived by inverting the distribution function into U, where U is distributed as U(0,1). Then the quantile function of LTTL family of distribution is given as

$$x_q = G^{-1}\left[1 - \left[1 - (2^u - 1)^{\frac{1}{\alpha}}\right]^{\frac{1}{2}}\right]. \tag{49}$$

Table 1 presents a collection of special sub-models of the LTTL family derived from various baseline distributions. To facilitate a detailed investigation of the statistical characteristics of the proposed family, the LTTL-Lindley distribution is selected as a representative sub-model for further analytical and empirical examination.

5. Logarithmic Transformed Topp-Leone Lindley Distribution

Let X be the random variable distributed as Lindley distribution with PDF

$$g(x) = \frac{\theta^2}{(\theta+1)}(1+x)e^{-\theta x} \quad x > 0,\ \theta > 0 \tag{50}$$

and the corresponding associated CDF is

$$G(x) = 1 - \frac{\theta + 1 + \theta x}{\theta + 1}e^{-\theta x} \quad x > 0,\ \theta > 0 \tag{51}$$

Therefore, substituting (50) and (51) as the baseline PDF and CDF in (6) and (5) respectively. The CDF of the logarithmic transformed Topp-Leone Lindley distribution is given by

$$W_{LTTLL}(x) = \frac{\log\left[1 + \left\{1 - \left(\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2\right\}^{\alpha}\right]}{\log(2)} \quad ; x > 0,\ \alpha > 0,\ \theta > 0 \tag{52}$$

and the corresponding PDF is

$$w_{LTTLL}(x) = \frac{2\theta^2\alpha(1+x)e^{-\theta x}\left[\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]\left[1 - \left(\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2\right]^{\alpha-1}}{(\theta+1)\left[1 + \left\{1 - \left(\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2\right\}^{\alpha}\right]\log(2)} \tag{53}$$

The survival and hazard functions respectively of the LTTL-Lindley distribution are given as

$$S_{LTTLL}(x) = 1 - \frac{\log\left[1 + \left\{1 - \left(\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2\right\}^{\alpha}\right]}{\log(2)} \tag{54}$$

and

$$h_{LTTLL}(x) = \frac{2\theta^2\alpha(1+x)e^{-\theta x}\left[\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]\left[1 - \left(\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2\right]^{\alpha-1}}{(\theta+1)\left[1 + \left\{1 - \left(\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2\right\}^{\alpha}\right]\log\left[\frac{2}{[1+\{1-(\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x})^2\}^{\alpha}]}\right]} \tag{55}$$

In the subsequent subsections, the fundamental statistical properties of the LTTLL distribution are examined based on the expression established in Sections 3 and 4.

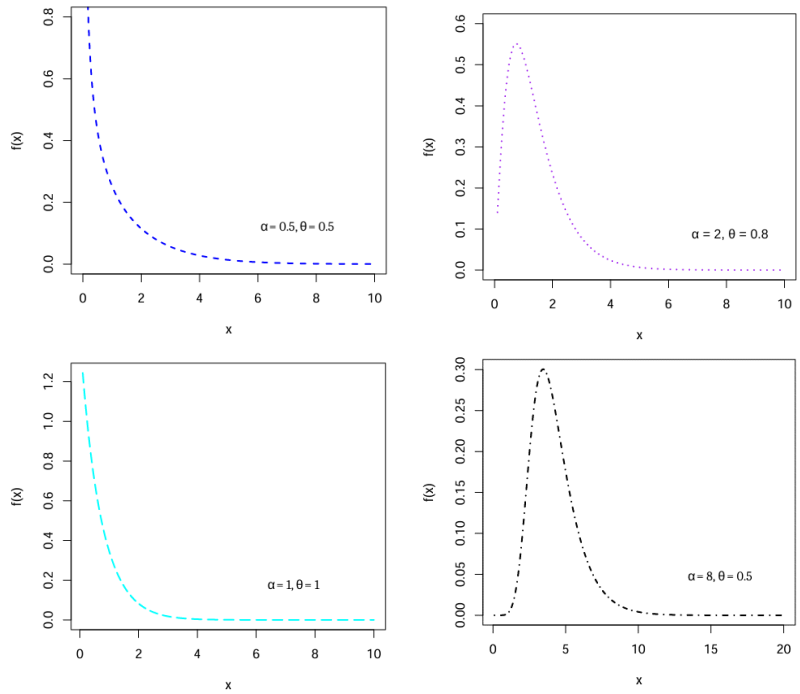


Figure 1 Graphical representation of PDF of the LTTLL distribution for various values of parameters

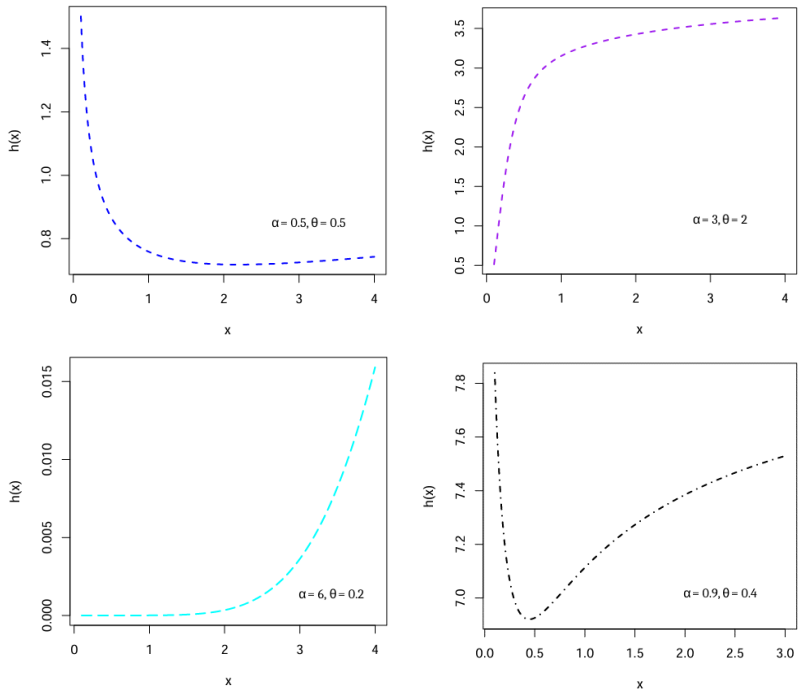


Figure 2 Graphical representation of hazard of the LTTLL distribution for various values of parameters

5.1. Linear representation of the logarithmic transformed Topp-leone Lindley distribution

In this subsection, the linear representation of the LTTLL distribution is obtained by inserting the PDF and CDF of Lindley distribution in (12) and (14) respectively. The linear representation of the PDF of LTTL distribution is given as

$$w_{LTTLL}(x) = \frac{2\alpha}{\log(2)}\left(\frac{\theta^2}{\theta+1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{k=0}^{2j+1}\Omega_{(i,j,k,l,m)}(1+x)x^m e^{-(1+l)\theta x} \tag{56}$$

where

$$\Omega_{(i,j,k,l,m)} = (-1)^{i+j+k+l}\binom{k}{l}\binom{l}{m}\binom{\alpha(i+1)-1}{j}\binom{2j+1}{k}\left(\frac{\theta}{\theta+1}\right)^m$$

and associated CDF is

$$W_{LTTLL}(x) = \frac{1}{\log(2)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{k=0}^{2j}\Phi_{(i,j,k,l,m)}x^m e^{-\theta l x} \tag{57}$$

where

$$\Phi_{(i,j,k,l,m)} = \frac{(-1)^{i+j+k+l}}{n+1}\binom{k}{l}\binom{l}{m}\binom{(i+1)\alpha}{j}\binom{2j}{k}\left(\frac{\theta}{\theta+1}\right)^m.$$

Based on the given PDF and CDF of LTTLL distribution the shape of hazard function is given in Figure 2.

5.2. Odds function

From Equation (20), the odds function for the LTTLL distribution is given as

$$O_{LTTLL}(x) = \sum_{l=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{2j+1}\sum_{n=0}^{m}(-1)^m C_{h,l+1}\binom{k+1}{m}\binom{m}{n}\left(\frac{\theta x}{\theta+1}\right)^n e^{-\theta m x}. \tag{58}$$

5.3. Cumulative hazard function

The cumulative hazard function for the LTTLL distribution from Equation (23) is given as

$$\begin{aligned}-\log(1-W_{LTTLL}(x)) = &\sum_{l=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\sum_{k=0}^{2j+1}\frac{(-1)^m}{(l+1)}C_{h,l+1}\binom{k+1}{m}\binom{m}{n}\\ &\times\left(\frac{\theta x}{\theta+1}\right)^n e^{-\theta m x}.\end{aligned} \tag{59}$$

5.4. Moments of LTTLL distribution function

The r^{th} moment of logarithmic transformed Topp-Leone Lindley distribution is given as

$$\begin{aligned}\mu_r^{'} = &\frac{2\alpha}{\log(2)}\left(\frac{\theta^2}{\theta+1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{2j+1}\sum_{m=0}^{l}(-1)^{i+j+k+l}\binom{k}{l}\binom{l}{m}\binom{\alpha(i+1)-1}{j}\\ &\times\binom{2j+1}{k}\left(\frac{\theta}{\theta+1}\right)^m\left[\frac{\Gamma(m+r+1)}{(\theta(1+i))^{m+r+1}}+\frac{\Gamma(m+r+2)}{(\theta(1+i))^{m+r+2}}\right].\end{aligned} \tag{60}$$

5.5. Moment generating function of LTTLL distribution

From Equation (31) moment generating function of LTTLL distribution is obtained as

$$M_x(t) = \left(\frac{2\alpha}{\log(2)}\right)\left(\frac{\theta^2}{\theta+1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{2j+1}\sum_{m=0}^{l}(-1)^{i+j+k+l}\binom{k}{l}\binom{l}{m}\binom{\alpha(i+1)-1}{j} \times \binom{2j+1}{k}\left(\frac{\theta}{\theta+1}\right)^m\left[\frac{\Gamma(m+1)}{(\theta(1+l)-t)^{m+1}} + \frac{\Gamma(m+2)}{(\theta(1+l)-t)^{m+2}}\right]. \tag{61}$$

5.6. Moments of residual life and reverse residual life

The s^{th} moment of residual life and reverse residual life, say $h_s(t)$ and $B_s(t)$ of LTTLL distribution is obtained by using the expression in Section 4 respectively. The s^{th} moment of residual life of LTTLL distribution is given by

$$h_s(t) = \frac{1}{1-W_{LTTLL}(x)}\left(\frac{2\alpha}{\log(2)}\right)\left(\frac{\theta^2}{\theta+1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{2j+1}\sum_{m=0}^{l}\sum_{n=0}^{s}(-1)^{i+j+k+l+s-n} \times \binom{k}{l}\binom{l}{m}\binom{s}{n}\binom{\alpha(i+1)-1}{j}\binom{2j+1}{k}\left(\frac{\theta}{\theta+1}\right)^m t^{s-n} \times \left[\frac{\Gamma(m+n+1)}{((1+l)\theta)^{m+n+1}} + \frac{\Gamma(m+n+2)}{((1+l)\theta)^{m+n+2}}\right], \tag{62}$$

and the s^{th} moment of reversed residual life of LTTLL distribution is given as

$$B_s(t) = \frac{1}{W_{LTTLL}(x)}\left(\frac{2\alpha}{\log(2)}\right)\left(\frac{\theta^2}{\theta+1}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{2j+1}\sum_{m=0}^{l}\sum_{n=0}^{s}(-1)^{i+j+k+l+n} \times \binom{k}{l}\binom{l}{m}\binom{s}{n}\binom{\alpha(i+1)-1}{j}\binom{2j+1}{k}\left(\frac{\theta}{\theta+1}\right)^m t^{s-n} \times \int_0^t (1+x)x^{m+n}e^{-(1+l)\theta x}dx. \tag{63}$$

5.7. Rényi entropy of LTTLL distribution

From Equation (42) Rényi entropy of LTTLL distribution is obtained as

$$J_R(\gamma) = \frac{1}{1-\gamma}\log\left[\left(\frac{2\alpha}{\log(2)}\right)^{\gamma}\left(\frac{\theta^2}{\theta+1}\right)^{\gamma}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\gamma}(-1)^{i+j+k+l} \times \frac{\gamma(\gamma+1)...(\gamma+i-1)}{i\,!}\binom{k}{l}\binom{l}{m}\binom{\gamma}{n}\binom{\gamma(2\alpha-1)}{j}\binom{2j+\gamma}{k}\left(\frac{\theta}{\theta+1}\right)^m \times \frac{\Gamma(m+n+1)}{((\gamma+l)\theta)^{m+n+1}}\right]. \tag{64}$$

5.8. Order statistics of LTTLL distribution

The r^{th} order statistics of LTTLL distribution is given as from Equations (46), (50), and (51).

$$w_{r:n}(x_r) = \frac{1}{B(r, n-r+1)} \sum_{l=0}^{n-r} (-1)^{l+m} \binom{n-r}{l} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{m} \sum_{k=0}^{2j+1} t_h C_{h,r+l-1} \times (k+1)(1+x) \binom{2k+1}{m} \binom{m}{n} \left(\frac{\theta^2}{\theta+1}\right) \left(\frac{\theta x}{\theta+1}\right)^n e^{-(m+1)\theta x}. \tag{65}$$

5.9. Probability weighted moment of LTTLL distribution

The $(s,r)^{th}$ probability weighted moment of the LTTLL distribution, (say) $\rho_{s,r}$, is obtained by combining Equations (48), (50), and (51).

$$\rho_{s,r} = \left(\frac{\theta^2}{\theta+1}\right) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \sum_{k=0}^{2j+1} (-1)^l t_h C_{h,r} (k+1) \binom{2k+1}{l} \binom{l}{m} \left(\frac{\theta}{\theta+1}\right)^m \times \left[\frac{\Gamma(s+m+1)}{((l+1)\theta)^{s+m+1}} + \frac{\Gamma(s+m+2)}{((l+1)\theta)^{s+m+2}}\right]. \tag{66}$$

5.10. Quantile function of LTTLL distribution

The quantile function x_q of LTTLL distribution is derived as

$$x_q = -\left[1 + \frac{1}{\theta} + \frac{1}{\theta} W_{-1}\left\{ -(1+\theta)\left(1 - \left(2^u - 1\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{2}} e^{-(1+\theta)} \right\}\right] \tag{67}$$

where, W_{-1} is the negative branch of the Lambert W function see Corless (1996) for details. Equation (67) can be used for random number generation from the LTTLL distribution.

6. Parameter Estimation

In the following subsection, two classical methods namely, ML, and MPS method of estimation are discussed to estimate the unknown parameters of the LTTLL distribution.

6.1. Maximum likelihood method of estimation

In this subsection, we obtain the MLEs of the parameters of the logarithmic transformed Topp Leone Lindley (LTTLL) distribution. Let $x_1, x_2, \ldots, x_n$ be a random sample of size n drawn from the LTTLL distribution. We assume that $x_1, x_2, \ldots, x_n$ are independent and identically distributed (i.i.d.) random variables with PDF given in Equation (53). Then, the log-likelihood function is written as,

$$\begin{aligned} l = n\ \log(2\alpha) - n\ \log(\log(2)) + 2n\ \log\theta - n\ \log(\theta+1) + \sum_{i=1}^{n} \log[S(x_i)] \\ + (\alpha-1) \sum_{i=1}^{n} \log[U(x_i)] + \sum_{i=1}^{n} \log(1+x_i) - \sum_{i=1}^{n} \log[V(x_i)] - \theta \sum_{i=1}^{n} x_i \end{aligned} \tag{68}$$

where

$$S(x_i) = \left(\frac{\theta+1+\theta x_i}{\theta+1} e^{-\theta x_i}\right), \qquad U(x_i) = \left[1 - \left(\frac{\theta+1+\theta x_i}{\theta+1} e^{-\theta x_i}\right)^2\right],$$

and

$$V(x_i) = \left[1 + \left\{1 - \left(\frac{\theta+1+\theta x_i}{\theta+1} e^{-\theta x_i}\right)^2\right\}^{\alpha}\right].$$

For ML estimates differentiating (68) with respect to parameter α and θ, the likelihood Equations are given by

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log[U(x_i)] - \sum_{i=1}^{n} \frac{V_{i\alpha}^{'}}{V_i} = 0, \tag{69}$$

and

$$\frac{\partial l}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{\theta+1} + \sum_{i=1}^{n} \frac{S_{i\theta}^{'}}{S_i} + (\alpha-1)\sum_{i=1}^{n} \frac{U_{i\theta}^{'}}{U_i} - \sum_{i=1}^{n} \frac{V_{i\theta}^{'}}{V_i} - \sum_{i=1}^{n} x_i = 0, \tag{70}$$

respectively. The above likelihood Equations cannot be solved analytically. Therefore, the maximum likelihood estimates of the parameters are obtained using numerical optimization techniques. In this study, the maximization of the log-likelihood function is carried out using the `optim()` function in the R software.

6.2. Maximum product spacing

Let $x_{(1)}, x_{(2)}, ..., x_{(n)}$ be the ordered sample from the LTTLL distribution then, the distance $D's$ are defined as

$$D_i = \frac{\log\left[1+\left\{1-\left(\frac{\theta+1+\theta x_{(i)}}{\theta+1}e^{-\theta x_{(i)}}\right)^2\right\}^{\alpha}\right]}{\log(2)} - \frac{\log\left[1+\left\{1-\left(\frac{\theta+1+\theta x_{(i-1)}}{\theta+1}e^{-\theta x_{(i-1)}}\right)^2\right\}^{\alpha}\right]}{\log(2)}. \tag{71}$$

For $i = 1, 2, ..., n$. The maximum product spacing (MPS) estimates of parameters are obtained by maximizing the geometric mean of differences

$$G = \left(\prod_{i=1}^{n+1} D_i\right)^{1/(n+1)} \tag{72}$$

after taking the logarithm of (72), we have

$$\log G = \left(\frac{1}{(n+1)}\right)\sum_{i=1}^{n+1} \log D_i \tag{73}$$

inserting the value of $D's$ from (71) in (73),we have

$$\begin{aligned}\log G = \left(\frac{1}{(n+1)}\right)\sum_{i=1}^{n+1} \log\Bigg[&\frac{\log\left[1+\left\{1-\left(\frac{\theta+1+\theta x_{(i)}}{\theta+1}e^{-\theta x_{(i)}}\right)^2\right\}^{\alpha}\right]}{\log(2)} \\ &- \frac{\log\left[1+\left\{1-\left(\frac{\theta+1+\theta x_{(i-1)}}{\theta+1}e^{-\theta x_{(i-1)}}\right)^2\right\}^{\alpha}\right]}{\log(2)}\Bigg].\end{aligned} \tag{74}$$

To obtain the estimator of the parameters, we have to differentiate (74) with respect to parameters and set it equal to zero. Thus, the resulting Equations cannot be solved analytically. Therefore, numerical methods are required to obtain the parameter estimates. In this study, the `optim()` function in the R software is employed to obtain the solutions.

7. Simulation Study

In this section, a Monte Carlo simulation study is conducted to assess the finite-sample performance of the ML and MPS estimators in terms of Mean Squared Errors (MSEs). The simulation is carried out under various parameter configurations and sample sizes; specifically, $n = 10, 20, 30, 50, 100$ with $(\alpha = 1,\ \theta = 0.25, 0.50, 0.75, 1, 2)$ and $(\alpha = 0.25, 0.50, 0.75, 1, 2,\ \theta = 1)$.

The average estimates and their corresponding MSEs for both estimation methods are computed over $N = 5000$ replications and reported in Tables 2 and 3. The numerical results reveal that both estimators exhibit the consistency property, as the MSEs decrease monotonically with increasing sample size. Furthermore, a comparative evaluation of the two methods indicates that the MPS estimator outperforms the ML estimator across all considered parameter configurations. The average estimates and MSEs are computed using the following formula.

Average estimates:

$$\hat{\alpha}_{AE} = \frac{1}{N}\sum_{i=1}^{N}\hat{\alpha}_i\,, \qquad \hat{\theta}_{AE} = \frac{1}{N}\sum_{i=1}^{N}\hat{\theta}_i$$

Mean square error:

$$\hat{\alpha}_{MSE} = \frac{1}{N}\sum_{i=1}^{N}(\alpha_i - \hat{\alpha})^2\,, \qquad \hat{\theta}_{MSE} = \frac{1}{N}\sum_{i=1}^{N}\left(\theta_i - \hat{\theta}\right)^2$$

respectively.

Table 2 Average estimate and corresponding MSEs (in brackets) of parameters

			MLE		MPSE	
n	α	θ	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$
10	1	0.25	1.4404(1.9579)	0.2983(0.0144)	0.9429(0.3784)	0.2249(0.0082)
		0.50	1.4088(1.3371)	0.5895(0.0665)	0.9412(0.3398)	0.4463(0.0382)
		0.75	1.4209(4.6632)	0.9045(0.1505)	0.9461(0.7206)	0.6774(0.0821)
		1.00	1.4104(3.7027)	1.2134(0.2858)	0.9429(0.5828)	0.9025(0.1557)
		2.00	1.3710(0.9603)	2.4719(1.2875)	0.9370(0.2671)	1.8044(0.6488)
20	1	0.25	1.1624(0.2046)	0.2721(0.0048)	0.9177(0.1014)	0.2289(0.0037)
		0.50	1.1623(0.1886)	0.5468(0.0213)	0.9242(0.0939)	0.4605(0.0153)
		0.75	1.1409(0.1739)	0.8135(0.0458)	0.9112(0.0925)	0.6810(0.0361)
		1.00	1.1482(0.1652)	1.0987(0.0878)	0.9202(0.0861)	0.9176(0.0643)
		2.00	1.1490(0.1630)	2.2291(0.4383)	0.9254(0.0851)	1.8396(0.2994)
30	1	0.25	1.1011(0.1017)	0.2648(0.0028)	0.9266(0.0636)	0.2328(0.0024)
		0.50	1.0944(0.0896)	0.5298(0.0118)	0.9256(0.0570)	0.4655(0.0101)
		0.75	1.0979(0.0980)	0.7941(0.0277)	0.9310(0.0620)	0.6962(0.0237)
		1.00	1.0929(0.0924)	1.0585(0.0492)	0.9289(0.0600)	0.9255(0.0423)
		2.00	1.0886(0.0826)	2.1343(0.2319)	0.9299(0.0543)	1.8512(0.1889)
50	1	0.25	1.0617(0.0492)	0.2587(0.0015)	0.9453(0.0366)	0.2370(0.0014)
		0.50	1.0607(0.0467)	0.5193(0.0065)	0.9474(0.0349)	0.4755(0.0059)
		0.75	1.0532(0.0431)	0.7755(0.0150)	0.9429(0.0334)	0.7092(0.0140)
		1.00	1.0528(0.0404)	1.0365(0.0269)	0.9439(0.0315)	0.9465(0.0245)
		2.00	1.0521(0.0402)	2.0807(0.1227)	0.9462(0.0314)	1.8888(0.1103)
100	1	0.25	1.0282(0.0203)	0.2545(0.0007)	0.9614(0.0179)	0.2418(0.0007)
		0.50	1.0279(0.0190)	0.5079(0.0028)	0.9628(0.0168)	0.4824(0.0028)
		0.75	1.0238(0.0183)	0.7620(0.0065)	0.9603(0.0166)	0.7233(0.0065)
		1.00	1.0245(0.0184)	1.0172(0.0122)	0.9619(0.0166)	0.9647(0.0122)
		2.00	1.0203(0.0167)	2.0361(0.0564)	0.9597(0.0155)	1.9248(0.0543)

Table 3 Average estimate and corresponding MSEs (in brackets) of parameters

			MLE		MPSE	
n	α	θ	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$
10	0.25	1	0.3005(0.0223)	1.3988(3.5652)	0.2381(0.0094)	1.0280(1.1918)
	0.50		0.6523(0.2898)	1.3170(0.6307)	0.4765(0.0787)	0.9166(0.2648)
	0.75		1.0172(0.4701)	1.2629(0.3778)	0.7114(0.1408)	0.9162(0.1842)
	1.00		1.4030(1.3505)	1.2233(0.2863)	0.9438(0.3454)	0.9101(0.1507)
	2.00		3.1796(18.212)	1.1749(0.1807)	1.9060(2.7479)	0.9132(0.1024)
20	0.25	1	0.2614(0.0067)	0.9083(0.8129)	0.2370(0.0036)	0.9181(0.2386)
	0.50		0.5557(0.0277)	1.1294(0.1869)	0.4650(0.0165)	0.9127(0.1042)
	0.75		0.8489(0.0801)	1.1111(0.1078)	0.6931(0.0443)	0.9140(0.0748)
	1.00		1.1479(0.1847)	1.0976(0.0895)	0.9197(0.0960)	0.9169(0.0660)
	2.00		2.4348(1.3260)	1.0796(0.0612)	1.8415(0.5506)	0.9249(0.0481)
30	0.25	1	0.2463(0.0051)	0.7348(0.6441)	0.2383(0.0024)	0.9079(0.1251)
	0.50		0.5347(0.0155)	1.0711(0.1126)	0.4700(0.0108)	0.9217(0.0662)
	0.75		0.8105(0.0395)	1.0762(0.0640)	0.6987(0.0270)	0.9306(0.0493)
	1.00		1.0966(0.0857)	1.0651(0.0506)	0.9322(0.0547)	0.9316(0.0417)
	2.00		2.2305(0.5401)	1.0483(0.0345)	1.8240(0.3168)	0.9342(0.0309)
50	0.25	1	0.2304(0.0052)	0.6033(0.6353)	0.2433(0.0016)	0.9202(0.0743)
	0.50		0.5209(0.0086)	1.0358(0.0774)	0.4786(0.0065)	0.9443(0.0396)
	0.75		0.7871(0.0202)	1.0421(0.0332)	0.7123(0.0159)	0.9451(0.0282)
	1.00		1.0532(0.0414)	1.0375(0.0269)	0.9442(0.0322)	0.9472(0.0247)
	2.00		2.1378(0.2585)	1.0276(0.0191)	1.8668(0.1858)	0.9501(0.0186)
100	0.25	1	0.2054(0.0076)	0.4328(0.7306)	0.2475(0.0013)	0.9201(0.0499)
	0.50		0.5061(0.0040)	0.9959(0.0528)	0.4839(0.0030)	0.9571(0.0181)
	0.75		0.7689(0.0092)	1.0234(0.0156)	0.7257(0.0082)	0.9667(0.0142)
	1.00		1.0268(0.0181)	1.0189(0.0123)	0.9641(0.0161)	0.9664(0.0121)
	2.00		2.0677(0.1088)	1.0150(0.0090)	1.9115(0.0927)	0.9696(0.0090)

8. Real Data Application

This section demonstrates the empirical flexibility and practical relevance of the LTTLL distribution through application to two real-life datasets. Parameter estimation for all fitted models is carried out using the maximum likelihood method, with all computations performed in R statistical software (R Core Team, 2023) using the `optim()` optimization routine. To rigorously assess and compare the goodness-of-fit of the competing models, several discrimination criteria are employed, namely the Kolmogorov–Smirnov statistic (D) and its associated p-value, the negative log-likelihood, the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC), the Hannan–Quinn Information Criterion (HQIC), and the Consistent Akaike Information Criterion (CAIC).

The goodness-of-fit of the LTTLL distribution is assessed against the following competing models:

1. Exponentiated Weibull (EW) distribution with PDF;

$$f(x) = \alpha\beta\theta^{\beta}x^{\beta-1}\left(1-e^{-(\theta x)^{\beta}}\right)^{\alpha-1}e^{-(\theta x)^{\beta}} \quad ;\, x>0,\, \alpha>0,\, \theta>0,\, \beta>0$$

2. Exponentiated Pareto (EP) distribution with PDF;

$$f(x) = \alpha\theta\left(1-(1+x)^{-\alpha}\right)^{\theta-1}(1+x)^{-(\alpha+1)} \quad ;\, x>0,\, \alpha>0,\, \theta>0$$

3. Exponentiated exponential (EE) distribution with PDF;

$$f(x) = \alpha\theta e^{-(\theta x)} \left(1 - e^{-(\theta x)}\right)^{\alpha-1} \quad ; x > 0,\ \alpha > 0,\ \theta > 0$$

4. Lindley exponential (Lindley-Exp) distribution with PDF;

$$f(x) = \frac{\theta\alpha^2 e^{-\theta x}\left(1 - e^{-\theta x}\right)^{\alpha-1}\left(1 - \log(1 - e^{-\theta x})\right)}{\alpha + 1} \quad ; x > 0,\ \alpha > 0,\ \theta > 0$$

Table 4 Summary of the considered data sets

Data	Min.	Q_1	Median	Mean	Q_3	SD	Skewness	Kurtosis	Max.
I	0.080	1.080	1.560	1.837	2.303	1.215	1.754	7.151	7.000
II	1.516	2.789	3.178	3.282	3.637	0.998	1.213	6.151	6.869

Table 5 ML estimates, AIC, CAIC, BIC, HQIC and KS statistics with p-values for I data set

	ML Estimates		KS-Test					
Models	parameters	-LogL	D	p	AIC	CAIC	BIC	HQIC
LTTLL	$\hat{\alpha}$= 2.1997 $\hat{\theta}$= 0.6540	102.4524	0.09621	0.5177	208.9047	209.0786	213.4580	210.7174
EW	$\hat{\alpha}$= 2.8777 $\hat{\theta}$= 1.0000 $\hat{\beta}$= 0.9818	103.0523	0.1071	0.3797	212.1045	212.4575	218.9345	214.8236
EP	$\hat{\alpha}$= 2.3708 $\hat{\theta}$= 5.3613	110.2878	0.1519	0.0720	224.5755	224.7494	229.1288	226.3882
EE	$\hat{\alpha}$= 2.6999 $\hat{\theta}$= 0.9472	102.9874	0.1065	0.3871	209.9748	210.1487	214.5282	211.7875
Lindley-Exp	$\hat{\alpha}$= 3.1927 $\hat{\theta}$= 0.9104	103.3388	0.1092	0.3563	210.6777	210.8516	215.231	212.4904

Table 6 ML estimates, AIC, CAIC, BIC, HQIC and KS statistics with p-values for II data set

	ML Estimates		KS-Test					
Models	parameters	-LogL	D	p	AIC	CAIC	BIC	HQIC
LTTLL	$\hat{\alpha}$= 15.9573 $\hat{\theta}$= 0.7352	48.4232	0.1186	0.6910	100.8465	101.2102	104.0136	101.9519
EW	$\hat{\alpha}$= 22.1842 $\hat{\theta}$= 1.0001 $\hat{\beta}$= 1.0938	48.5089	0.1232	0.6447	102.7380	103.488	107.4885	104.3961
EP	$\hat{\alpha}$= 4.4757 $\hat{\theta}$= 366.269	51.2654	0.1587	0.3247	106.5308	106.8945	109.6979	107.6362
EE	$\hat{\alpha}$= 29.0723 $\hat{\theta}$= 1.1984	48.5134	0.1235	0.6414	101.0268	101.3905	104.1939	102.1322
Lindley-Exp	$\hat{\alpha}$= 29.9788 $\hat{\theta}$= 1.1980	48.5187	0.1237	0.6400	101.0376	101.4012	104.2046	102.1429

The first dataset comprises the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, as originally observed and reported by Bjerkedal (1960). The second dataset consists of COVID-19 mortality rate data for Canada, recorded over a period of 36 consecutive days from April 10 to May 15, 2020, sourced from the World Health Organization database (WHO, 2020; https://covid19.who.int/). A descriptive summary of both datasets is presented in Table 4. The total time on test (TTT) plots of the considered data sets shown in Figure 3 lie above the diagonal line, indicating that the data exhibit a non decreasing hazard rate behaviour.

Tables 5 and 6 report the maximum likelihood estimates of the model parameters along with the goodness-of-fit statistics, including the AIC, CAIC, BIC, HQIC, and Kolmogorov–Smirnov (KS) statistic with its associated p-value, for all competing models. The KS statistic and corresponding p-value collectively indicate that the LTTLL distribution provides a superior fit relative to all other considered models. Further, the conclusion is supported by the information-theoretic criteria, as the LTTLL distribution attains the smallest values of AIC, CAIC, BIC, and HQIC among all fitted models. Consistent with these findings, the LTTLL distribution also yields the lowest negative log-likelihood value across both datasets, collectively affirming its distributional adequacy and competitive superiority.

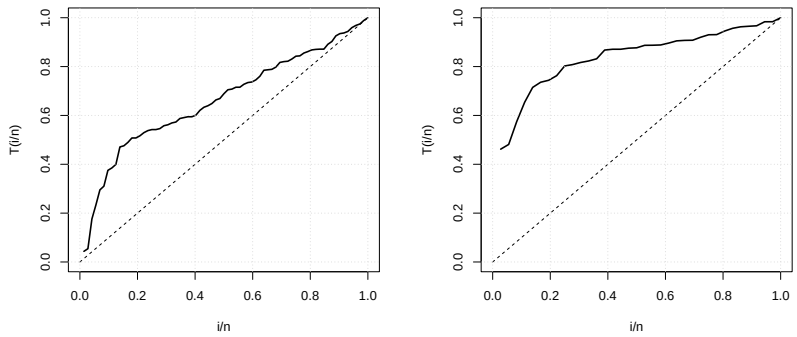


Figure 3 TTT plot for I and II data sets, respectively

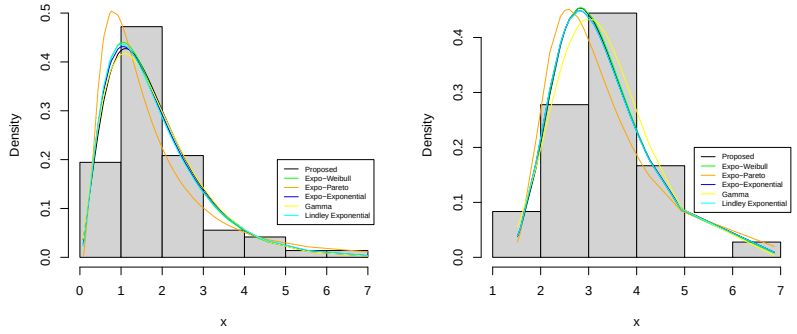


Figure 4 Observed histogram and fitted densities of the LTTLL and other models for I and II data sets, respectively

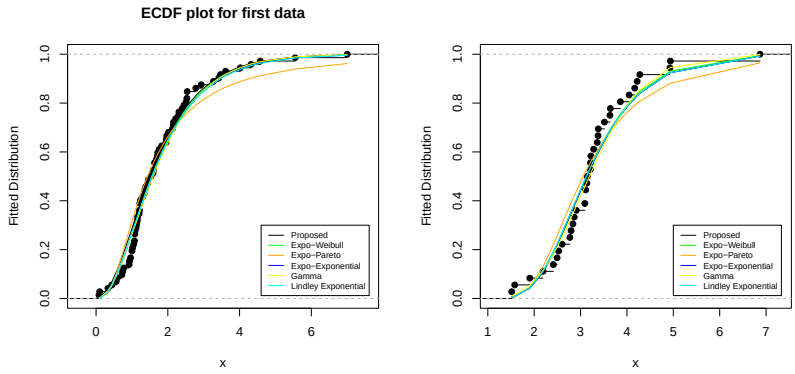


Figure 5 ECDF and fitted CDF plot of LTTLL distribution and other models for I and II data sets, respectively

Figures 4 and 5 display the observed histograms with superimposed fitted density curves and the empirical versus fitted CDF plots, respectively, for both datasets. The visual evidence presented in these figures further corroborates the superior goodness-of-fit of the LTTLL distribution relative to all competing models. On the basis of both graphical and numerical assessments, it can be concluded that the LTTLL distribution constitutes a viable and compelling alternative to several established models in the literature.

9. Conclusion

Presented in logical sequence in the text, with tables and illustrations. All the results of the tables should not be repeated in the text; the most important results should be emphasized.In this manuscript, a new two-parameter family of distributions, namely the logarithmic transformed Topp-Leone (LTTL) family, is constructed by integrating a Topp-Leone-G family into a logarithmic transformation, yielding a more flexible class of lifetime distributions. A comprehensive set of statistical properties of the proposed family is derived and examined, encompassing the odds function, cumulative hazard function, moments, moment generating function, moments of the residual and reversed residual life, Rényi entropy, order statistics, probability weighted moments, and quantile function. The PDF and CDF of the LTTL family are expressed as infinite linear combinations of baseline density and distribution functions, facilitating analytical tractability in the derivation of these properties. To illustrate the practical applicability of the proposed family, the Lindley distribution is adopted as a baseline model, yielding the logarithmic transformed Topp-Leone Lindley (LTTLL) distribution as a special sub-model. The hazard rate function of the LTTLL distribution accommodates increasing, decreasing, and bathtub-shaped failure rates, demonstrating its flexibility in modeling diverse lifetime phenomena. The unknown parameters of the LTTLL distribution are estimated under a complete sample using both the maximum likelihood estimation (MLE) and maximum product spacing estimation (MPSE) methods. A Monte Carlo simulation study is conducted to assess the finite-sample performance of the proposed estimators in terms of MSEs. The empirical superiority of the LTTLL distribution is demonstrated through comparative analysis of two real-life datasets, wherein it outperforms several well-established competing models, including the exponentiated Weibull, exponentiated Pareto, exponentiated exponential, and Lindley exponential distributions.

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