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A Power Generalized DUS–Kumaraswamy Distribution with Applications to Parallel System Reliability

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Abstract

In reliability engineering, modeling parallel systems requires lifetime distributions that combine hazard rate flexibility with structural tractability for reliability analysis. In this article, we introduce a new lifetime distribution obtained by applying the Power Generalized DUS transformation to the Kumaraswamy distribution, referred to as the PGDUS-Kumaraswamy (PGDUS-K) distribution. The proposed model provides enhanced flexibility for modeling parallel systems and accommodates various monotone and non-monotone hazard rate shapes. Several statistical properties of the PGDUS-K distribution are derived, including moments, entropy measures, quantile function, and stress-strength reliability expressions for single- and multi- component systems. The model is invariant and closed under exponentiation, and preserves its form under maxima, making it suitable for parallel systems. Parameter estimation is carried out using method of maximum likelihood, method of maximum product of spacings, method of moments, method of least squares, and Bayesian approaches. A simulation study is conducted to evaluate the performance of the estimators. The practical utility of the model is illustrated through real data analysis, where it demonstrates superior goodness-of-fit compared with competing models based on standard model selection criteria. In addition, the derived stress–strength expressions enable direct reliability computation, demonstrating practical applicability.

Keywords: Kumaraswamy distribution, invariance, Bayesian estimation, MPS estimation, multi-component stress-strength model.

1. Introduction

In reliability analysis, accurate modeling of parallel systems is essential for evaluating system reliability and comparing performance across competing designs. When system components operate independently and identically, the distributional form of component lifetimes directly determines the behavior of the overall system, influencing both parameter estimation and subsequent inference. Classical models such as the exponential and Weibull distributions are frequently employed; however, bounded lifetime data and complex hazard rate behaviors often require more flexible alternatives.

To address the limitations of classical models, transformation-based approaches have emerged as an effective mechanism for constructing flexible distributional families from well-known baseline models. Such transformations enhance model characteristics while preserving analytical tractability, making them particularly suitable for reliability applications. Among these, the DUS transformation introduced by Kumar et al. (2015) has gained attention for improving model fit without introducing

additional parameters. In recent years, several extensions and applications of the DUS framework have been developed to enhance its flexibility in reliability and distributional modeling (see Hassan and Aslam (2023), Akkanphudit (2023), Elgarhy et al. (2025)). Subsequently, Thomas and Chacko (2021) proposed the Power Generalized DUS (PGDUS) transformation, offering greater adaptability, especially for parallel system modeling. Later, Thomas and Chacko (2023) developed PGDUS-based models with Weibull and Lomax baselines. More recently, Amrutha and Chacko (2024a) and Amrutha and Chacko (2024b) investigated PGDUS extensions of the inverse Kumaraswamy (IK) and inverse Rayleigh distributions, including applications to stress–strength reliability.

Here we used the Kumaraswamy (K) distribution as the baseline distribution for modeling our parallel system. The K distribution (Kumaraswamy (1980)) is widely used in reliability analysis due to its suitability for bounded data. Its flexible parameterization allows intuitive control over shape characteristics (Nadarajah (2008)), and unlike the Beta distribution, it possesses a closed-form cumulative distribution function (CDF) and explicit quantile expression, enhancing computational efficiency and making it well suited for transformation-based extensions. An extensive amount of work has been done related to K distribution, see for example, Nadar et al. (2013), Iqbal et al. (2017), Anakha and Chacko (2021), Baharith (2022), Alotaibi et al. (2023), Alizadeh et al. (2025).

Despite the growing literature on K distribution and PGDUS-based models, the development of a PGDUS construction built directly upon the K baseline with explicit emphasis on parallel system reliability remains limited. In particular, it is essential to develop a model capable of capturing diverse hazard rate behaviors while maintaining its structural properties under operations intrinsic to parallel systems, including exponentiation and maximization. Furthermore, obtaining tractable stress–strength reliability expressions within such a framework is essential for practical reliability assessment. These considerations motivate the development of the PGDUS-K distribution, which integrates the flexibility of the PGDUS transformation with the bounded and analytically convenient structure of the K distribution.

PGDUS Transformation:

We consider a parallel system consisting of δ independent and identically distributed (i.i.d.) components, each following a DUS-transformed distribution based on a baseline random variable (RV), Z with probability density function (PDF) $f(z)$ and CDF $F(z)$. The CDF of the resulting parallel system, denoted by $G(z)$ and referred to as the PGDUS distribution, is obtained from the distribution of the maximum of component lifetime. Although δ can be interpreted as the number of parallel components, it is not restricted to integer values; instead, $\delta > 0$ is defined on the positive real line and acts as a continuous shape parameter, thereby enhancing the flexibility of the model. The CDF $G(z)$ of the PGDUS distribution is given by

$$G(z) = \left[\frac{e^{F(z)} - 1}{e - 1}\right]^{\delta}, \ \ \delta > 0, z > 0. \tag{1}$$

The PDF $g(z)$ and failure rate function $h(z)$ of PGDUS transformed distribution are respectively,

$$g(z) = \frac{\delta}{(e-1)^{\delta}}(e^{F(z)} - 1)^{\delta-1} e^{F(z)} f(z), \ \ \delta > 0, \ z > 0, \tag{2}$$

$$h(z) = \frac{\delta f(z) e^{F(z)} (e^{F(z)} - 1)^{\delta-1}}{(e-1)^{\delta} - (e^{F(z)} - 1)^{\delta}}, \ \ \delta > 0, z > 0. \tag{3}$$

Parallel systems enhance reliability by ensuring that the system continues to operate as long as at least one component is functioning. For example, a server network can maintain continuous operation despite server failures through a parallel configuration. In the PGDUS framework, system reliability can be computed directly by modeling each component's reliability with a DUS-transformed distribution. Moreover, the PGDUS transformation simplifies formulas for CDF and quantile, making them

more manageable compared to DUS transformation especially while dealing with applications like stress-strength analysis.

The main contributions of this paper are summarized as follows:

- Construction of a New Model: We introduce the PGDUS-K distribution by applying the Power Generalized DUS transformation to the K baseline distribution, specifically motivated by modeling parallel systems with i.i.d. components.
- Obtained expressions for important statistical characteristics of the proposed model, including moments, generating functions, entropy measures, skewness, and kurtosis.
- Establish important theoretical properties of the proposed model, including invariance under linear and exponentiation transformations, preservation under exponentiation, and preservation of the distributional family under maximization. These structural results ensure that reliability characteristics remain within the same family for parallel systems, enhancing analytical tractability and practical applicability.
- Comparative Performance Evaluation: We investigate multiple parameter estimation procedures, including maximum likelihood (ML) method, maximum product spacing (MPS) method, method of moments, method of least squares, and Bayesian approaches, and conduct a comparative assessment of their finite-sample performance through simulation studies. The practical effectiveness of the proposed model is further demonstrated by using real data analysis.
- Stress–Strength Reliability Development: Explicit expressions for stress–strength reliability are obtained for both single-component and multi-component systems under the PGDUS-K framework.

This research paper is structured as follows. Development of PGDUS-K model and its fundamental properties are given in Sections 2 and 3, respectively. Various characterization properties are discussed in Section 4. Section 5 provided an in-depth discussion of estimation methods, covering the ML method, method of moments, MPS method, method of least squares and Bayesian approach. Section 6 provided the execution of simulation tests across different sample sizes and parameter configurations. Real-data applications are given in Section 7. In Section 8, an examination of both single and multi-component stress-strength reliability has been undertaken, utilizing the ML technique for comprehensive analysis. Finally, Section 9 encapsulates the findings of the undertaken journey.

2. PGDUS - K Distribution

This section outlines the construction of PGDUS - K distribution. For $0 < z < 1$, the PDF $f(z)$ and CDF $F(z)$ of K distribution with parameters λ and β are respectively,

$$f(z) = \lambda\beta z^{\lambda-1}(1-z^{\lambda})^{\beta-1}, \lambda > 0,\ \beta > 0, \tag{4}$$

$$F(z) = 1-(1-z^{\lambda})^{\beta}, \lambda > 0,\ \beta > 0. \tag{5}$$

Figure 1, depicts a parallel system with δ components, each modeled using DUS-Kumaraswamy (DUS-K) distribution with parameters λ and β. The overall reliability of this parallel system is computed directly using the PGDUS-K distribution, eliminating the complexities often associated with reliability calculations. This simplifies the modeling process and enables efficient parameter estimation, making it a good choice for analyzing parallel systems in various applications.

Although δ may be interpreted in the context of parallel systems as the number of components connected in parallel in the system, it is not restricted to integer values in the mathematical formulation. In the proposed model, δ is defined on the positive real line ($\delta > 0$) and functions as a continuous shape parameter. This generalization enhances flexibility, allowing smoother control over

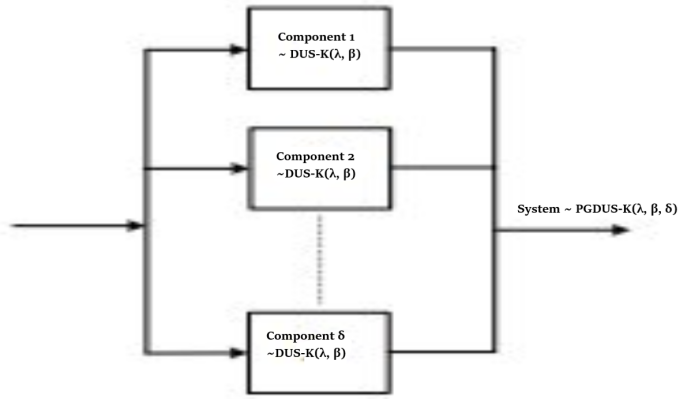


Figure 1 Parallel system configuration with δ components following $DUS-K(\lambda,\beta)$ distribution

distributional characteristics while preserving the structural properties of the model. The parallel system interpretation provides motivation for the transformation mechanism, whereas the real-valued specification of δ ensures broader applicability in statistical modeling.

The CDF $G(z)$ of PGDUS-K(λ, β, δ) is given by

$$G(z)=\left[\frac{e^{1-(1-z^{\lambda})^{\beta}}-1}{e-1}\right]^{\delta}, 0<z<1,\ \lambda>0,\ \beta>0,\ \delta>0. \tag{6}$$

Corresponding PDF $g(z)$ and failure rate function $h(z)$ of PGDUS-K(λ, β, δ) are respectively,

$$g(z)=\frac{\lambda\beta\delta}{(e-1)^{\delta}}z^{\lambda-1}(1-z^{\lambda})^{\beta-1}e^{1-(1-z^{\lambda})^{\beta}}(e^{1-(1-z^{\lambda})^{\beta}}-1)^{\delta-1}, \tag{7}$$

$$h(z)=\frac{\lambda\beta\delta z^{\lambda-1}(1-z^{\lambda})^{\beta-1}e^{1-(1-z^{\lambda})^{\beta}}(e^{1-(1-z^{\lambda})^{\beta}}-1)^{\delta-1}}{(e-1)^{\delta}-(e^{1-(1-z^{\lambda})^{\beta}}-1)^{\delta}}, \tag{8}$$

where, $0<z<1,\ \lambda>0,\ \beta>0,\ \delta>0$.

Remark 1 When $\delta=1$, the proposed PGDUS-K distribution reduces to the corresponding DUS-K distribution, confirming that the model extends the existing framework.

3. Statistical Properties

We derived various statistical properties like moments, generating functions, quantiles, skewness, kurtosis and entropy, to give a clear picture of the distribution's behavior. This foundational analysis is key to using the distribution effectively in real-world applications.

3.1. Shapes

PDF plots of the PGDUS-K distribution for some specified values of $(\lambda,\beta,\ \delta)$ are shown in Figures 2, 3 and 4. Clearly, λ, β and δ are all shape parameters. Plots illustrates that the PGDUS-K distribution is uni-modal and its parameters offer intuitive control over the shape of the distribution, showcasing its remarkable adaptability.

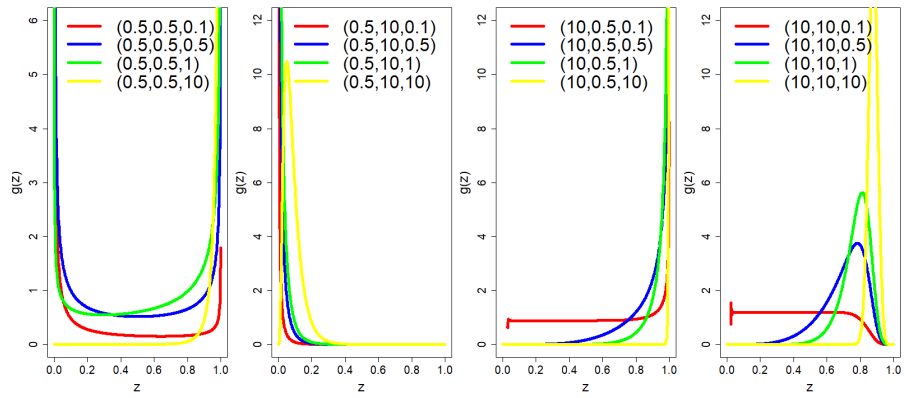


Figure 2 Plot of the PDF of PGDUS-K distribution at some specified values of $(\lambda,\ \beta, \delta)$ by keeping λ and β fixed and varying δ

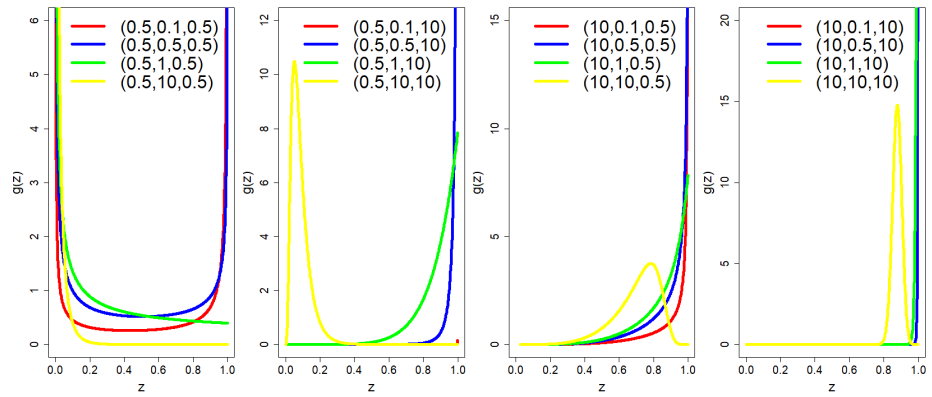


Figure 3 Plot of the PDF of PGDUS-K distribution at some specified values of (λ, β, δ) by keeping λ and δ fixed and varying β

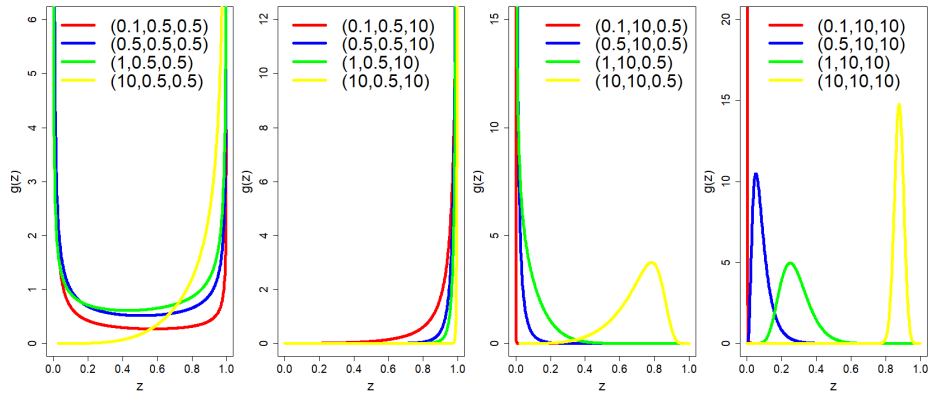


Figure 4 Plot of the PDF of PGDUS-K distribution at some specified values of $(\lambda,\ \beta, \delta)$ by keeping β and δ fixed and varying λ

Parameter λ primarily controls the steepness and behavior near $z = 0$, while β controls the tail behavior as z approaches 1. The parameter δ acts as a shape-controlling parameter, influencing skewness, tail weight, and the overall concentration of probability mass. In particular, small values of δ produce more pronounced skewness and boundary concentration, whereas larger values modify the curvature and allow for more balanced or interior-peaked shapes, thereby enhancing distributional flexibility. Since these parameters affect different structural components of the model, they can be uniquely determined from the likelihood function under standard regularity conditions.

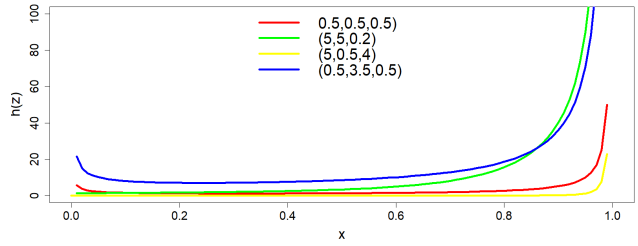


Figure 5 Hazard rate function of the PGDUS-K distribution for selected parameter combinations: (λ, β, δ) = (0.5, 0.5, 0.5), (5, 5, 0.2), (5, 0.5, 4), (0.5, 3.5, 0.5)

The illustrations in Figure 5 reveal that the PGDUS-K distribution can exhibit both increasing and bathtub-shaped hazard rates. This flexibility enhances its adaptability to various failure patterns, improves risk assessment, and ensures better model fitting across diverse fields such as reliability engineering, hydrology, and environmental studies, making it highly versatile and practical.

3.2. Moments

Moments provide essential information for describing the distributional shape and are useful in statistical inference and model comparison. The r^{th} raw moment of the PGDUS-K(λ, β, δ) distribution is given in (9), and its derivation is provided in Appendix A.

$$\mu_r^{'} = E(z^r) = \frac{e\beta\delta}{(e-1)^{\delta}} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{m!} \binom{\delta-1}{k} e^{\delta-k-1}(\delta-k)^m B(\frac{r}{\lambda}+1, \beta+m\beta). \quad (9)$$

3.2.1 Mean and variance

From (9), we get the mean and variance, respectively, as

$$E(z) = \mu_1^{'} = \frac{e\beta\delta}{(e-1)^{\delta}} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{m!} \binom{\delta-1}{k} e^{\delta-k-1}(\delta-k)^m B(\frac{1}{\lambda}+1, \beta+m\beta)$$

$$\begin{aligned} Var(Z) &= \mu_2^{'} - \mu_1^{'2} \\ &= [\frac{e\beta\delta}{(e-1)^{\delta}} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{m!} \binom{\delta-1}{k} e^{\delta-k-1}(\delta-k)^m B(\frac{2}{\lambda}+1, \beta+m\beta)] \\ &\quad - [\frac{e\beta\delta}{(e-1)^{\delta}} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{m!} \binom{\delta-1}{k} e^{\delta-k-1}(\delta-k)^m B(\frac{1}{\lambda}+1, \beta+m\beta)]^2. \end{aligned}$$

3.3. Generating functions

Generating functions of RVs are significant as they offer various methods to compute the moments. In addition, each distribution has a unique generating function, which distinguishes one distribution from another.

3.3.1 Moment generating function (MGF)

The MGF of PGDUS-K(λ, β, δ) is given by (10) and its derivation is provided in Appendix A.

$$M_Z(t) = E(e^{tZ}) = \frac{e\beta\delta}{(e-1)^\delta}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{k+m}}{m!n!}t^n\binom{\delta-1}{k}e^{\delta-k-1}(\delta-k)^m B(\frac{n}{\lambda}+1,\beta+m\beta). \quad (10)$$

3.3.2 Characteristic function

The Characteristic Function of PGDUS-K(λ, β, δ) is obtained as

$$\phi_Z(t) = E(e^{itZ}) = \frac{e\beta\delta}{(e-1)^\delta}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{k+m}}{m!n!}(it)^n\binom{\delta-1}{k}e^{\delta-k-1}(\delta-k)^m B(\frac{n}{\lambda}+1,\beta+m\beta),$$

where $i=\sqrt{-1}$ is the unit imaginary number.

3.4. Quantiles

The p^{th} quantile function or the inverse CDF, denoted by $Q(p)$ of PGDUS-K(λ,β,δ) distribution is defined by $Q(p)=G^{-1}(p)$. It is non-decreasing and gives the probability assignment in terms of the quantiles of the distribution. It is useful in RV generation and percentile analysis. It is obtained by solving $G(Q(p))=p$, where $0<p<1$. i.e,

$$\left[\frac{e^{1-(1-Q(p)^\lambda)^\beta}-1}{e-1}\right]^\delta = p \implies Q(p)=[1-[1-log[1+p^{\frac{1}{\delta}}(e-1)]]^{\frac{1}{\beta}}]^{\frac{1}{\lambda}}. \quad (11)$$

Median and Inter-Quartile Range (IQR) offer robust measures of location and variability for skewed reliability data. Median and IQR of PGDUS-K(λ,β,δ) distribution defined by $Q(0.5)$ and $Q(0.75)-Q(0.25)$, respectively, is given by

$$Q(0.5) = [1-[1-log[1+(0.5)^{\frac{1}{\delta}}(e-1)]]^{\frac{1}{\beta}}]^{\frac{1}{\lambda}}$$
$$IQR = [1-[1-log[1+(0.75)^{\frac{1}{\delta}}(e-1)]]^{\frac{1}{\beta}}]^{\frac{1}{\lambda}} - [1-[1-log[1+(0.25)^{\frac{1}{\delta}}(e-1)]]^{\frac{1}{\beta}}]^{\frac{1}{\lambda}}.$$

3.5. Skewness and kurtosis

The shapes of PGDUS-K can be studied using Galton's skewness and Moors' kurtosis see (Moors, 1988) given by

$$Skewness = \frac{Q(0.75)-2Q(0.5)+Q(0.25)}{Q(0.75)-Q(0.25)} \quad (12)$$

$$Kurtosis = \frac{Q(0.875)-Q(0.625)+Q(0.375)-Q(0.125)}{Q(0.75)-Q(0.25)}. \quad (13)$$

Figure 6 plots the Galton's skewness for PGDUS-K at different values of λ and β when $\delta=2.5$ (red plot), β and δ when $\lambda=10$ (purple plot) and λ and δ when $\beta=0.1$ (green plot). It can be seen that the skewness can be positive, negative and zero. The red plot illustrates that the skewness is highly stable over a range of λ and β values, with only minor variations, indicating that the distribution's asymmetry does not change markedly within this parameter range, providing robustness in modeling scenarios where these parameters vary. The purple figure shows a significant decrease in skewness as δ increases, especially when β is low. This illustrates a sensitive region where changes in δ have a major impact on the distribution's asymmetry. This shows that precise adjustment of δ is required for accurate modeling. The third yellow plot shows an initial steady zone followed by a sharp fall in skewness as λ and δ approaches higher levels.

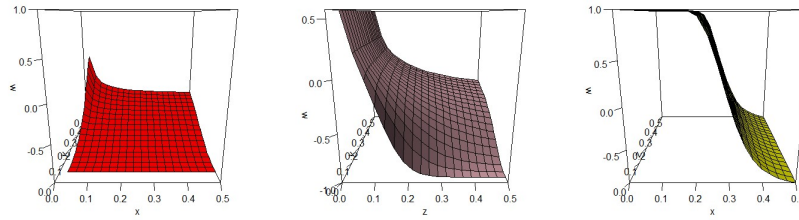


Figure 6 Plot of the skewness function of PGDUS-K distribution when $\delta = 2.5$ (red plot), $\lambda = 10$ (purple plot) and $\beta = 0.1$ (green plot), respectively

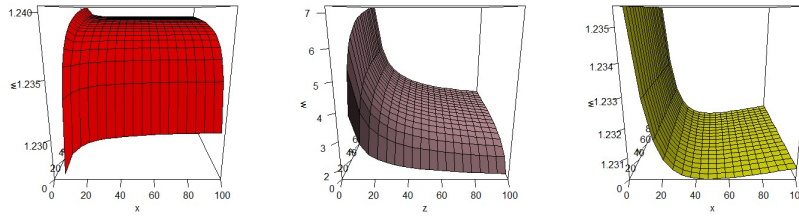


Figure 7 Plot of the kurtosis function of PGDUS-K distribution when $\delta = 10$ (red plot), $\lambda = 0.1$ (purple plot) and $\beta = 10$ (green plot) respectively

Figure 7 presents the Moors' Kurtosis for PGDUS-K at different values of λ and β when $\delta = 10$ (red plot), β and δ when $\lambda = 0.1$ (purple plot) and λ and δ when $\beta = 10$ (green plot). Areas where the kurtosis increases dramatically as the two parameters on the two axes increase indicate that the distribution gets more peaked and has heavier tails. In the context of reliability engineering, this could suggest a higher possibility of extreme values (failures). Understanding these areas aids risk assessment and management. Regions with a sharp fall in kurtosis indicate that the distribution grows flatter and has lighter tails. The curvature in the plot reveals how the interaction between the two varying parameters influences kurtosis. A pronounced curvature indicates strong interaction effects. A flat zone on the kurtosis plot implies that changing the parameters within this range has no significant effect on the kurtosis.

3.6. Entropy

The entropy of a RV Z with PDF $g(z)$ is a measure of variation of the uncertainty.

3.6.1 Rényi entropy

Rényi entropy measures distributional uncertainty and offers a flexible information-theoretic tool for evaluating randomness and reliability characteristics of the model. It is defined by

$$\tau_R(\gamma) = \frac{1}{1-\gamma} \log \int_0^1 g^\gamma(z)\, dz,\ \gamma > 0,\ \gamma \neq 1.$$

Then Rényi entropy is obtained as (for proof, see Appendix A),

$$\tau_R(\gamma) = \frac{1}{1-\gamma} \log \left[\frac{(e\lambda\beta\delta)^\gamma}{\lambda(e-1)^{\delta\gamma}} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{m!} \binom{\gamma(\delta-1)}{k} e^{\gamma(\delta-1)-k} (\gamma\delta - k)^m \right.$$
$$\left. B(\gamma + \frac{1-\gamma}{\lambda}, \gamma(\beta-1) + m\beta + 1) \right], \text{where } \gamma > 0,\ \gamma \neq 1.$$

4. Characterization Properties

A characterization is a statistical property that is unique to the associated model. Three characterizations of the PGDUS-K distribution are discussed here. The first one is the invariance property under linear and exponential transformation with K as base. The second property is closure under exponentiation, and the third is that the distribution of maximum of PGDUS-K variables follows PGDUS-K distribution. These properties preserves some of the distribution and relationship traits, allowing for more accurate analysis, modeling and effective problem solving.

4.1. Invariance property

Theorem 1 establishes that the PGDUS-K distribution is invariant under linear and exponentiation transformations, ensuring preservation of its structural form under linear and power modifications. Linear invariance implies stability under location and scale changes, while exponentiation invariance guarantees that power-transformed lifetimes retain the same transformation structure. These properties enhance the robustness and applicability of the model, particularly in rescaling and system-level aggregation contexts. The proofs are straightforward, and only one direction of implication is presented.

Theorem 1 *$Z \sim K(\lambda, \beta)$ and $W \sim PGDUS-K(\lambda, \beta, \delta)$, iff the distribution of*

(i) $W' = a + bW$ is obtained as the PGDUS transformation of $Z' = a + bZ$. [Invariant under linear transformation]

(ii) W^a is obtained as the PGDUS transformation of Z^a. [Invariant under exponentiation]

Proof:

(i) Given $Z \sim K(\lambda, \beta)$ and $W \sim PGDUS-K(\lambda, \beta, \delta)$. Then CDF of $Z' = a + bZ$ is

$$F_{Z'}(z) = F_{a+bZ}(z) = F_Z(\frac{z-a}{b}) = 1 - (1 - (\frac{z-a}{b})^{\lambda})^{\beta}, \lambda > 0,\ \beta > 0$$

Then the CDF corresponding to PGDUS transformation of Z' is given by

$$F(z) = \left[\frac{e^{1-(1-(\frac{z-a}{b})^{\lambda})^{\beta}} - 1}{e-1}\right]^{\delta},\ \lambda > 0,\ \beta > 0,\ \delta > 0. \tag{14}$$

Now $W \sim PGDUS-K(\lambda, \beta, \delta)$. Then the distribution of $W' = a + bW$ is given by

$$F_{W'}(w) = F_{a+bW}(w) = F_W(\frac{w-a}{b}) = \left[\frac{e^{1-(1-(\frac{w-a}{b})^{\lambda})^{\beta}} - 1}{e-1}\right]^{\delta}, \lambda, \beta, \delta > 0. \tag{15}$$

(ii) Given $Z \sim K(\lambda, \beta)$ and $W \sim PGDUS-K(\lambda, \beta, \delta)$. Then the CDF of Z^a is

$$F_{Z^a}(z) = F_Z(z^{1/a}) = (1 - z^{\lambda/a})^{\beta},\ \lambda > 0,\ \beta > 0$$

Then the CDF corresponding to PGDUS transformation of Z^a is given by

$$F(z) = \left[\frac{e^{1-(1-z^{\lambda/a})^{\beta}} - 1}{e-1}\right]^{\delta},\ \lambda > 0,\ \beta > 0,\ \delta > 0. \tag{16}$$

Now $W \sim PGDUS-K(\lambda, \beta, \delta)$. Then the distribution of W^a is given by:

$$F_{W^a}(w) = F_W(w^{1/a}) = \left[\frac{e^{1-(1-w^{\lambda/a})^{\beta}} - 1}{e-1}\right]^{\delta},\ \lambda > 0,\ \beta > 0, \delta > 0. \tag{17}$$

4.2. Closure under exponentiation

Unlike variables distributed according to the Beta distribution, Kumaraswamy variables possess a unique property: they remain within the K distribution even under exponentiation (see Mitnik (2013)). This distinctive feature extends to the PGDUS-K distribution and facilitates analytical tractability in reliability modeling, particularly in contexts involving power transformations of lifetimes, stress adjustments, or system aggregation mechanisms.

Theorem 2 *If $W = Z^p$ and $p > 0$, then $Z \sim$ PGDUS-K(λ, β, δ) if and only if $W \sim$ PGDUS-K$(\frac{\lambda}{p}, \beta, \delta)$.*

Proof: First to prove the sufficient part, given that $W = Z^p$ and $Z \sim$ PGDUS-K(λ, β, δ) . Then by using change of variable technique, we have

$$\begin{aligned} g_W(w) =& g_Z(w^{\frac{1}{p}}) \left|\tfrac{dz}{dw}\right| \\ =& \frac{\lambda\beta\delta}{(e-1)^{\delta}} w^{\frac{\lambda-1}{p}} (1-w^{\frac{\lambda}{p}})^{\beta-1} e^{1-(1-w^{\frac{\lambda}{p}})^{\beta}} (e^{1-(1-w^{\frac{\lambda}{p}})^{\beta}} - 1)^{\delta-1} . \frac{1}{p} w^{\frac{1}{p}-1} \\ =& \frac{\frac{\lambda}{p}\beta\delta}{(e-1)^{\delta}} w^{\frac{\lambda}{p}-1} (1-w^{\frac{\lambda}{p}})^{\beta-1} e^{1-(1-w^{\frac{\lambda}{p}})^{\beta}} (e^{1-(1-w^{\frac{\lambda}{p}})^{\beta}} - 1)^{\delta-1}. \end{aligned}$$

That is, we have $W \sim$ PGDUS-K$(\frac{\lambda}{p}, \beta, \delta)$.

Similarly, we can prove that if $W \sim$ PGDUS-K$(\frac{\lambda}{p}, \beta, \delta)$, then the distribution of $Z = W^{1/p} \sim$ PGDUS-K(λ, β, δ) and prove the necessary part.

4.3. Distribution of the order statistics

In reliability theory, finding the distribution of order statistics proves essential for studying serial, parallel and k-out-of-n systems. In series systems, the minimum order statistic determines the system's reliability, whereas in parallel systems, the maximum order statistic does. In k-out-of-n systems, reliability involves intermediate order statistics. Hence understanding order statistics improves system reliability and guides maintenance and design decisions.

4.3.1 The distribution of single order statistics

Proposition 1 *Let $Z_{(1)}, Z_{(2)}, ..., Z_{(n)}$ be the order statistic of i.i.d. RVs $Z_1, Z_2, ..., Z_n \sim PGDUS-K(\lambda, \beta, \delta)$, then the distribution of r^{th} order statistic is given by*

$$g_{Z_{(r)}}(z) = \tfrac{n!}{(r-1)!(n-r)!} \; \tfrac{\delta\lambda\beta}{(e-1)^{n\delta}} \; z^{\lambda-1}(1-z^{\lambda})^{\beta-1} \; e^{1-(1-z^{\lambda})^{\beta}} \left[e^{1-(1-z^{\lambda})^{\beta}} - 1\right]^{r\delta-1} \left[(e-1)^{\delta} - (e^{1-(1-z^{\lambda})^{\beta}} - 1)^{\delta}\right]^{n-r},$$

where $0 < z < 1, \; \lambda > 0, \; \beta > 0, \; \delta > 0$.

Corollary 1 *If $r = n$, we get the distribution of $Z_{(n)} = max(Z_1, Z_2, ..., Z_n)$ as*

$$g_{Z_{(n)}}(z) = \frac{\lambda\beta n\delta}{(e-1)^{n\delta}} \; z^{\lambda-1}(1-z^{\lambda})^{\beta-1} \; e^{1-(1-z^{\lambda})^{\beta}} \left[e^{1-(1-z^{\lambda})^{\beta}} - 1\right]^{n\delta-1},$$

where $0 < z < 1, \; \lambda > 0, \; \beta > 0, \; \delta > 0$.

That is $Z_{(n)} \sim$ PGDUS-K $(\lambda, \beta, n\delta)$.

Remark 2 In the context of parallel systems with independent components following the PGDUS-K distribution, the reliability distribution of the system can be expressed as the maximum of the individual component distributions. Remarkably, this maximum distribution remains within the PGDUS-K family, thereby preserving the characteristics of the individual component distributions.

Corollary 2 *If $r = 1$, we get the distribution of $Z_{(1)} = min(Z_1, Z_2, ..., Z_n)$ as*

$$g_{Z_{(1)}}(z) = \tfrac{\lambda\beta n\delta}{(e-1)^{n\delta}} \; z^{\lambda-1}(1-z^{\lambda})^{\beta-1} \; e^{1-(1-z^{\lambda})^{\beta}} \left[e^{1-(1-z^{\lambda})^{\beta}} - 1\right]^{\delta-1} \left[(e-1)^{\delta} - (e^{1-(1-z^{\lambda})^{\beta}} - 1)^{\delta}\right]^{n-1},$$

where $0 < z < 1, \; \lambda > 0, \; \beta > 0, \; \delta > 0$.

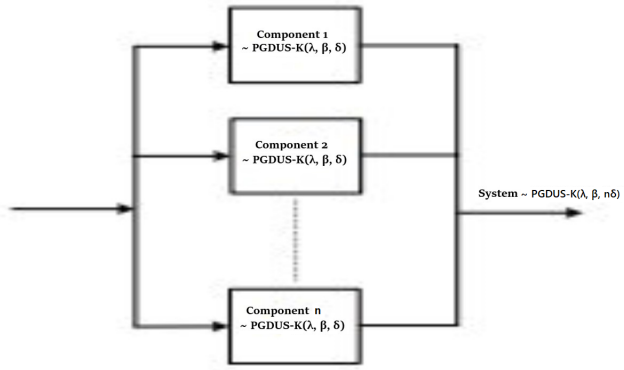


Figure 8 Parallel PGDUS-K system

4.3.2 Joint distribution of two order statistics

Proposition 2 *The joint PDF of two order statistics $Z_{(r)}$ and $Z_{(s)}$, $1 \leq r < s \leq n$ is*

$$g_{Z_{(r)},Z_{(s)}}(z_r, z_s) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \frac{(\lambda\beta\delta)^2}{(e-1)^{n\delta}} (z_r z_s)^{\lambda-1}((1-z_r^{\lambda})(1-z_s^{\lambda}))^{\beta-1}$$
$$e^{2-(1-z_r^{\lambda})^{\beta}-(1-z_s^{\lambda})^{\beta}}[e^{1-(1-z_r^{\lambda})^{\beta}}-1]^{r\delta-1}\left[e^{1-(1-z_s^{\lambda})^{\beta}}-1\right]^{\delta-1}$$
$$[(e^{1-(1-z_r^{\lambda})^{\beta}}-1)^{\delta}-(e^{1-(1-z_s^{\lambda})^{\beta}}-1)^{\delta}]^{s-r-1}[(e-1)^{\delta}-(e^{1-(1-z_s^{\lambda})^{\beta}}-1)^{\delta}]^{n-s},$$

where $0 < z_r, z_s < 1,\ \lambda > 0,\ \beta > 0,\ \delta > 0.$

Corollary 3 *If we take $s = r+1$, we get the distribution of two consecutive order statistics as*

$$g_{Z_{(r)},Z_{(r+1)}}(z, w) = \frac{(\lambda\beta\delta)^2}{(e-1)^{n\delta}} \binom{n}{r-1} (zw)^{\lambda-1}((1-z^{\lambda})(1-w^{\lambda}))^{\beta-1} e^{2-(1-z^{\lambda})^{\beta}-(1-w^{\lambda})^{\beta}}$$
$$[e^{1-(1-z^{\lambda})^{\beta}}-1]^{r\delta-1}[e^{1-(1-z^{\lambda})^{\beta}}-1]^{r\delta-1}\left[e^{1-(1-w^{\lambda})^{\beta}}-1\right]^{\delta-1}$$
$$[(e-1)^{\delta}-(e^{1-(1-w^{\lambda})^{\beta}}-1)^{\delta}]^{n-r-1}$$

where $0 < z, w < 1,\ \lambda > 0,\ \beta > 0,\ \delta > 0.$

Corollary 4 *For $r = 1$ and $s = n$, we obtain the joint distribution of extreme order statistics as*

$$g_{Z_{(1)},Z_{(n)}}(z, w) = \frac{n(n-1)(\lambda\beta\delta)^2}{(e-1)^{n\delta}}(zw)^{\lambda-1}((1-z^{\lambda})(1-w^{\lambda}))^{\beta-1}\ e^{2-(1-z^{\lambda})^{\beta}-(1-w^{\lambda})^{\beta}}$$
$$\left[[e^{1-(1-z^{\lambda})^{\beta}}-1][e^{1-(1-w^{\lambda})^{\beta}}-1]\right]^{\delta-1}[(e^{1-(1-z^{\lambda})^{\beta}}-1)^{\delta}-(e^{1-(1-w^{\lambda})^{\beta}}-1)^{\delta}]^{n-2}$$

where $0 < z, w < 1,\ \lambda > 0,\ \beta > 0,\ \delta > 0.$

5. Estimation

Estimation of parameters is essential for making conclusions about a population from sample data, which helps in decision-making. This section discuss five estimation methods namely, the method of moments, method of least squares, ML estimation, MPS and Bayesian estimation methods, and provided detailed descriptions for each.

5.1. Method of moment estimation

Let $Z_1, Z_2, \ldots, Z_n$ be a random sample of size n drawn from the PGDUS-K(λ, β, δ) distribution. The r^{th} raw moment of PGDUS-K(λ, β, δ) distribution is given by (9). Since we have to estimate three parameters λ, β and δ, we have to equate the first three raw moments, $\mu_r^{'}, r = 1, 2, 3$ to corresponding sample moments, as given below

$$\mu_1^{'} = \frac{1}{n}\sum_{i=1}^{n} Z_i; \;\; \mu_2^{'} = \frac{1}{n}\sum_{i=1}^{n} Z_i^2; \;\; \mu_3^{'} = \frac{1}{n}\sum_{i=1}^{n} Z_i^3. \tag{18}$$

To obtain the moment estimators, we solve the system of nonlinear equations in (18). Since explicit closed-form solutions are not available, numerical methods, such as iterative schemes based on the Newton–Raphson algorithm or general-purpose optimization routines like `optim()` function in R, are employed. The convergence of the numerical solution depends on the stability of the nonlinear system and the choice of starting values. The method is computationally efficient, as it avoids repeated evaluation of complex likelihood derivatives. Although the method of moments may exhibit lower statistical efficiency than the maximum likelihood estimates (MLE), particularly in small samples, it provides stable preliminary estimates and is therefore used to generate initial values for subsequent likelihood-based optimization.

5.2. Method of least squares estimation

Let $Z_1, Z_2, \ldots, Z_n$ be a random sample of size n drawn from the PGDUS-K(λ, β, δ) distribution. Let $Z_{(1)}, Z_{(2)}, \ldots, Z_{(n)}$ denote the corresponding order statistics. The least squares estimators are obtained by minimizing the objective function

$$L_s = \sum_{i=1}^{n} \left(F(Z_{(i)}) - \frac{i}{n+1} \right)^2, \tag{19}$$

with respect to parameters, where $F(\cdot)$ is the theoretical CDF of the PGDUS-K distribution and $\frac{i}{n+1}$ represents the empirical distribution function.

In order to minimize L_s, we find the partial derivatives of L_s with respect to the parameters λ, β and δ.

$$\frac{\partial L_s}{\partial \lambda} = \sum_{i=1}^{n} \left[\left(\frac{e^{1-(1-z^\lambda)^\beta} - 1}{e-1}\right)^\delta - \frac{i}{n+1} \right] (e^{1-(1-z^\lambda)^\beta} - 1)^{\delta-1} e^{1-(1-z^\lambda)^\beta} (1-z^\lambda)^{\beta-1} z^\lambda \log z \tag{20}$$

$$\frac{\partial L_s}{\partial \beta} = \sum_{i=1}^{n} \left[\left(\frac{e^{1-(1-z^\lambda)^\beta} - 1}{e-1}\right)^\delta - \frac{i}{n+1} \right] (e^{1-(1-z^\lambda)^\beta} - 1)^{\delta-1} e^{1-(1-z^\lambda)^\beta} (1-z^\lambda)^{\beta} \log(1-z^\lambda) \tag{21}$$

$$\frac{\partial L_s}{\partial \delta} = \sum_{i=1}^{n} \left[\left(\frac{e^{1-(1-z^\lambda)^\beta} - 1}{e-1}\right)^\delta - \frac{i}{n+1} \right] \left(\frac{e^{1-(1-z^\lambda)^\beta} - 1}{e-1}\right)^\delta \log\left(\frac{e^{1-(1-z^\lambda)^\beta} - 1}{e-1}\right) \tag{22}$$

Equating to zero and solving (20) - (22), we get the least square estimators of λ, β and δ. Since the objective function L_s is nonlinear in the parameters, the least squares estimates are obtained using numerical optimization procedures, such as iterative methods based on the Newton–Raphson algorithm or general-purpose optimization routines (e.g., `optim()` function in R). Convergence depends on the smoothness of the objective function and the choice of suitable starting values. Although least squares estimation is computationally straightforward and comparable to other optimization-based methods, it may exhibit lower statistical efficiency than ML estimation, particularly in small samples. Nevertheless, it provides a simple and useful alternative for distributional fitting.

5.3. Maximum likelihood (ML) estimation

If $Z_1, Z_2, ..., Z_n$ is a random sample from PGDUS-K(λ, β, δ) distribution, then its log-likelihood function is given by

$$\log L(\lambda,\beta,\delta) = n\log\lambda + n\log\beta + n\log\delta - n\delta\log(e-1) + (\lambda-1)\sum_{i=1}^{n}\log z_i + (\beta-1)\sum_{i=1}^{n}\log(1-z_i^{\lambda}) + (\delta-1)\sum_{i=1}^{n}\log(e^{1-(1-z_i^{\lambda})^{\beta}}-1) + \sum_{i=1}^{n}1-(1-z_i^{\lambda})^{\beta}.$$

The ML estimates of the parameters can be obtained by maximizing $\log L(\lambda,\beta,\delta)$. Partial derivatives of $logL$ with respect to λ,β and δ respectively are:

$$\frac{\partial\log L(\lambda,\beta,\delta)}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\log z_i - \beta\sum_{i=1}^{n}(1-z_i^{\lambda})^{\beta-1}z_i^{\lambda}\log z_i - (\beta-1)\sum_{i=1}^{n}\frac{z_i^{\lambda}\log z_i}{1-z_i^{\lambda}} + \beta(\delta-1)\sum_{i=1}^{n}\frac{e^{1-(1-z_i^{\lambda})^{\beta}}(1-z_i^{\lambda})^{\beta-1}z_i^{\lambda}\log z_i}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)} \tag{23}$$

$$\frac{\partial\log L(\lambda,\beta,\delta)}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}(1-z_i^{\lambda})^{\beta}\log(1-z_i^{\lambda}) + \sum_{i=1}^{n}\log(1-z_i^{\lambda}) - (\delta-1)\sum_{i=1}^{n}\frac{e^{1-(1-z_i^{\lambda})^{\beta}}(1-z_i^{\lambda})^{\beta}\log(1-z_i^{\lambda})}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)} \tag{24}$$

$$\frac{\partial\log L(\lambda,\beta,\delta)}{\partial\delta} = \frac{n}{\delta} + n\log(e-1) + \sum_{i=1}^{n}\log(e^{1-(1-z_i^{\lambda})^{\beta}}-1) \tag{25}$$

The MLE, denoted by $\hat{\boldsymbol{\theta}} = (\hat{\lambda},\hat{\beta},\hat{\delta})$, is obtained by equating Equations (23)-(25) to zero and solving the resulting nonlinear system of equations. Since closed-form solutions are not available, numerical optimization techniques are employed. The method of moments estimates are used as initial values to facilitate stable convergence, and the `nlm()` function in R is applied to obtain $\hat{\boldsymbol{\theta}}$. Convergence depends on the smoothness of the log-likelihood surface and the choice of suitable starting values; in this study, method of moments estimates are used to ensure stable initialization. In general, likelihood-based estimation is computationally more demanding than moment-based methods but provides superior statistical efficiency under standard regularity conditions.

To construct the asymptitoc confidence intervals for λ, β, and δ, we use the Fisher information matrix,

$$I(\boldsymbol{\theta}) = -E\left[\begin{pmatrix} \frac{\partial^2\ell}{\partial\lambda^2} & \frac{\partial^2\ell}{\partial\lambda\partial\beta} & \frac{\partial^2\ell}{\partial\lambda\partial\delta} \\ \frac{\partial^2\ell}{\partial\beta\partial\lambda} & \frac{\partial^2\ell}{\partial\beta^2} & \frac{\partial^2\ell}{\partial\beta\partial\delta} \\ \frac{\partial^2\ell}{\partial\delta\partial\lambda} & \frac{\partial^2\ell}{\partial\delta\partial\beta} & \frac{\partial^2\ell}{\partial\delta^2} \end{pmatrix}\right],$$

where $\ell = \log L$ denotes the log-likelihood function. Expansion of second derivative terms are given in Appendix B.

Under standard regularity conditions, the MLE $\hat{\boldsymbol{\theta}} = (\hat{\lambda},\hat{\beta},\hat{\delta})$ is consistent and asymptotically normal, i.e.

$$\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \xrightarrow{d} N\left(0, I^{-1}(\boldsymbol{\theta})\right).$$

In practice, the observed Fisher information matrix evaluated at $\hat{\boldsymbol{\theta}}$ is used to estimate the asymptotic covariance matrix. Thus, the approximate $100(1-\eta)\%$ asymptotic confidence intervals are given by

$$\hat{\lambda} \pm z_{\eta/2}\sqrt{\mathrm{Var}(\hat{\lambda})}, \quad \hat{\beta} \pm z_{\eta/2}\sqrt{\mathrm{Var}(\hat{\beta})}, \quad \hat{\delta} \pm z_{\eta/2}\sqrt{\mathrm{Var}(\hat{\delta})},$$

where $z_{\eta/2}$ is the upper $\eta/2$ quantile of the standard normal distribution.

While ML estimation is a more popular method for estimating parameters, we could see that it doesn't work for every parametric combinations of the PGDUS-K model. While conducting a thorough examination of the existence of MLEs based on findings from the paper Popović et al. (2016), we could see that the ML estimation technique fails to give results due to issues like flat likelihood surfaces. A numerical exploration of the score equations revealed instability in certain regions of the parameter space. For extreme parameter values, evaluation of the score functions resulted in overflow or undefined numerical values, indicating severe ill-conditioning of the likelihood surface. In other regions, the score functions exhibited weak curvature and failed to produce stable interior roots during iterative optimization. These behaviors suggest that the likelihood surface may become nearly flat or boundary-dominated for some parameter configurations, thereby hindering reliable computation of the MLE. Consequently, the existence of stable MLEs cannot be uniformly guaranteed from a numerical perspective.

To mitigate these issues, the MPS method is adopted, as it is generally less sensitive to flat or ill-conditioned likelihood surfaces and tends to provide more stable parameter estimates in practice.

5.4. Maximum product spacing estimation

MPS method introduced by Cheng and Amin (1983) and Ranneby (1984), guarantees consistent estimators regardless of the existence of the ML method. Let $Z_1, Z_2, \ldots, Z_n$ be a random sample from the PGDUS-K(λ, β, δ) distribution, and let $Z_{(1)} \leq Z_{(2)} \leq \cdots \leq Z_{(n)}$ denote the corresponding order statistics. The MPS estimates are obtained by maximizing the geometric mean of the spacings or equivalently by maximizing the log-geometric mean of the spacings given by

$$\log G(\boldsymbol{\theta}) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log D_i,$$

where the spacings, $D_i = F_Z\big(Z_{(i)}; \boldsymbol{\theta}\big) - F_Z\big(Z_{(i-1)}; \boldsymbol{\theta}\big), \quad i = 1, 2, \ldots, n+1,$, $\boldsymbol{\theta} = (\lambda, \beta, \delta)$, $F_Z(\cdot; \boldsymbol{\theta})$ is the CDF given in (6), $F_Z(Z_{(0)}; \boldsymbol{\theta}) = 0$, and $F_Z(Z_{(n+1)}; \boldsymbol{\theta}) = 1$. Substituting the expression of F_Z from (6), we obtain

$$\log G(\boldsymbol{\theta}) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log \left[\left(\frac{e^{1-(1-Z_{(i)}^{\lambda})^{\beta}} - 1}{e-1} \right)^{\delta} - \left(\frac{e^{1-(1-Z_{(i-1)}^{\lambda})^{\beta}} - 1}{e-1} \right)^{\delta} \right].$$

To maximize $\log G$, we find its partial derivatives with respect to λ, β, δ as follows:

$$\begin{aligned} \frac{\partial \log G}{\partial \lambda} =& \frac{\delta\beta}{(n+1)(e-1)^{\delta}} \sum_{i=1}^{n+1} \frac{(e^{1-(1-z_i^{\lambda})^{\beta}} - 1)^{\delta-1} e^{1-(1-z_i^{\lambda})^{\beta}} (1-z_i^{\lambda})^{\beta-1} z_i^{\lambda} \log z_i}{\left(\frac{e^{1-(1-z_i^{\lambda})^{\beta}}-1}{e-1}\right)^{\delta} - \left(\frac{e^{1-(1-z_{i-1}^{\lambda})^{\beta}}-1}{e-1}\right)^{\delta}} \\ & - \frac{(e^{1-(1-z_{i-1}^{\lambda})^{\beta}} - 1)^{\delta-1} e^{1-(1-z_{i-1}^{\lambda})^{\beta}} (1-z_{i-1}^{\lambda})^{\beta-1} z_{i-1}^{\lambda} \log z_{i-1}}{\left(\frac{e^{1-(1-z_i^{\lambda})^{\beta}}-1}{e-1}\right)^{\delta} - \left(\frac{e^{1-(1-z_{i-1}^{\lambda})^{\beta}}-1}{e-1}\right)^{\delta}} \end{aligned} \tag{26}$$

$$\begin{aligned} \frac{\partial \log G}{\partial \beta} =& \frac{-\delta}{(n+1)(e-1)^{\delta}} \sum_{i=1}^{n+1} \frac{(e^{1-(1-z_i^{\lambda})^{\beta}} - 1)^{\delta-1} e^{1-(1-z_i^{\lambda})^{\beta}} (1-z_i^{\lambda})^{\beta} \log(1-z_i^{\lambda})}{\left(\frac{e^{1-(1-z_i^{\lambda})^{\beta}}-1}{e-1}\right)^{\delta} - \left(\frac{e^{1-(1-z_{i-1}^{\lambda})^{\beta}}-1}{e-1}\right)^{\delta}} \\ & - \frac{(e^{1-(1-z_{i-1}^{\lambda})^{\beta}} - 1)^{\delta-1} e^{1-(1-z_{i-1}^{\lambda})^{\beta}} (1-z_{i-1}^{\lambda})^{\beta} \log(1-z_{i-1}^{\lambda})}{\left(\frac{e^{1-(1-z_i^{\lambda})^{\beta}}-1}{e-1}\right)^{\delta} - \left(\frac{e^{1-(1-z_{i-1}^{\lambda})^{\beta}}-1}{e-1}\right)^{\delta}} \end{aligned} \tag{27}$$

$$\frac{\partial \log G}{\partial \delta} = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{\left(\frac{e^{1-(1-z_i^{\lambda})^{\beta}}-1}{e-1}\right)^{\delta} \log\left(\frac{e^{1-(1-z_i^{\lambda})^{\beta}}-1}{e-1}\right) - \left(\frac{e^{1-(1-z_{i-1}^{\lambda})^{\beta}}-1}{e-1}\right)^{\delta} \log\left(\frac{e^{1-(1-z_{i-1}^{\lambda})^{\beta}}-1}{e-1}\right)}{\left(\frac{e^{1-(1-z_i^{\lambda})^{\beta}}-1}{e-1}\right)^{\delta} - \left(\frac{e^{1-(1-z_{i-1}^{\lambda})^{\beta}}-1}{e-1}\right)^{\delta}} \tag{28}$$

Equating Equations (26)-(28) to zero yields the MPS estimates, which are obtained numerically using the `optim()` function in R. The spacing-based objective function is generally smoother and less sensitive to flat or boundary-dominated regions than the log-likelihood, thereby improving numerical stability. Although convergence depends on appropriate starting values, MPS typically exhibits better curvature properties and computational stability while maintaining complexity comparable to likelihood-based methods.

5.5. Bayesian estimation

In this section, parameter estimation is carried out within a Bayesian framework. Let $Z = (Z_1, Z_2, \ldots, Z_n)$ be a random sample from the PGDUS-K distribution with parameter vector $\boldsymbol{\theta} = (\lambda, \beta, \delta)$. Bayesian estimation incorporates prior information about the parameters and combines it with the sample information through the likelihood function to obtain the posterior distribution.

The joint posterior density of $\boldsymbol{\theta}$ is given by

$$\phi(\boldsymbol{\theta} \mid Z) = \frac{L(\boldsymbol{\theta} \mid Z)\, f(\boldsymbol{\theta})}{\int_{\boldsymbol{\theta}} L(\boldsymbol{\theta} \mid Z)\, f(\boldsymbol{\theta})\, d\boldsymbol{\theta}},$$

where $L(\boldsymbol{\theta} \mid Z)$ denotes the likelihood function and $f(\boldsymbol{\theta})$ is the joint prior density.

Under the squared error loss function, the Bayes estimator of any function $h(\boldsymbol{\theta})$ is the posterior mean, given by

$$\hat{h}_B = E_{\boldsymbol{\theta}|Z}(h(\boldsymbol{\theta})) = \frac{\int_{\boldsymbol{\theta}} h(\boldsymbol{\theta}) L(\boldsymbol{\theta} \mid Z) f(\boldsymbol{\theta})\, d\boldsymbol{\theta}}{\int_{\boldsymbol{\theta}} L(\boldsymbol{\theta} \mid Z) f(\boldsymbol{\theta})\, d\boldsymbol{\theta}}.$$

Lindley's Approximation

Since the Bayes estimator $\hat{h}_B = E_{\boldsymbol{\theta}|Z}[h(\boldsymbol{\theta})]$ does not admit a closed-form expression, the approximation proposed by Lindley (1980) is employed to approximate the ratio of integrals. For a three-parameter model (Ali and Al-Kanani (2021)) $\boldsymbol{\theta} = (\lambda, \beta, \delta)$, Lindley's approximation to $I(Z) = E_{\boldsymbol{\theta}|Z}[u(\boldsymbol{\theta})]$ is given by

$$I(Z) \approx u + \sum_{i=1}^{3} u_i \theta_i + \theta_4 + \theta_5 + \frac{1}{2} \sum_{i=1}^{3} A_i \left(\sum_{j=1}^{3} u_j \sigma_{ij} \right),$$

where all quantities are evaluated at the MLE $\hat{\boldsymbol{\theta}}$ and

$$u_i = \frac{\partial u}{\partial \theta_i}, \quad u_{ij} = \frac{\partial^2 u}{\partial \theta_i \partial \theta_j}, \quad \rho = \log f(\boldsymbol{\theta}), \quad \rho_i = \frac{\partial \rho}{\partial \theta_i},$$

$$D_{ij} = \frac{\partial^2 L}{\partial \theta_i \partial \theta_j}, \quad D_{ijk} = \frac{\partial^3 L}{\partial \theta_i \partial \theta_j \partial \theta_k}.$$

Denote $\Sigma = (\sigma_{ij})$ the inverse of the observed information matrix $(-D_{ij})$, i.e., $\Sigma = (-D_{ij})^{-1}$.

Further,

$$\theta_i = \sum_{j=1}^{3} \rho_j \sigma_{ij}, \quad i = 1, 2, 3,$$

$$\theta_4 = u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23},$$

$$\theta_5 = \frac{1}{2} \left(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33} \right),$$

and

$$A_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \sigma_{jk} D_{jki}, \quad i = 1, 2, 3.$$

Since the parameters λ, β, and δ are strictly positive, independent Gamma priors are assumed for each parameter. Specifically, $\lambda \sim \text{Gamma}(a, r)$, $\beta \sim \text{Gamma}(b, s)$, $\delta \sim \text{Gamma}(c, t)$, with joint prior density

$$f(\boldsymbol{\theta}) \propto \lambda^{a-1}\beta^{b-1}\delta^{c-1}\exp(-r\lambda - s\beta - t\delta), \text{ where } a, b, c > 0 \text{ and } r, s, t > 0.$$

The independent Gamma priors are adopted due to their support on $(0, \infty)$ and their flexibility in modeling various shapes through the choice of hyperparameters. The independence assumption simplifies computation and reflects the absence of strong prior dependence structure among the parameters. In the absence of substantive prior information, weakly informative hyperparameter values are selected so that the posterior inference is primarily driven by the likelihood.

Using Lindley's approximation, the Bayes estimators of the parameters under squared error loss are given by

$$\hat{\lambda}_B = \hat{\lambda} + \theta_1 + \frac{1}{2}(A_1\sigma_{11} + A_2\sigma_{21} + A_3\sigma_{31}),$$
$$\hat{\beta}_B = \hat{\beta} + \theta_2 + \frac{1}{2}(A_1\sigma_{12} + A_2\sigma_{22} + A_3\sigma_{32}),$$
$$\hat{\delta}_B = \hat{\delta} + \theta_3 + \frac{1}{2}(A_1\sigma_{13} + A_2\sigma_{23} + A_3\sigma_{33}),$$

where the detailed expressions of the all the terms are provided in Appendix B.

The Metropolis - Hastings (MH) algorithm

The MH algorithm (Metropolis et al. (1953); Hastings (1970)) is a Markov Chain Monte Carlo method used to generate samples from complex posterior distributions. It is widely applied in Bayesian inference when the posterior distribution cannot be obtained in closed form.

Target Distribution ($\pi(\boldsymbol{\theta} \mid \mathbf{z})$): The posterior distribution, given by

$$\pi(\boldsymbol{\theta} \mid \mathbf{z}) \propto L(\boldsymbol{\theta} \mid \mathbf{z})\pi(\boldsymbol{\theta}),$$

where $L(\boldsymbol{\theta} \mid \mathbf{z})$ is the likelihood function and $\pi(\boldsymbol{\theta})$ is the prior distribution.

Proposal Distribution ($q(\boldsymbol{\theta}^* \mid \boldsymbol{\theta}^{(t)})$): A distribution used to generate candidate values at each iteration.

The steps of the MH algorithm are as follows:

1. Choose an initial value $\boldsymbol{\theta}^{(0)}$.
2. At iteration t, generate a candidate value $\boldsymbol{\theta}^*$ from the proposal distribution $q(\boldsymbol{\theta}^* \mid \boldsymbol{\theta}^{(t)})$.
3. Compute the acceptance probability

$$\alpha = \min\left\{1, \frac{\pi(\boldsymbol{\theta}^* \mid \mathbf{z})\, q(\boldsymbol{\theta}^{(t)} \mid \boldsymbol{\theta}^*)}{\pi(\boldsymbol{\theta}^{(t)} \mid \mathbf{z})\, q(\boldsymbol{\theta}^* \mid \boldsymbol{\theta}^{(t)})}\right\}.$$

4. Generate $u \sim U(0, 1)$.
   - If $u \leq \alpha$, set $\boldsymbol{\theta}^{(t+1)} = \boldsymbol{\theta}^*$.
   - Otherwise, set $\boldsymbol{\theta}^{(t+1)} = \boldsymbol{\theta}^{(t)}$.
5. Repeat Steps 2–4 for a sufficiently large number of iterations.
6. After discarding an initial burn-in period, the remaining samples $\{\boldsymbol{\theta}^{(t)}\}$ are used to approximate posterior summaries.

6. Simulation

In this section, a simulation study is conducted to evaluate the performance of the ML, MPS and Bayesian estimates for the PGDUS-K distribution. Samples of sizes 100, 200, 500, and 1000 are generated from the PGDUS-K distribution using the inversion method, with parameters set to $\lambda = 0.5$, $\beta = 0.5$, $\delta = 0.5$ (SET 1) and $\lambda = 5$, $\beta = 0.2$, $\delta = 3.5$ (SET 2). For each sample size, the mean estimates, biases and mean squared errors (MSEs) are calculated. For the ML estimates, coverage probabilities (CPs) and standard errors (SEs) are also computed.

Table 1 ML Estimate, Bias, MSE, SE and CP values for PGDUS-K model at SET 1 ($\lambda = 0.5$, $\beta = 0.5$, $\delta = 0.5$) and SET 2 ($\lambda = 5$, $\beta = 0.2$, $\delta = 3.5$)

Parameter	l	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\delta}$	MSE			SE			CP		
Set		(Bias)	(Bias)	(Bias)	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\delta}$
	100	0.6588	0.5025	0.7107	0.2611	0.0070	0.6997	0.4258	0.0834	0.7645	0.8764	0.9211	0.8535
		(0.1588)	(0.0025)	(0.2107)									
SET 1	200	0.5683	0.5021	0.6100	0.1097	0.0037	0.2447	0.3057	0.0602	0.3995	0.9104	0.9258	0.8947
		(0.0683)	(0.0021)	(0.1100)									
	500	0.5319	0.4992	0.5289	0.0429	0.0015	0.0334	0.1937	0.0388	0.1710	0.9308	0.9429	0.9078
		(0.0319)	(-0.0007)	(0.0289)									
	1000	0.5143	0.5001	0.5115	0.0183	0.0007	0.0119	0.1362	0.0275	0.1088	0.9509	0.9500	0.9380
		(0.0143)	(0.0001)	(0.0115)									
	100	5.6111	0.2049	3.9336	0.8215	0.0214	0.8854	0.6039	0.0150	1.0434	0.9130	0.9070	0.9380
		(0.6111)	(0.0049)	(0.4336)									
SET 2	200	5.0283	0.2012	3.5328	0.1170	0.0033	0.0385	0.1046	0.0056	0.1937	0.9480	0.9339	0.9285
		(0.0283)	(0.0012)	(0.0328)									
	500	5.0084	0.2003	3.506	0.0015	0.0007	0.0066	0.0384	0.0026	0.0814	0.955	0.941	0.944
		(0.0084)	(0.0003)	(0.0061)									
	1000	5.0006	0.20004	3.5003	0.00007	0.00004	0.0005	0.0086	0.0007	0.0224	0.9499	0.9552	0.9504
		(0.0.0006)	(0.00004)	(0.0003)									

The results, presented in Table 1, indicate that as the sample size l increases, the bias and MSE of the estimators decrease, and the estimated values of the parameters move closer to their corresponding true values. This behavior suggests that the estimators possess desirable large-sample properties and provide more accurate inference when larger samples are available. In addition, the SEs of the estimates also decrease as the sample size increases, reflecting a reduction in the variability of the estimators. Furthermore, the CP of the confidence intervals shows a tendency to approach the nominal confidence level as the sample size becomes larger.

Table 2 MPS Estimate, Bias and MSE values for PGDUS-K model at SET 1 ($\lambda = 0.5$, $\beta = 0.5$, $\delta = 0.5$) and SET 2 ($\lambda = 5$, $\beta = 0.2$, $\delta = 3.5$)

Parameter	l	MPS Estimates			Bias			MSE		
Set		$\hat{\lambda}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\delta}$
	100	0.5326	0.5131	0.7169	0.0326	0.0131	0.2169	0.3348	0.0073	2.0752
SET 1	200	0.4748	0.5102	0.6708	-0.0251	0.0102	0.1708	0.1232	0.0039	1.9090
	500	0.5786	0.4915	0.5472	0.0786	0.0084	0.0427	0.0779	0.0061	0.2060
	1000	0.4818	0.5001	0.5443	-0.0018	0 .0001	0.0103	0.0124	0.0007	0.0190
	100	4.8939	0.2004	3.9156	-0.1060	0.0004	0.4156	4.2286	0.0137	2.2287
SET 2	200	4.9469	0.2003	3.5317	-0.0530	0.0003	0.0317	0.8210	0.0079	0.4409
	500	4.999	0.2001	3.4999	-0.0002	0.0001	0.00009	0.0122	0.0020	0.0648
	1000	5.000001	0.20001	3.5002	0.000001	0 .00001	0.00023	0.0014	0.0003	0.0099

Table 3 Bayesian Estimates, Biases and MSEs for PGDUS-K model at SET 1 ($\lambda = 0.5$, $\beta = 0.5$, $\delta = 0.5$) and SET 2 ($\lambda = 5$, $\beta = 0.2$, $\delta = 3.5$)

Parameter	l	Bayesian Estimates			Bias			MSE		
Set		$\hat{\lambda}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\delta}$
SET 1	100	2.5340	1.8370	0.2681	2.0340	1.3370	-0.2319	4.1374	1.0751	0.0538
	200	1.8524	1.2328	0.3679	1.3524	0.7328	-0.1321	3.0259	0.5378	0.0174
	500	1.4857	0.3384	0.3696	0.9857	-0.1615	-0.1303	1.8295	0.0261	0.0169
	1000	0.6995	0.6235	0.4881	0.1995	0.1235	-0.0119	0.0398	0.0152	0.0001
SET 2	100	3.3727	0.1851	4.1046	-1.6272	-0.0148	0.6046	3.4691	0.0003	0.6321
	200	3.3874	0.1897	3.9064	-1.1256	0.0102	0.4064	2.2478	0.0002	0.3512
	500	4.2493	0.1929	3.7725	-0.7506	-0.0071	0.2725	1.3188	0.0001	0.1903
	1000	4.4846	0.1952	3.6814	-0.5153	-0.0047	0.1814	0.8339	0.00007	0.0982

The simulation results obtained using the MPS and Bayesian estimation methods are presented in Tables 2 and 3, respectively. The results show that the bias and MSE decrease as the sample size l increases, indicating that the parameter estimates become more accurate and stable for larger samples. In the Bayesian approach, a relatively larger bias is observed for small sample sizes due to the stronger influence of the prior distribution when data are limited. In such cases, the posterior represents a balance between prior information and the likelihood, which may lead to slight shrinkage of the estimates away from the true values. However, as the sample size increases, the likelihood dominates the posterior distribution, resulting in reduced bias and improved estimation accuracy. Overall, both the MPS and Bayesian methods provide reliable parameter estimates, particularly for moderate to large sample sizes.

7. Application

To illustrate the applicability and performance of the proposed PGDUS distribution, we analyze two real datasets in this section.

7.1. Dataset 1

Dataset 1 in Table 4 consists of 34 observations of vinyl chloride concentrations (µg/L) collected from clean upgradient groundwater monitoring wells. The data, originally reported in Bhaumik and Gibbons (2006), were analyzed in the context of approximate prediction intervals for environmental monitoring. Due to the positive, continuous, and moderately skewed nature of the measurements, this dataset provides an appropriate empirical setting for evaluating the flexibility and goodness-of-fit characteristics of the proposed distribution.

Table 4 Dataset 1

5.1	1.2	1.3	0.6	0.5	2.4	0.5	1.1	8.0	0.8	0.4	0.6	0.9	0.4	2.0	0.5	5.3
3.2	2.7	2.9	2.5	2.3	1.0	0.2	0.1	0.1	1.8	0.9	2.0	4.0	6.8	1.2	0.4	0.2

We fitted PGDUS-K (λ, β, δ) distribution to the data set in Table 4, using ML, MPS and Bayesian methods and compared the results with existing distributions like DUS-K (Anakha and Chacko (2021)), K distribution Kumaraswamy (1980) and other parallel models like PGDUS-Inverse Kumaraswamy (PGDUS-IK) (Amrutha and Chacko Amrutha and Chacko (2024b) and PGDUS-Weibull (PGDUS-W) (Thomas and Chacko (2023)). Various goodness-of-fit measures like Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) and statistical tests like Kolmogorov–Smirnov (K-S) test are employed to evaluate the performance of each distribution.

Table 5 Comparison table of ML estimates for Dataset 1

Distribution	Estimates	K-S value	LL value	p-value	AIC	BIC
PGDUS-K	$\hat{\lambda} = 0.015$ $\hat{\beta} = 2.077$ $\hat{\delta} = 565.866$	0.073	23.942	0.992	-43.884	-40.832
PGDUS-W	$\hat{\lambda} = 0.483$ $\hat{\beta} = 39.630$ $\hat{\delta} = 3.454$	0.076	23.283	0.988	-42.566	-39.513
PGDUS-IK	$\hat{\lambda} = 7.139$ $\hat{\beta} = 0.023$ $\hat{\delta} = 34.137$	0.088	23.048	0.948	-40.097	-35.517
K	$\hat{\lambda} = 0.836$ $\hat{\beta} = 3.146$	0.120	21.999	0.703	-39.999	-36.946
DUS-K	$\hat{\lambda} = 0.698$ $\hat{\beta} = 3.211$	0.124	21.309	0.671	-38.618	-35.565

Table 6 Comparison table of MPS estimates for Dataset 1

Distribution	Estimates	K-S value	LL value	p-value	AIC	BIC
PGDUS-K	$\hat{\lambda} = 0.037$ $\hat{\beta} = 1.655$ $\hat{\delta} = 1.039$	0.190	21.979	0.172	-37.958	-33.379
PGDUS-W	$\hat{\lambda} = 0.788$ $\hat{\beta} = 6.784$ $\hat{\delta} = 1.240$	0.192	21.317	0.162	-36.635	-32.056
PGDUS-IK	$\hat{\lambda} = 5.852$ $\hat{\beta} = 0.792$ $\hat{\delta} = 1.204$	0.191	21.243	0.164	-36.486	-31.907
K	$\hat{\lambda} = 0.797$ $\hat{\beta} = 2.168$	0.210	20.461	0.099	-36.919	-32.866
DUS-K	$\hat{\lambda} = 0.662$ $\hat{\beta} = 2.289$	0.213	19.729	0.089	-35.458	-32.405

Table 7 Comparison table of Bayesian estimates for Dataset 1

Distribution	Estimates	K-S value	LL value	p-value	AIC	BIC
PGDUS-K	$\hat{\lambda} = 0.456$ $\hat{\beta} = 3.261$ $\hat{\delta} = 1.848$	0.112	19.457	0.784	-32.840	-28.261
PGDUS-W	$\hat{\lambda} = 0.372$ $\hat{\beta} = 2.443$ $\hat{\delta} = 1.619$	0.120	19.355	0.729	-32.710	-28.131
DUS-K	$\hat{\lambda} = 0.451$ $\hat{\beta} = 7.530$	0.118	17.129	0.711	-30.258	-27.205
K	$\hat{\lambda} = 0.531$ $\hat{\beta} = 1.481$	0.164	18.370	0.322	-30.740	-26.687
PGDUS-IK	$\hat{\lambda} = 7.991$ $\hat{\beta} = 1.927$ $\hat{\delta} = 0.862$	0.204	13.420	0.115	-29.914	-24.335

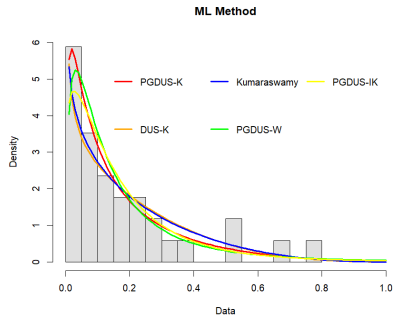


Figure 9 Fitted density plot for Dataset 1 using ML estimates

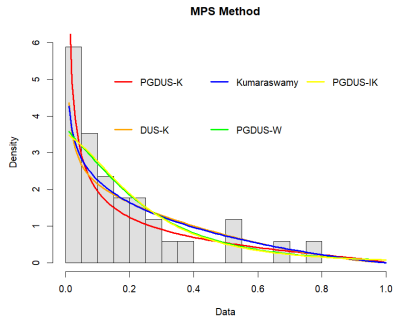


Figure 10 Fitted density plot for Dataset 1 using MPS estimates

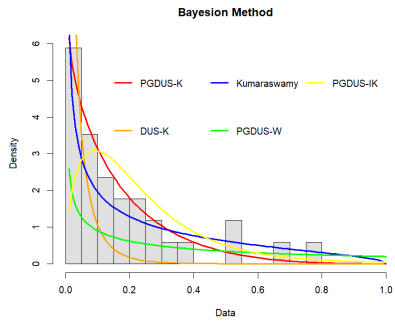


Figure 11 Fitted density plot for Dataset 1 using Bayesian estimates

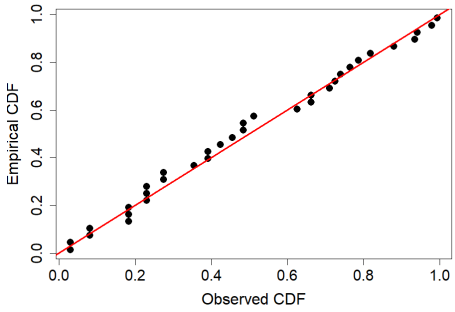


Figure 12 P-P Plot for Dataset 1

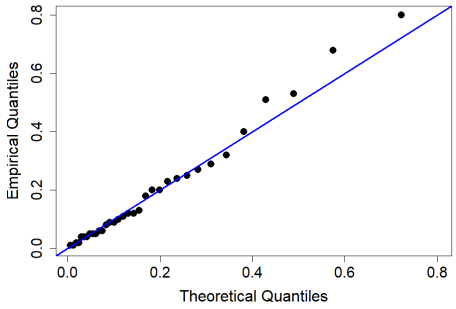


Figure 13 Q-Q Plot for Dataset 1

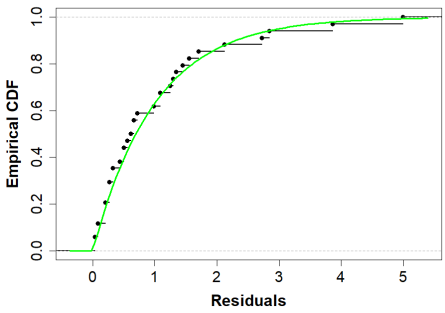


Figure 14 Cox-Snell Residual Plot for Dataset 1

The results in Tables 5, 6 and 7, shows that PGDUS-K(λ, β, δ) distribution has a superior fit to the data, capturing the underlying patterns and trends more effectively than the alternative distributions using all the three estimation techniques discussed in this paper. The findings from this analysis not only validate the efficiency of the PGDUS-K(λ, β, δ) distribution in this specific context but also suggest its potential applicability to other types of environmental and meteorological data. Figures

9-11 depicts the graphical representation of the data analysis.

The adequacy of the fitted PGDUS model was assessed using P–P plot, Q–Q plot, and Cox–Snell residual plot diagnostics. The P–P plot in Figure 12 further demonstrates strong agreement between the empirical and fitted cumulative probabilities, with the plotted points lying approximately along the 45° reference line, indicating that the model provides an adequate fit across most of the distributional range. The Q–Q plot in Figure 13 similarly shows that the empirical quantiles align well with the theoretical quantiles, with only minor deviations observed in the upper tail, suggesting a slight tendency of the model to under- or over-estimate extreme observations.

The Cox–Snell residual plot in Figure 14 indicates that the empirical distribution of the residuals closely follows the theoretical Exp(1) distribution, as the empirical cumulative curve lies in close agreement with the reference curve. This suggests that the overall model specification is appropriate and that no substantial systematic deviation remains in the transformed residual space. Overall, these graphical diagnostics collectively confirm that the proposed model provides a satisfactory fit to the data, with only negligible discrepancies at the distributional tails.

To assess whether the additional shape parameter δ significantly improves model fit, a likelihood ratio (LR) test was conducted to compare the full PGDUS-K model with its sub-model DUS-K obtained by setting $\delta = 1$. The LR test statistic used to compare two models is defined as

$$LR = 2\left(\ell_{\text{PGDUS-K}} - \ell_{\text{DUS-K}}\right),$$

where $\ell_{\text{PGDUS-K}}$ denotes the maximized log-likelihood under the full model (PGDUS-K), which contains three parameters and $\ell_{\text{DUS-K}}$ denotes the maximized log-likelihood under the restricted (sub-model) model (DUS-K), obtained by imposing the constraint $\delta = 1$. Under regularity conditions, the LR statistic (5.244, 1 d.f., p = 0.022) leads to rejection of $H_0 : \delta = 1$, indicating that the additional shape parameter significantly improves model fit. This conclusion is further supported by lower AIC and BIC values for the three-parameter model.

Table 8 Likelihood Ratio Test and Model Comparison

Model	Parameters	LL Value	LR	p-value	AIC	BIC
PGDUS-K	3	23.931			-41.862	-37.283
DUS-K (Sub-model - $\delta = 1$)	2	21.309	5.244	0.022	-38.618	-35.565

7.2. Dataset 2

Dataset 2 in Table 9, originally reported by Ghitany et al. (2008), contains the waiting times (in minutes) of 100 bank customers and is used to assess the performance of the proposed distribution.

Table 9 Dataset 2

0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1	2.6	2.7	2.9	3.1	3.2	3.3	3.5	4.0	4.1
4.2	4.2	4.3	4.3	4.4	4.4	4.7	4.8	4.9	4.9	3.6	4.6	4.7	5.0	5.3	5.5	5.7
5.7	6.1	6.2	6.2	6.2	6.3	6.7	6.9	7.1	7.1	7.1	7.1	7.4	7.6	7.7	8.0	8.2
8.6	8.6	8.6	8.8	8.8	8.9	8.9	9.5	9.6	9.7	9.8	10.7	10.9	11.0	11.0	11.1	11.2
11.2	11.5	11.9	12.4	12.5	12.9	13.0	13.1	13.3	13.6	13.7	13.9	14.1	15.4	15.4	17.3	17.3
18.1	18.2	18.4	18.9	19.0	19.9	20.6	21.3	21.4	21.9	23.0	27.0	31.6	33.1	38.5		

We fitted PGDUS-K (λ, β, δ) distribution to the dataset 2 in Table 9, using ML, MPS and Bayesian methods and compared the results with existing distributions like DUS-K, K, IK introduced by Abd Al-Fattah et al. (2017) and other parallel models like PGDUS-W and PGDUS-IK. Various goodness-of-fit measures like AIC, BIC and statistical tests like K-S test are employed to evaluate the performance of each distribution.

Table 10 Comparison table of ML estimates for Dataset 2

Distribution	Estimates	K-S value	LL value	p-value	AIC	BIC
PGDUS-K	$\hat{\lambda} = 0.530$ $\hat{\beta} = 7.771$ $\hat{\delta} = 7.920$	0.035	143.560	0.999	-281.120	-273.305
PGDUS-W	$\hat{\lambda} = 0.793$ $\hat{\beta} = 26.972$ $\hat{\delta} = 2.961$	0.038	143.395	0.998	-280.790	-272.98
DUS-K	$\hat{\lambda} = 1.237$ $\hat{\beta} = 19.549$	0.065	140.468	0.785	-276.936	-271.725
K	$\hat{\lambda} = 1.421$ $\hat{\beta} = 22.323$	0.062	141.317	0.828	-278.634	-273.424
PGDUS-IK	$\hat{\lambda} = 17.897$ $\hat{\beta} = 0.055$ $\hat{\delta} = 27.406$	0.037	143.405	0.999	-280.8112	-272.995
IK	$\hat{\lambda} = 17.781$ $\hat{\beta} = 2.378$	0.036	143.416	0.999	-282.832	-277.622

Table 11 Comparison table of MPS estimates for Dataset 2

Distribution	Estimates	K-S value	LL value	p-value	AIC	BIC
PGDUS-K	$\hat{\lambda} = 0.489$ $\hat{\beta} = 6.469$ $\hat{\delta} = 4.279$	0.069	142.582	0.726	-279.165	-271.350
PGDUS-W	$\hat{\lambda} = 0.780$ $\hat{\beta} = 23.622$ $\hat{\delta} = 2.340$	0.064	142.427	0.794	-278.855	-271.039
DUS-K	$\hat{\lambda} = 1.152$ $\hat{\beta} = 14.672$	0.094	139.329	0.333	-274.657	-269.447
K	$\hat{\lambda} = 1.325$ $\hat{\beta} = 16.265$	0.093	140.203	0.343	-276.405	-271.194
PGDUS-IK	$\hat{\lambda} = 15.395$ $\hat{\beta} = 0.073$ $\hat{\delta} = 17.947$	0.064	142.447	0.792	-278.894	-271.077
IK	$\hat{\lambda} = 15.246$ $\hat{\beta} = 0.013$	0.064	142.469	0.795	-280.939	-275.728

Table 12 Comparison table of Bayesian estimates for Dataset 2

Distribution	Estimates	K-S value	LL value	p-value	AIC	BIC
PGDUS-K	$\hat{\lambda} = 0.814$ $\hat{\beta} = 9.282$ $\hat{\delta} = 1.519$	0.072	139.522	0.677	-273.044	-265.228
PGDUS-W	$\hat{\lambda} = 0.719$ $\hat{\beta} = 8.512$ $\hat{\delta} = 1.615$	0.109	138.620	0.184	-271.240	-263.425
DUS-K	$\hat{\lambda} = 1.102$ $\hat{\beta} = 7.530$	0.073	138.452	0.651	-270.186	-264.976
K	$\hat{\lambda} = 1.174$ $\hat{\beta} = 12.334$	0.087	138.439	0.424	-272.879	-267.669
PGDUS-IK	$\hat{\lambda} = 18.264$ $\hat{\beta} = 1.142$ $\hat{\delta} = 1.914$	0.132	140.717	0.061	-275.435	-267.619
IK	$\hat{\lambda} = 11.387$ $\hat{\beta} = 1.453$	0.116	136.644	0.131	-269.288	-264.078

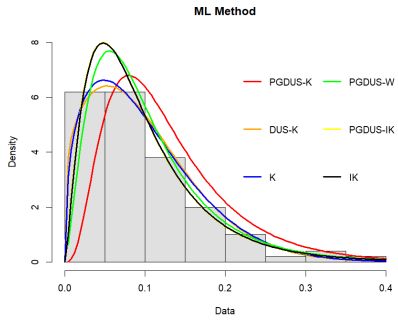


Figure 15 Fitted density plot for Dataset 2 using ML estimates

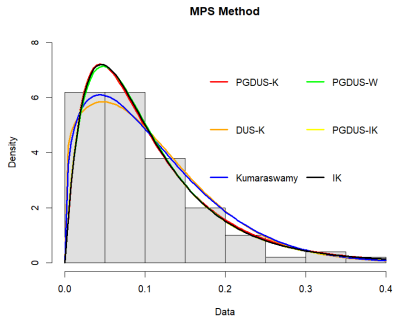


Figure 16 Fitted density plot for Dataset 2 using MPS estimates

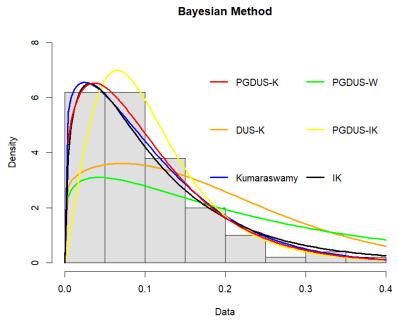


Figure 17 Fitted density plot for Dataset 2 using Bayesian estimates

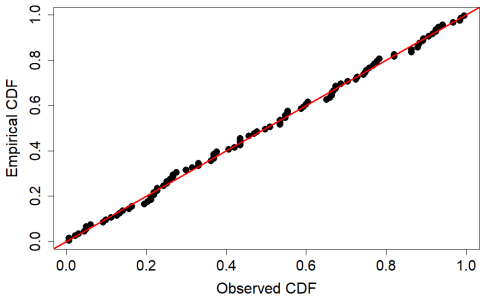


Figure 18 P-P Plot for Dataset 2

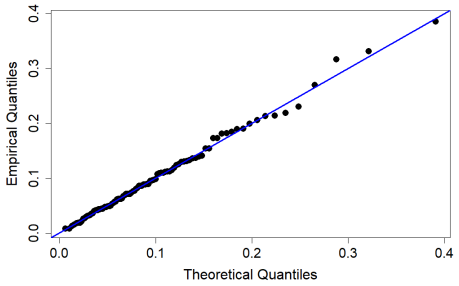


Figure 19 Q-Q Plot for Dataset 2

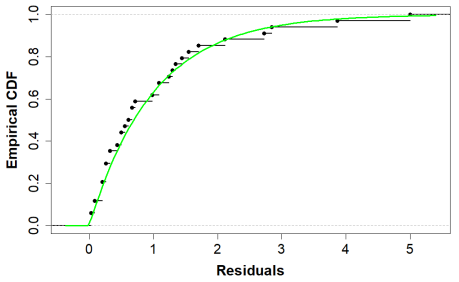


Figure 20 Cox-Snell Residual Plot for Dataset 2

Model adequacy was further assessed using P–P, Q–Q, and Cox–Snell residual plots. The P–P plot (Figure 18) shows close alignment of empirical and fitted cumulative probabilities along the reference line, indicating a good fit across most of the distribution. The Q–Q plot (Figure 19) demonstrates strong agreement between empirical and theoretical quantiles. The Cox–Snell residual plot (Figure 20) confirms that the residuals closely follow the Exp(1) distribution, indicating appropriate model specification and absence of substantial systematic deviation. Collectively, these diagnostics

confirm that the proposed PGDUS-K model provides an adequate fit.

Under standard regularity conditions, the LR statistic was calculated as 6.184. Since the p-value ($p = 0.012$) is below the 5% significance level, the null hypothesis $H_0 : \delta = 1$ is rejected, indicating that the additional shape parameter significantly enhances model flexibility and fit. This finding is further supported by the lower AIC and BIC values observed for the PGDUS-K model as shown in Table 13.

Table 13 Likelihood Ratio Test and Model Comparison for Dataset 2

Model	Parameters	LL Value	LR	p-value	AIC	BIC
PGDUS-K	3	143.560	–	–	-281.120	-273.305
DUS-K (Sub-model - $\delta = 1$)	2	140.468	6.184	0.012	-276.936	-271.725

8. Stress-Strength Reliability

In statistical literature, estimating stress-strength reliability is a common concern, where the relationship between a system's strength and the stress it encounters describes its reliability. If the system's strength is inadequate to withstand the stress, it fails. The first to suggest the Mann-Whitney statistic, a non-parametric estimator of Reliability, was Birnbaum (1956). Church and Harris (1970) were the first to introduce the idea of stress-strength reliability estimation.

8.1. Single component stress-strength reliability

In the stress-strength reliability framework, the reliability of a component or system is defined as the probability that its random strength exceeds the applied random stress. Here the strength and stress systems are represented by Z and W, respectively. Each system is modeled as a parallel configuration, where every component follows a DUS-K(λ, β) distribution. Specifically, the strength system contains δ_1 DUS-K components and the stress system contains δ_2 DUS-K components. Consequently, the system-level stress-strength reliability is given by

$$R = P(Z > W) = \int_0^\infty g_Z(z) G_W(z) dz$$

where $g_Z(z)$ and $G_W(z)$ denote the PDF and CDF of the corresponding PGDUS-K parallel systems.

$$R = \frac{\lambda\beta\delta_1}{(e-1)^{\delta_1+\delta_2}} \int_0^1 z^{\lambda-1}(1-z^\lambda)^{\beta-1} e^{1-(1-z^\lambda)^\beta} (e^{1-(1-z^\lambda)^\beta} - 1)^{\delta_1+\delta_2-1} = \frac{\delta_1}{(\delta_1+\delta_2)}.$$

This expression provides a direct and interpretable measure of reliability: the probability that the strength system exceeds the stress system depends solely on the relative number of components in each system. Specifically, increasing the number of strength components (δ_1) relative to the stress components (δ_2) enhances the system reliability, while a higher number of stress components reduces it. Thus, δ_1 and δ_2 can be viewed as weights of robustness and vulnerability, respectively, in the overall stress-strength performance of the system.

8.1.1 Estimation of stress-strength reliability

Here we use the ML technique to find the estimates of the unknown parameters. Let us consider the log of joint likelihood of the stress-strength sample $Z_1, Z_2, ..., Z_n, W_1, W_2, ..., W_m$, which is given by

$$
\begin{aligned}
logL = &(n+m)\log\lambda + (n+m)\log\beta + n\log\delta_1 + m\log\delta_2 - (n\delta_1 + m\delta_2)\log(e-1) \\
&+ (\lambda-1)(\sum_{i=1}^{n}\log z_i + \sum_{j=1}^{m}\log w_j) + (\beta-1)(\sum_{i=1}^{n}\log(1-z_i^{\lambda}) + \sum_{j=1}^{m}\log(1-w_j^{\lambda})) \\
&+ (\delta_1-1)\sum_{i=1}^{n}\log(e^{1-(1-z_i^{\lambda})^{\beta}}-1) + (\delta_2-1)\sum_{j=1}^{m}\log(e^{1-(1-w_j^{\lambda})^{\beta}}-1) + \\
&\sum_{i=1}^{n}(1-(1-z_i^{\lambda})^{\beta}) + \sum_{j=1}^{m}(1-(1-w_j^{\lambda})^{\beta}).
\end{aligned}
$$

The ML estimates of the parameters can be obtained by maximizing the logL. Partial derivatives of log-likelihood with respect to λ, β, δ_1 and δ_2 respectively are

$$
\begin{aligned}
\frac{\partial \log L}{\partial\lambda} =& \frac{(n+m)}{\lambda} - (\beta-1)\big[\sum_{i=1}^{n}\frac{z_i^{\lambda}\log z_i}{1-z_i^{\lambda}} + \sum_{j=1}^{m}\frac{w_j^{\lambda}\log w_j}{1-w_j^{\lambda}}\big] + \\
&+ \sum_{i=1}^{n}\log z_i + \beta\big[(\delta_1-1)\sum_{i=1}^{n}\frac{e^{1-(1-z_i^{\lambda})^{\beta}}(1-z_i^{\lambda})^{\beta-1}z_i^{\lambda}\log z_i}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)} + \\
&\delta_2-1)\sum_{j=1}^{m}\frac{e^{1-(1-w_j^{\lambda})^{\beta}}(1-w_j^{\lambda})^{\beta-1}w_i^{\lambda}\log w_j}{(e^{1-(1-w_j^{\lambda})^{\beta}}-1)}\big] + \sum_{j=1}^{m}\log w_j \\
&+ \beta\big[\sum_{i=1}^{n}(1-z_i^{\lambda})^{\beta-1}z_i^{\lambda}\log z_i + \sum_{j=1}^{m}(1-w_j^{\lambda})^{\beta-1}w_j^{\lambda}\log w_j\big] \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \log L}{\partial\beta} =& \frac{(n+m)}{\beta} + \sum_{i=1}^{n}\log(1-z_i^{\lambda}) + (\delta_1-1)\sum_{i=1}^{n}\frac{e^{1-(1-z_i^{\lambda})^{\beta}}(1-z_i^{\lambda})^{\beta}\log(1-z_i^{\lambda})}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)} \\
&+ \sum_{j=1}^{m}\log(1-w_j^{\lambda}) + (\delta_2-1)\sum_{j=1}^{m}\frac{e^{1-(1-w_j^{\lambda})^{\beta}}(1-w_j^{\lambda})^{\beta}\log(1-w_j^{\lambda})}{(e^{1-(1-w_j^{\lambda})^{\beta}}-1)} \\
&- \sum_{i=1}^{n}(1-z_i^{\lambda})^{\beta}\log(1-z_i^{\lambda}) - \sum_{j=1}^{m}(1-w_j^{\lambda})^{\beta}\log(1-w_j^{\lambda}) \qquad (30)
\end{aligned}
$$

$$
\frac{\partial \log L}{\partial\delta_1} = \frac{n}{\delta_1} - n\log(e-1) + \sum_{i=1}^{n}\log(e^{1-(1-z_i^{\lambda})^{\beta}}-1) \qquad (31)
$$

$$
\frac{\partial \log L}{\partial\delta_2} = \frac{m}{\delta_2} - m\log(e-1) + \sum_{j=1}^{m}\log(e^{1-(1-w_j^{\lambda})^{\beta}}-1) \qquad (32)
$$

MLEs of δ_1 and δ_2 denoted by $\hat{\delta}_1$ and $\hat{\delta}_2$ respectively, can be obtained by equating the equations (29), (30), (31) and (32) to zero and solving using `nlm()` function in R software. Then ML estimate of R is given by

$$
\hat{R} = \frac{\hat{\delta}_1}{(\hat{\delta}_1 + \hat{\delta}_2)}
$$

8.1.2 Asymptotic distribution and confidence interval

In this section, the asymptotic distribution and asymptotic confidence interval of the MLE of R. The variance–covariance matrix is:

$$\begin{bmatrix} var(\hat{\lambda}) & Cov(\hat{\lambda},\hat{\beta} & Cov(\hat{\lambda},\hat{\delta}_1) & Cov(\hat{\lambda},\hat{\delta}_2) \\ & var(\hat{\beta} & Cov(\hat{\beta},\hat{\delta}_1) & Cov(\hat{\beta},\hat{\delta}_2) \\ & & var(\hat{\delta}_1 & Cov(\hat{\delta}_1,\hat{\delta}_2) \\ & & & var(\hat{\delta}_2) \end{bmatrix}$$

This can be approximated by inverted observed Fisher information matrix $I(\lambda,\beta,\delta_1,\delta_2)$ at $(\hat{\lambda},\hat{\beta},\hat{\delta}_1,\hat{\delta}_2)$ i.e, by $I^{-}(\hat{\lambda},\hat{\beta},\hat{\delta}_1,\hat{\delta}_2)$.

Fisher Information matrix, I is given by

$$I = E\begin{bmatrix} \frac{-\partial^2 logL}{\partial\lambda^2} & \frac{-\partial^2 logL}{\partial\lambda\partial\beta} & \frac{-\partial^2 logL}{\partial\lambda\partial\delta_1} & \frac{-\partial^2 logL}{\partial\lambda\partial\delta_2} \\ \frac{-\partial^2 logL}{\partial\lambda\partial\beta} & \frac{-\partial^2 logL}{\partial\beta^2} & \frac{-\partial^2 logL}{\partial\beta\partial\delta_1} & \frac{-\partial^2 logL}{\partial\beta\partial\delta_2} \\ \frac{-\partial^2 logL}{\partial\lambda\partial\delta_1} & \frac{-\partial^2 logL}{\partial\beta\partial\delta_1} & \frac{-\partial^2 logL}{\partial\delta_1^2} & \frac{-\partial^2 logL}{\partial\delta_1\partial\delta_2} \\ \frac{-\partial^2 logL}{\partial\lambda\partial\delta_2} & \frac{-\partial^2 logL}{\partial\beta\partial\delta_2} & \frac{-\partial^2 logL}{\partial\delta_1\partial\delta_2} & \frac{-\partial^2 logL}{\partial\delta_2^2} \end{bmatrix}$$

The expansions of the terms in the matrix is given in Appendix C. In order to prove the asymptotic normality of R, we define the partial derivative of R with respect to the parameters as $B(\boldsymbol{\theta}) = (\frac{\partial R}{\partial\lambda}, \frac{\partial R}{\partial\beta}, \frac{\partial R}{\partial\delta_1}, \frac{\partial R}{\partial\delta_2})^T$, $\boldsymbol{\theta} = (\lambda,\beta,\delta_1,\delta_2)$ where

$$\frac{\partial R}{\partial\lambda} = \frac{\partial R}{\partial\beta} = 0, \quad \frac{\partial R}{\partial\delta_1} = \frac{\delta_2}{(\delta_1+\delta_2)^2}, \quad \frac{\partial R}{\partial\delta_2} = \frac{-\delta_1}{(\delta_1+\delta_2)^2}$$

We obtain the asymptotic distribution of $\hat{R}$ as

$$\sqrt{n+m}(\hat{R}-R) \xrightarrow{d} N(0, B^T(\boldsymbol{\theta})I^{-1}(\boldsymbol{\theta})B(\boldsymbol{\theta}))$$

Asymptotic variance of $\hat{R}$ is obtained as

$$AV(\hat{R}) = \frac{1}{n+m}B^T(\delta)I^{-1}(\boldsymbol{\theta})B(\boldsymbol{\theta})$$

Then Asymptotic $100(1-\eta)\%$ confidence interval for R can be obtained as

$$\hat{R} \pm Z_{\eta/2}\sqrt{AV(\hat{R})}$$

where $Z_{\eta/2}$ is the upper $(\eta/2)th$ quantile of the standard normal distribution.

8.2. Multi-component stress-strength (MSS) reliability

With technological advancements, products have evolved from basic single-component systems to complex multicomponent subsystems, leading to a shift in reliability research towards the MSS models. For instance, readers can refer to Rao et al. (2015), Kizilaslan and Nadar (2015), Dey et al. (2017), Wang et al. (2022), Kayal et al. (2020), Hassan et al. (2024). Wang et al. (2020) studied the inference for a MSS model under censored data, where strength and stress RVs follow K distributions with common shape parameters. Consider a multi-component system with k i.i.d. strength components $Z_1,\ Z_2, ...,\ Z_k$ with common distribution function $G(.)$ following PGDUS-K(λ,β,δ_1), each of which is subjected to a common random stress $W \sim$ PGDUS-K(λ,β,δ_2) with CDF $F(.)$. Then the reliability is given by

$$\begin{aligned} R_{s,k} &= P(\text{at least s of the } (Z_1,\ Z_2,\ ...,\ Z_k) \text{ exceed } W) \\ &= \sum_{i=s}^{k}\binom{k}{i}\int_0^1 (1-G(z))^i\, G(z)^{k-i}\, dF(z) \end{aligned}$$

$$= \frac{\lambda\beta\delta_2}{(e-1)^{\delta_2}} \sum_{i=s}^{k} \binom{k}{i} \int_0^1 \left[1 - \left(\frac{e^{1-(1-z^\lambda)^\beta} - 1}{e-1}\right)^{\delta_1}\right]^i \left[\frac{e^{1-(1-z^\lambda)^\beta} - 1}{e-1}\right]^{\delta_1(k-i)}$$
$$z^{\lambda-1}(1-z^\lambda)^{\beta-1} e^{1-(1-z^\lambda)^\beta} (e^{1-(1-z^\lambda)^\beta} - 1)^{\delta_2-1} dz$$
$$= \frac{\lambda\beta\delta_2}{(e-1)^{k\delta_1+\delta_2}} \sum_{i=s}^{k} \binom{k}{i} \sum_{j=0}^{i} \binom{i}{j} (-1)^j (e-1)^{(i-j)\delta_1}$$
$$\int_0^1 z^{\lambda-1}(1-z^\lambda)^{\beta-1} e^{1-(1-z^\lambda)^\beta} (e^{1-(1-z^\lambda)^\beta} - 1)^{\delta_1(k+j-i)+\delta_2-1} dz$$
$$= \sum_{i=s}^{k} \sum_{j=0}^{i} \binom{k}{i} \binom{i}{j} \frac{(-1)^j \delta_2}{\delta_1(k+j-i)+\delta_2}, \ \ \delta_1 > 0, \ \delta_2 > 0$$

8.2.1 Estimation of stress-strength reliability

Assume that the sample of the multi-component system given by $Z_{i1}, Z_{i2}, ..., Z_{ik}$ and $W_i, i = 1, 2, ..., n$ follows PGDUS-K$(\lambda, \beta, \delta_1)$ and PGDUS-K$(\lambda, \beta, \delta_2)$,respectively. Then its log-likelihood function is given by

$$\begin{aligned}
\log L &= \log \prod_{i=1}^{n} \big[\prod_{j=1}^{k} g(z_{ij}) \big] f(w_i) \\
&= n(k+1)(\log\lambda + \log\beta) + nk\log\delta_1 + n\log\delta_2 - nk\delta_1 \log(e-1) - n\delta_2 \log(e-1) \\
&+ (\lambda-1) \sum_{i=1}^{n}\sum_{j=1}^{k} \log z_{ij} + (\beta-1) \sum_{i=1}^{n}\sum_{j=1}^{k} \log(1-z_{ij}^\lambda) + \sum_{i=1}^{n}\sum_{j=1}^{k} (1-(1-z_{ij}^\lambda)^\beta) \\
&+ (\delta_1-1) \sum_{i=1}^{n}\sum_{j=1}^{k} \log(e^{1-(1-z_{ij}^\lambda)^\beta} - 1) + (\lambda-1) \sum_{i=1}^{n} \log w_i + \sum_{i=1}^{n} (1-(1-w_i^\lambda)^\beta) \\
&+ (\delta_2-1) \sum_{i=1}^{n} \log(e^{1-(1-w_i^\lambda)^\beta} - 1) + (\beta-1) \sum_{i=1}^{n} \log(1-w_i^\lambda)
\end{aligned}$$

The estimate of $R_{s,k}$ can be obtained by finding the estimates of δ_1 and δ_2. Here we use ML estimation technique. For that, we maximize $\log L$ by finding first derivatives of $\log L$ with respect to λ, β, δ_1 and δ_2 and equating them to zero and solving for estimates as described below:

$$\begin{aligned}
\frac{\partial \log L}{\partial \lambda} =& \frac{n(k+1)}{\lambda} + \sum_{i=1}^{n}\sum_{j=1}^{k} \log z_{ij} + \beta \sum_{i=1}^{n}\sum_{j=1}^{k} (1-z_{ij}^\lambda)^{\beta-1} z_{ij}^\lambda \log z_{ij} + \sum_{i=1}^{n} \log w_i \\
&+ \beta \sum_{i=1}^{n} (1-w_i^\lambda)^{\beta-1} w_i^\lambda \log w_i - (\beta-1) \sum_{i=1}^{n}\sum_{j=1}^{k} \frac{z_{ij}^\lambda \log z_{ij}}{1-z_{ij}^\lambda} - (\beta-1) \sum_{i=1}^{n} \frac{w_i^\lambda \log w_i}{1-w_i^\lambda} \\
&+ \beta(\delta_1-1) \sum_{i=1}^{n}\sum_{j=1}^{k} \frac{e^{1-(1-z_{ij}^\lambda)^\beta} (1-z_{ij}^\lambda)^{\beta-1} z_{ij}^\lambda \log z_{ij}}{(e^{1-(1-z_{ij}^\lambda)^\beta} - 1)} \\
&+ \beta(\delta_2-1) \sum_{i=1}^{n} \frac{e^{1-(1-w_i^\lambda)^\beta} (1-w_i^\lambda)^{\beta-1} w_i^\lambda \log w_i}{(e^{1-(1-w_i^\lambda)^\beta} - 1)}
\end{aligned} \tag{33}$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n(k+1)}{\beta} + \sum_{i=1}^{n}\sum_{j=1}^{k}\log(1-z_{ij}^{\lambda}) - \sum_{i=1}^{n}\sum_{j=1}^{k}(1-z_{ij}^{\lambda})^{\beta}\log(1-z_{ij}^{\lambda})$$
$$+ (\delta_1 - 1)\sum_{i=1}^{n}\sum_{j=1}^{k}\frac{e^{1-(1-z_{ij}^{\lambda})^{\beta}}(1-z_{ij}^{\lambda})^{\beta}\log(1-z_{ij}^{\lambda})}{(e^{1-(1-z_{ij}^{\lambda})^{\beta}}-1)} + \sum_{i=1}^{n}\log(1-w_i^{\lambda})$$
$$+ (\delta_2 - 1)\sum_{i=1}^{m}\frac{e^{1-(1-w_i^{\lambda})^{\beta}}(1-w_i^{\lambda})^{\beta}\log(1-w_i^{\lambda})}{(e^{1-(1-w_i^{\lambda})^{\beta}}-1)} - \sum_{i=1}^{m}(1-w_i^{\lambda})^{\beta}\log(1-w_i^{\lambda}) \quad (34)$$

$$\frac{\partial \log L}{\partial \delta_1} = \frac{nk}{\delta_1} - nk\log(e-1) + \sum_{i=1}^{n}\sum_{j=1}^{k}\log(e^{1-(1-z_{ij}^{\lambda})^{\beta}}-1) \quad (35)$$

$$\frac{\partial \log L}{\partial \delta_2} = \frac{n}{\delta_2} - n\log(e-1) + \sum_{i=1}^{n}\log(e^{1-(1-w_i^{\lambda})^{\beta}}-1). \quad (36)$$

By equating Equations (33), (34), (35) and (36) to zero and solving using `nlm()` function in R software, we obtain $\hat{\lambda}$, $\hat{\beta}$, $\hat{\delta}_1$ and $\hat{\delta}_2$. Then using invariance property of MLE, we obtain the MSS reliability estimate as:

$$R_{s,k} = \sum_{i=s}^{k}\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j}\frac{(-1)^j\hat{\delta}_2}{\hat{\delta}_1(k+j-i)+\hat{\delta}_2}.$$

8.2.2 Asymptotic confidence interval of $R_{s,k}$

To prove the asymptotic notmality of $R_{s,k}$, we define the partial derivative of $R_{s,k}$ with respect to the parameters $\boldsymbol{\theta} = (\lambda, \beta, \delta_1, \delta_2)$ as

$$B(\boldsymbol{\theta}) = (\frac{\partial R_{s,k}}{\partial \lambda}, \frac{\partial R_{s,k}}{\partial \beta}, \frac{\partial R_{s,k}}{\partial \delta_1}, \frac{\partial R_{s,k}}{\partial \delta_2})^T = (b_1, b_2, b_3, b_4)^T$$

where

$$\frac{\partial R_{s,k}}{\partial \lambda} = \frac{\partial R_{s,k}}{\partial \beta} = 0,$$

$$\frac{\partial R_{s,k}}{\partial \delta_1} = \sum_{i=s}^{k}\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j}\frac{(-1)^{j+1}\delta_2(k+j-i)}{(\delta_1(k+j-i)+\delta_2)^2},$$

$$\frac{\partial R_{s,k}}{\partial \delta_2} = \sum_{i=s}^{k}\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j}\frac{\delta_1(k+j-i)}{(\delta_1(k+j-i)+\delta_2)^2}$$

Asymptotic variance of $\hat{R}$ is obtained as

$$AV(\hat{R}_{s,k}) = \sum_{i=1}^{4}\sum_{j=1}^{4}\frac{\partial R_{s,k}}{\partial \theta_2}\frac{\partial R_{s,k}}{\partial \theta_2}I^{-1}(\boldsymbol{\theta}), \;\; \theta_1, \theta_2 \in \{\lambda, \beta, \delta_1, \delta_2\}$$

where $I^{-1}(\boldsymbol{\theta})$ is the Fisher Information matrix.Then $100(1-\eta)\%$ asymptotic confidence interval (ACI) for$R_{s,k}$ is obtained as

$$\hat{R}_{s,k} \pm Z_{\eta/2}\sqrt{AV(\hat{R}_{s,k})}$$

where $Z_{\eta/2}$ is the upper $(\eta/2)th$ quantile of the standard notmal distribution.

8.2.3 Simulation study

The behavior of proposed estimates of R and $R_{s,k}$ derived using the ML estimation approach is examined in this section through a simulation study. All the estimates are computed when sample sizes are 50, 100, 200 and 500. For each sample size, estimated value of R and $R_{s,k}$, Bias, MSE and 95% ACI are obtained. Stress and strength populations for single component case is generated using two sets of parameter values, say

$$\text{Set 1: } \lambda = 5.0, \beta = 0.2, \delta_1 = 3.0,\ \delta_2 = 4.0 \text{ and Set 2: } \lambda = 0.5, \beta = 0.5, \delta_1 = 1, \delta_2 = 0.1$$

Table 14 Estimates, Biases and MSEs for R at Set 1 and Set 2

n,m	R	Estimate $\hat{R}$	Bias	MSE	95% ACI
50		0.3999	-0.0286	0.0008	(0.3941722, 0.4056322)
100	0.4285	0.4181	-0.0104	0.0001	(0.4179939, 0.4183281)
200	(Set1	0.4238	-0.0047	2.26 e-05	(0.4237526, 0.4238814)
500		0.4277	-0.0008	7.027 e-07	(0.4277613, 0.4277621)
50		0.8539	-0.0551	0.0030	(0.8475248, 0.86042460)
100	0.90909	0.9074	-0.0016	2.57 e-06	(0.9074526, 0.9075168)
200	(Set 2	0.9087	-0.0003	9.1928 e-08	(0.9087781, 0.9087973)
500		0.90904	-4.1965 e-05	1.7611 e-09	(0.9090462, 0.9090516)

In MSS, stress and strength populations are generated assuming a set of parameter values, say

$$\text{Set 1: } \lambda = 0.5, \beta = 0.3, \delta_1 = 4, \delta_2 = 0.5$$

Table 15 Estimates, Biases and MSEs for $R_{s,k}$ for Set 1

s,k	n,m	$R_{s,k}$	Estimate $\hat{R}_{s,k}$	Bias	MSE	95% ACI
	50		0.9243	-0.0065	4.315e-05	(0.9153,0.9333)
2,4	100	0.9309	0.9260	-0.0048	2.338e-05	(0.9176,0.9344)
	200	(Set 1)	0.9331	0.0022	4.97e-06	(0.9249,0.9413)
	500		0.9316	0.0007	5.33e-07	(0.9237,0.9395)

It is observed from Table 14 and Table 15 that the MSE and bias decreases as the sample size increases and the MLEs of R become closer to its true value.

8.2.4 Real data illustration

For application purposes, a real-life instance is provided in this section. We have used the below two data sets in Table 16, that was used by Xia et al. (2009) which tabulates the breaking strengths of jute fibre at two different gauge lengths, say 10 mm and 20 mm, denoted by Z and W, respectively.

Table 16 Breaking strengths of jute fibre at 10 mm (Z) and 20 mm (W) gauge length

10 mm														
693.73	704.66	323.83	778.17	123.06	637.66	383.43	151.48	108.94	50.16	671.49	183.16	257.44	727.23	291.27
101.15	376.42	163.40	141.38	700.74	262.90	353.24	422.11	43.93	590.48	212.13	303.90	506.60	530.55	177.25
20 mm														
71.46	419.02	284.64	585.57	456.60	113.85	187.85	688.16	662.66	45.58	578.62	756.70	594.29	166.49	99.72
707.36	765.14	187.13	145.96	350.70	547.44	116.99	375.81	581.60	119.86	48.01	200.16	36.75	244.53	83.55

PGDUS-K fits the above two data sets given in Table 16. The corresponding estimated parameters, p-value associated to K-S test, Log-likelihood values, AIC, and BIC values are given in Table 17 and the MLE of R is $\hat{R} = \frac{\hat{\delta}_1}{(\hat{\delta}_1+\hat{\delta}_2)} = 0.5606$.

Table 17 Estimates, K-S(p-value), LL, AIC, BIC values

Data	Estimates	K-S(p-value)	LL	AIC	BIC
Z	$\hat{\lambda} = 6.5290$ $\hat{\beta} = 7.8206$ $\hat{\delta}_1 = 0.3518$	0.9294	3.5708	-3.1416	-0.3392
W	$\hat{\lambda} = 6.5290$ $\hat{\beta} = 7.8206$ $\hat{\delta}_2 = 0.2756$	0.3622	2.7467	-1.4934	1.3089

Table 18 Comparison Table

Models	R value
PGDUS-K	0.5606
K	0.5256
DUS-K	0.4701
Weibull	0.5222
Exponential	0.4971
Normal	0.5057

The reliability estimates for different models are presented in Table 18. It is observed that the proposed PGDUS-K distribution provides the highest reliability value (0.5606). The K (0.5256) and Weibull (0.5222) models yield comparable reliability estimates, while the Normal (0.5057) and Exponential (0.4971) distributions provide slightly lower values. Among the considered models, the DUS-K distribution gives the smallest reliability estimate (0.4701). Overall, the results suggest that the PGDUS-K model performs better than the competing distributions.

Table 19 Estimates and Confidence Interval (CI) for $R_{s,k}$

(s,k)	$\hat{R}_{s,k}$	95% ACI
(1,3)	0.75	(0.7499922, 0.7500078)
(2,4)	0.6	(0.5999796, 0.6000204)
(3,6)	0.5714	(0.5712446, 0.5715554)

The MLEs of $R_{s,k}$, along with their 95% approximate confidence intervals (ACI), are presented in Table 19. The results show a decreasing trend in reliability as (s, k) increase, while the narrow confidence intervals indicate precise and stable estimates.

9. Conclusion

This study introduced the PGDUS–K distribution for modeling parallel systems and investigated its main statistical and reliability properties. Compared with the standard DUS–K model, the proposed PGDUS–K distribution incorporates an additional power parameter, which enhances shape flexibility and allows the model to capture a wider range of data behaviors. Moreover, the distribution preserves important structural properties such as closure under exponentiation, which maintains consistency under certain transformations and supports its applicability in reliability modeling. The model parameters were estimated using ML, MPS, and Bayesian methods. A simulation study under multiple parameter settings demonstrated satisfactory estimator performance, particularly for moderate and large sample sizes. The practical applicability of the model was further illustrated using two real datasets, where the proposed distribution was compared with several competing models through goodness-of-fit criteria and graphical diagnostics, confirming its competitive fitting capability. In addition, stress–strength reliability was examined for both single-component and multi-component parallel systems, and analytical reliability expressions were obtained. Numerical illustrations showed that the proposed model provides reliable and interpretable measures for system reliability analysis.

Despite its advantages, the model has certain limitations. The bounded support of the distribution may restrict its use to datasets that naturally lie within a finite range, and the relatively complex func-

tional form of the PDF makes the associated estimation procedures computationally intensive. Future research may consider extending the model to regression frameworks for incorporating covariate effects and to censored data settings commonly encountered in reliability and survival analysis.

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References

Abd Al-Fattah AM, El-Helbawy AA, Al-Dayian GR. Inverted Kumaraswamy distribution: properties and estimation. Pak J Stat. 2017; 33(1):37-61.

Akkanphudit T. Generalized DUS transformed Garima distribution: properties, simulations and applications. Lobachevskii J Math. 2023; 44(2):803-814.

Ali AH, Al-Kanani IH. Bayesian methods to estimate the parameters of exponentiated Weibull distribution. J Phys Conf Ser. 2021; 1818(1):012143.

Alizadeh M, Cordeiro GM, Rodrigues GM, Ortega EM, Yousof HM. The extended Kumaraswamy model: properties, risk indicators, risk analysis, regression model, and applications. Stats. 2025; 8(3):62.

Alotaibi N, Elbatal I, Shrahili M, Al-Moisheer AS, Elgarhy M, Almetwally EM. Statistical inference for the Kavya-Manoharan Kumaraswamy model under ranked set sampling with applications. Symmetry (Basel). 2023; 15(3):587.

Amrutha M, Chacko VM. PGDUS-powered inverse Rayleigh distribution: properties and stress-strength reliability analysis using maximum likelihood and maximum product spacing methods. Asian J Stat Sci. 2024; 4(2):155-179.

Amrutha M, Chacko VM. Power generalized DUS transformation of inverse Kumaraswamy distribution and stress-strength analysis. Stat Appl. 2024; 22(2):323-359.

Anakha KK, Chacko VM. DUS-Kumaraswamy distribution: a bathtub-shaped failure rate model. Int J Stat Reliab Eng. 2021; 8(3):359-367.

Baharith LA. Alpha-power Kumaraswamy-Burr III distribution with applications of COVID-19 data in Saudi Arabia. Results Phys. 2022; 35:105353.

Bhaumik DK, Gibbons RD. One-sided approximate prediction intervals for at least p of m observations from a gamma population at each of r locations. Technometrics. 2006; 48(1):112-119.

Birnbaum ZW. On a use of the Mann-Whitney statistic. In: Neyman J, editor. Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability. Vol 1. Berkeley (CA): University of California Press; 1956. p. 13-18.

Cheng RC, Amin NA. Estimating parameters in continuous univariate distributions with a shifted origin. J R Stat Soc B. 1983; 45(3):394-403.

Church JD, Harris B. The estimation of reliability from stress-strength relationships. Technometrics. 1970; 12(1):49-54.

Deepthi KS, Chacko VM. An upside-down bathtub-shaped failure rate model using a DUS transformation of the Lomax distribution. In: Cui L, Frenkel I, Lisnianski A, editors. Stochastic Models in Reliability Engineering. Boca Raton (FL): CRC Press; 2020. p. 81-100.

Dey S, Mazucheli J, Anis MZ. Estimation of reliability of multicomponent stress-strength for a Kumaraswamy distribution. Commun Stat Theory. 2017; 46(4):1560-1572.

Elgarhy M, Metwally DS, Ragab IE, Bashiru SO, Semary HE, Gemeay AM. The DUS-transformed generalized linear failure rate distribution: properties, estimation, and applications. Sci Afr. 2025; 29:e02891.

Ghitany ME, Atieh B, Nadarajah S. Lindley distribution and its application. Math Comput Simulat. 2008; 78(4):493-506.

Hassan MK, Aslam M. DUS-neutrosophic multivariate inverse Weibull distribution: properties and applications. Complex Intell Syst. 2023; 9(5):5679-5691.

Hassan AS, Alsadat N, Elgarhy M, Ahmad H, Nagy HF. On estimating multi-stress strength reliability for inverted Kumaraswamy under ranked set sampling with application in engineering. J Nonlinear Math Phy. 2024; 31(1):30.

Hastings WK. Monte Carlo sampling methods using Markov chains and their applications. Biometrika. 1970; 57(1):97-109.

Hinkley DV. On quick choice of power transformation. J R Stat Soc C-Appl. 1977; 26(1):67-69.

Iqbal Z, Tahir MM, Riaz N, Ali SA, Ahmad M. Generalized inverted Kumaraswamy distribution: properties and application. Open J Stat. 2017; 7(4):645-662.

Jones MC. Kumaraswamy's distribution: a beta-type distribution with some tractability advantages. Stat Methodol. 2009; 6(1):70-81.

Kayal T, Tripathi YM, Dey S, Wu SJ. On estimating the reliability in a multicomponent stress-strength model based on Chen distribution. Commun Stat Theory. 2020; 49(10):2429-2447.

Kizilaslan F, Nadar M. Classical and Bayesian estimation of reliability in multicomponent stress-strength model based on Weibull distribution. Rev Colomb Estad. 2015; 38(2):467-484.

Kumar D, Singh U, Singh SK. A method of proposing new distribution and its application to bladder cancer patients data. J Stat Appl Pro Lett. 2015; 2(3):235-245.

Kumaraswamy P. A generalized probability density function for double-bounded random processes. J Hydrol. 1980; 46(1-2):79-88.

Lindley DV. Approximate Bayesian methods. Trab Estad Investig Oper. 1980; 31:223-245.

Maurya SK, Kaushik A, Singh SK, Singh U. A new class of exponential transformed Lindley distribution and its application to yarn data. Int J Stat Econ. 2017; 18(2):135-151.

Metropolis N, Rosenbluth AW, Rosenbluth MN, Teller AH, Teller E. Equation of state calculations by fast computing machines. J Chem Phys. 1953; 21(6):1087-1092.

Mitnik PA. New properties of the Kumaraswamy distribution. Commun Stat Theor. 2013; 42(5):741-755.

Moors JJ. A quantile alternative for kurtosis. Statistician. 1988; 37(1):25-32.

Nadar M, Papadopoulos A, Kizilaslan F. Statistical analysis for Kumaraswamy's distribution based on record data. Stat Pap. 2013; 54:355-369.

Nadarajah S. On the distribution of Kumaraswamy. J Hydrol. 2008; 348(3):568-569.

Popović BV, Ristić MM, Cordeiro GM. A two-parameter distribution obtained by compounding the generalized exponential and exponential distributions. Mediterr J Math. 2016; 13(5):2935-2949.

Ranneby B. The maximum spacing method. An estimation method related to the maximum likelihood method. Scand J Stat. 1984; 11(2):93-112.

Rao GS, Aslam M, Kundu D. Burr-XII distribution parametric estimation and estimation of reliability of multicomponent stress-strength. Commun Stat Theory. 2015; 44(23):4953-4961.

Thomas B, Chacko VM. Power generalized DUS transformation. Int J Stat Reliab Eng. 2021; 9(1):150-157.

Thomas B, Chacko VM. Power generalized DUS transformation in Weibull and Lomax distributions. Reliab Theory Appl. 2023; 18(1):368-384.

Wang L, Dey S, Tripathi YM, Wu SJ. Reliability inference for a multicomponent stress-strength model based on Kumaraswamy distribution. J Comput Appl Math. 2020; 376:112823.

Wang L, Wu K, Tripathi YM, Lodhi C. Reliability analysis of multicomponent stress-strength reliability from a bathtub-shaped distribution. J Appl Stat. 2022; 49(1):122-142.

Xia ZP, Yu JY, Cheng LD, Liu LF, Wang WM. Study on the breaking strength of jute fibres using modified Weibull distribution. Compos A Appl Sci Manuf. 2009; 40(1):54-59.

Appendix A

- The r^{th} raw moment of PGDUS-K(λ, β, δ) distribution is given by

$$\begin{aligned}
\mu_r^{'} = E(z^r) &= \int_0^1 z^r \frac{\lambda\beta\delta}{(e-1)^\delta} z^{\lambda-1}(1-z^\lambda)^{\beta-1} e^{1-(1-z^\lambda)^\beta}(e^{1-(1-z^\lambda)^\beta}-1)^{\delta-1} dz \\
&= \frac{e\lambda\beta\delta}{(e-1)^\delta}\sum_{k=0}^{\infty}(-1)^k \binom{\delta-1}{k} e^{\delta-k-1}\int_0^1 z^{r+\lambda-1}(1-z^\lambda)^{\beta-1} e^{-(\delta-k)(1-z^\lambda)^\beta} dz \\
&= \frac{e\lambda\beta\delta}{(e-1)^\delta}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+m}}{m!} \binom{\delta-1}{k} e^{\delta-k-1}\int_0^1 z^{r+\lambda-1}(1-z^\lambda)^{\beta-1}[(\delta-k)(1-z^\lambda)^\beta]^m dz \\
&= \frac{e\beta\delta}{(e-1)^\delta}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+m}}{m!} \binom{\delta-1}{k} e^{\delta-k-1}(\delta-k)^m B(\frac{r}{\lambda}+1, \beta+m\beta)
\end{aligned}$$

- The moment generating function of PGDUS-K(λ, β, δ) is given by

$$\begin{aligned}
M_Z(t) = E(e^{tZ}) &= \int_0^1 e^{tz}\frac{\lambda\beta\delta}{(e-1)^\delta} z^{\lambda-1}(1-z^\lambda)^{\beta-1} e^{1-(1-z^\lambda)^\beta}(e^{1-(1-z^\lambda)^\beta}-1)^{\delta-1} dz \\
&= \frac{e\lambda\beta\delta}{(e-1)^\delta}\int_0^1 z^{\lambda-1}(1-z^\lambda)^{\beta-1} e^{tz-(1-z^\lambda)^\beta}\sum_{k=0}^{\infty}(-1)^k \binom{\delta-1}{k}[e^{1-(1-z^\lambda)^\beta}]^{\delta-k-1} dz \\
&= \frac{e\lambda\beta\delta}{(e-1)^\delta}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+m}}{m!} \binom{\delta-1}{k} e^{\delta-k-1}\int_0^1 e^{tz} z^{\lambda-1}(1-z^\lambda)^{\beta-1}[(\delta-k)(1-z^\lambda)^\beta]^m dz \\
&= \frac{e\lambda\beta\delta}{(e-1)^\delta}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{k+m}}{m!n!} t^n \binom{\delta-1}{k} e^{\delta-k-1}(\delta-k)^m\int_0^1 z^{\lambda+n-1}(1-z^\lambda)^{m\beta+\beta-1} dz \\
&= \frac{e\beta\delta}{(e-1)^\delta}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{k+m}}{m!n!} t^n \binom{\delta-1}{k} e^{\delta-k-1}(\delta-k)^m B(\frac{n}{\lambda}+1, \beta+m\beta)
\end{aligned}$$

- Rényi entropy is defined by $\tau_R(\gamma) = \frac{1}{1-\gamma}\log\int g^\gamma(z)\,dz,\ \gamma > 0,\ \gamma \neq 1.$

$$\begin{aligned}
\int_0^1 g^\gamma(z)dz &= \int_0^1 \left(\frac{\lambda\beta\delta}{(e-1)^\delta} z^{\lambda-1}(1-z^\lambda)^{\beta-1} e^{1-(1-z^\lambda)^\beta}(e^{1-(1-z^\lambda)^\beta}-1)^{\delta-1}\right)^\gamma dz \\
&= \frac{(e\lambda\beta\delta)^\gamma}{(e-1)^{\delta\gamma}}\int_0^1 (z^{(\lambda-1)}(1-z^\lambda)^{(\beta-1)} e^{-(1-z^\lambda)^\beta})^\gamma \sum_{k=0}^{\infty}(-1)^k \binom{\gamma(\delta-1)}{k}(e^{1-(1-z^\lambda)^\beta})^{\gamma(\delta-1)-k} dz \\
&= \frac{(e\lambda\beta\delta)^\gamma}{(e-1)^{\delta\gamma}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+m}}{m!} \binom{\gamma(\delta-1)}{k} e^{\gamma(\delta-1)-k}(\gamma\delta-k)^m\int_0^1 z^{\gamma(\lambda-1)}(1-z^\lambda)^{\gamma(\beta-1)+m\beta} dz \\
&= \frac{(e\lambda\beta\delta)^\gamma}{\lambda(e-1)^{\delta\gamma}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+m}}{m!} \binom{\gamma(\delta-1)}{k} e^{\gamma(\delta-1)-k}(\gamma\delta-k)^m B(\gamma+\frac{1-\gamma}{\lambda}, \gamma(\beta-1)+m\beta+1)
\end{aligned}$$

Appendix B

$$\begin{aligned}
\frac{\partial^2 l}{\partial\lambda^2} =& -\frac{n}{\lambda^2} + \beta\sum_{i=1}^{n}(1-z_i^\lambda)^{\beta-2} z_i^\lambda(\log z_i)^2(2-\beta-z_i^\lambda) \\
&+ \beta(\delta-1)\sum_{i=1}^{n}\frac{(\log z_i)^2 e^{1-(1-z_i^\lambda)^\beta}(1-z_i^\lambda)^{\beta-1} z_i^\lambda}{(e^{1-(1-z_i^\lambda)^\beta}-1)^2}\left[\frac{(e^{1-(1-z_i^\lambda)^\beta}-1)(1-\beta z_i^\lambda)}{(1-z_i^\lambda)}\right. \\
&\left.+ \beta z_i^\lambda(1-z_i^\lambda)^{\beta-1} e^{1-(1-z_i^\lambda)^\beta}(1-\log z_i)\right] - (\beta-1)\sum_{i=1}^{n}\frac{z_i^\lambda(\log z_i)^2}{(1-z_i^\lambda)^2} \\
\frac{\partial^2\ell}{\partial\lambda\partial\beta} =& \frac{\partial^2\ell}{\partial\beta\partial\lambda} = \sum_{i=1}^{n}(1-z_i^\lambda)^{\beta-1} z_i^\lambda\log z_i(1+\beta\log(1-z_i^\lambda)) \\
&- (\delta-1)\sum_{i=1}^{n}\frac{e^{1-(1-z_i^\lambda)^\beta} z_i^\lambda\log z_i(1-z_i^\lambda)^{\beta-1}}{(e^{1-(1-z_i^\lambda)^\beta}-1)^2}\left[(e^{1-(1-z_i^\lambda)^\beta}-1)(\beta-\beta\log(1-z_i^\lambda)-1)\right. \\
&\left.- \beta e^{1-(1-z_i^\lambda)^\beta}(1-z_i^\lambda)^\beta\log(1-z_i^\lambda)\right] - \sum_{i=1}^{n}\frac{z_i^\lambda\log z_i}{1-z_i^\lambda}
\end{aligned}$$

$$\frac{\partial^2 \ell}{\partial \beta^2} = -\frac{n}{\beta^2} + \beta(\delta-1)\sum_{i=1}^{n} \frac{\log(1-z_i^{\lambda}) e^{1-(1-z_i^{\lambda})^{\beta}} (1-z_i^{\lambda})^{\beta-1}}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)^2} \left[e^{1-(1-z_i^{\lambda})^{\beta}} - 1 - (1-z_i^{\lambda})^{\beta}\right]$$
$$- \sum_{i=1}^{n} \beta(1-z_i^{\lambda})^{\beta-1} \log(1-z_i^{\lambda})$$
$$\frac{\partial^2 \ell}{\partial \lambda \partial \delta} = \frac{\partial^2 \ell}{\partial \delta \partial \lambda} = \beta \sum_{i=1}^{n} \frac{e^{1-(1-z_i^{\lambda})^{\beta}} (1-z_i^{\lambda})^{\beta-1} z_i^{\lambda} \log z_i}{e^{1-(1-z_i^{\lambda})^{\beta}}-1}$$
$$\frac{\partial^2 \ell}{\partial \beta \partial \delta} = \frac{\partial^2 \ell}{\partial \delta \partial \beta} = -\sum_{i=1}^{n} \frac{e^{1-(1-z_i^{\lambda})^{\beta}} (1-z_i^{\lambda})^{\beta} \log(1-z_i^{\lambda})}{e^{1-(1-z_i^{\lambda})^{\beta}}-1}$$
$$\frac{\partial^2 \ell}{\partial \delta^2} = -\frac{n}{\delta^2}$$

Due to the symmetry of mixed partial derivatives, $D_{ijk} = D_{jik} = D_{ikj} = D_{kij} = D_{jki} = D_{kji}$, only the following unique elements are required:

$$\{D_{111}, D_{222}, D_{333}, D_{112}, D_{122}, D_{113}, D_{133}, D_{223}, D_{233}, D_{123}\}.$$

$$D_{111} = \frac{2n}{\lambda^3} + \beta \sum_{i=1}^{n} (\log z_i)^3 z_i^{\lambda} (1-z_i^{\lambda})^{\beta-3} (2-\beta-3\beta z_i^{\lambda} + z_i^{\lambda}\beta^2 + \beta z_i^{2\lambda}) - (\beta-1)\sum_{i=1}^{n} \frac{z_i^{\lambda}(1+z_i^{\lambda})(\log z_i)^3}{(1-z_i^{\lambda})^3} +$$
$$\beta(\delta-1)\sum_{i=1}^{n} (\log z_i)^2 \Bigg[\Bigg[z_i^{\lambda} \log z_i \, \frac{(1-z_i^{\lambda})(\beta e^{1-(1-z_i^{\lambda})^{\beta}} (1-z_i^{\lambda})^{\beta-1}(1-\beta z_i^{\lambda}) - \beta(e^{1-(1-z_i^{\lambda})^{\beta}}-1))}{(1-z_i^{\lambda})^2} +$$
$$\beta(1-\log z_i) z_i^{\lambda} \log z_i e^{1-(1-z_i^{\lambda})^{\beta}} (1-z_i^{\lambda})^{\beta-2} \left[1+\beta z_i^{\lambda}(1-z_i^{\lambda})^{\beta} - \beta z_i^{\lambda}\right] \Bigg] \left[\frac{e^{1-(1-z_i^{\lambda})^{\beta}} (1-z_i^{\lambda})^{\beta-1} z_i^{\lambda}}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)^2}\right] +$$
$$\left[\frac{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)(1-\beta z_i^{\lambda})}{(1-z_i^{\lambda})} + \beta z_i^{\lambda}(1-z_i^{\lambda})^{\beta-1} e^{1-(1-z_i^{\lambda})^{\beta}} (1-\log z_i)\right]$$
$$\left[\frac{z_i^{\lambda} \log z_i e^{1-(1-z_i^{\lambda})^{\beta}} (1-z_i^{\lambda})^{\beta-2}\left[1+\beta z_i^{\lambda}(1-z_i^{\lambda})^{\beta} - \beta z_i^{\lambda}\right] - \frac{2\beta \log z_i \, e^{2(1-(1-z_i^{\lambda})^{\beta})} (1-z_i^{\lambda})^{2\beta-2} z_i^{2\lambda}}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)}}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)^2}\right]\Bigg]$$

$$D_{222} = -\sum_{i=1}^{n} \log(1-z_i^{\lambda})(1-z_i^{\lambda})^{\beta-2}[\beta(\beta-1)+1-z_i^{\lambda}] + \frac{2n}{\beta^3} + (\delta-1)\sum_{i=1}^{n} \log(1-z_i^{\lambda}) \Bigg[\frac{e^{1-(1-z_i^{\lambda})^{\beta}} (1-z_i^{\lambda})^{\beta-1}}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)^2}$$
$$\Big[-\beta \log(1-z_i^{\lambda})(((1-z_i^{\lambda})^{\beta})[e^{1-(1-z_i^{\lambda})^{\beta}}+1] - (e^{1-(1-z_i^{\lambda})^{\beta}} - 1 - (1-z_i^{\lambda})^{\beta}) + (1-z_i^{\lambda})^{\beta}(e^{1-(1-z_i^{\lambda})^{\beta}} - 1 - (1-z_i^{\lambda})^{\beta}$$
$$+ (e^{1-(1-z_i^{\lambda})^{\beta}} - 1 - (1-z_i^{\lambda})^{\beta})\Big] + \frac{2\beta \log(1-z_i^{\lambda}) e^{2-2(1-z_i^{\lambda})^{\beta}} (1-z_i^{\lambda})^{2\beta-1} (e^{1-(1-z_i^{\lambda})^{\beta}} - 1 - (1-z_i^{\lambda})^{\beta})(e^{1-(1-z_i^{\lambda})^{\beta}} -}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)^4}$$

$$D_{333} = \frac{2n}{\delta^3}; \quad D_{133} = D_{233} = 0$$

$$D_{112} = \sum_{i=1}^{n} \frac{z_i^{\lambda}(\log z_i)^3}{(1-z_i^{\lambda})^2} \left[(1-z_i^{\lambda})^{\beta}\left(\beta(1-z_i^{\lambda})\log(1-z_i^{\lambda}) - (\beta+1)z_i^{\lambda}\right) - 1\right]$$
$$+ (\delta-1)\sum_{i=1}^{n} \Bigg[\frac{\left((e^{1-(1-z_i^{\lambda})^{\beta}}-1)(\beta - \beta\log(1-z_i^{\lambda}) - 1) - \beta e^{1-(1-z_i^{\lambda})^{\beta}} (1-z_i^{\lambda})^{\beta} \log(1-z_i^{\lambda})\right)}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)^4} (e^{1-(1-z_i^{\lambda})^{\beta}}-1)^2$$
$$\times \Big[z_i^{\lambda} \log z_i \, e^{1-(1-z_i^{\lambda})^{\beta}} (1-z_i^{\lambda})^{\beta-2}\left(1+\beta z_i^{\lambda}(1-z_i^{\lambda})^{\beta} - \beta z_i^{\lambda}\right) - e^{1-(1-z_i^{\lambda})^{\beta}} z_i^{\lambda} \log z_i (1-z_i^{\lambda})^{\beta-1}$$
$$\left(2\beta \log z_i (e^{1-(1-z_i^{\lambda})^{\beta}}-1) e^{1-(1-z_i^{\lambda})^{\beta}} z_i^{\lambda}(1-z_i^{\lambda})^{\beta-1}\right)\Big] + \frac{\beta e^{1-(1-z_i^{\lambda})^{\beta}} z_i^{2\lambda} (\log z_i)^2 (1-z_i^{\lambda})^{\beta-1}}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)^2}$$
$$\times \Big[\frac{e^{1-(1-z_i^{\lambda})^{\beta}}-1}{1-z_i^{\lambda}} + e^{1-(1-z_i^{\lambda})^{\beta}} (1-z_i^{\lambda})^{\beta-1}\left(1+\beta - \beta\log(1-z_i^{\lambda})(1-z_i^{\lambda})^{\beta}\right)\Big]\Bigg].$$

$$
\begin{aligned}
D_{122} =&\beta z_i^{\lambda}\log z_i(1-z_i^{\lambda})^{\beta-2}[1+(\beta-1)\log(1-z_i^{\lambda})]+\\
&\beta(\delta-1)\sum_{i=1}^{n}\Bigg[\frac{e^{1-(1-z_i^{\lambda})}z_i^{\lambda}\log z_i(1-z_i^{\lambda})^{\beta-2}}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)^2}\Big[\beta(1-z_i^{\lambda})^{\beta}\log(1-z_i^{\lambda})(e^{1-(1-z_i^{\lambda})}+1)-\\
&(e^{1-(1-z_i^{\lambda})^{\beta}}-1-(1-z_i^{\lambda})^{\beta})[(\beta-1)\log(1-z_i^{\lambda})-\beta(1-z_i^{\lambda})^{\beta}\log(1-z_i^{\lambda})+1]\Big]\\
&-\frac{2\beta\log(1-z_i^{\lambda})e^{2-2(1-z_i^{\lambda})^{\beta}}(1-z_i^{\lambda})^{2\beta-2}(e^{1-(1-z_i^{\lambda})^{\beta}}-1-(1-z_i^{\lambda})^{\beta})z_i^{\lambda}\log z_i^{\lambda}}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)^3}\Bigg]\\
D_{113} =&\beta\sum_{i=1}^{n}\frac{\log z_i e^{1-(1-z_i^{\lambda})^{\beta}}(1-z_i^{\lambda})^{\beta-1}z_i^{\lambda}\log z_i}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)^2}[(e^{1-(1-z_i^{\lambda})^{\beta}}-1)\\
&(1-\frac{(\beta-1)z_i^{\lambda}}{(1-z_i^{\lambda})}+\beta z_i^{\lambda}(1-z_i^{\lambda})^{\beta-1})-\beta e^{1-(1-z_i^{\lambda})^{\beta}}(1-z_i^{\lambda})^{\beta-1}z_i^{\lambda}\log z_i]\\
D_{223} =&\beta\sum_{i=1}^{n}\frac{\log(1-z_i^{\lambda})e^{1-(1-z_i^{\lambda})^{\beta}}(1-z_i^{\lambda})^{\beta-1}(e^{1-(1-z_i^{\lambda})^{\beta}}-1-(1-z_i^{\lambda})^{\beta})}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)^2}\\
D_{123} =&\sum_{i=1}^{n}\frac{e^{1-(1-z_i^{\lambda})^{\beta}}z_i^{\lambda}\log z_i(1-z_i^{\lambda})^{\beta-1}}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)^2}[(e^{1-(1-z_i^{\lambda})^{\beta}}-1)(\beta-\beta\log(1-z_i^{\lambda})-1)\\
&-\beta(e^{1-(1-z_i^{\lambda})^{\beta}}(1-z_i^{\lambda})^{\beta}\log(1-z_i^{\lambda}))]
\end{aligned}
$$

Appendix C

$$
\begin{aligned}
\frac{\partial^2 logL}{\partial\lambda^2} =&\frac{-(n+m)}{\lambda^2}+\beta\sum_{i=1}^{n}(1-z_i^{\lambda})^{\beta-2}z_i^{\lambda}(\log z_i)^2(2-\beta-z_i^{\lambda})-(\beta-1)\sum_{i=1}^{n}\frac{z_i^{\lambda}(\log z_i)^2}{(1-z_i^{\lambda})^2}-(\beta-1)\sum_{j=1}^{m}\frac{w_j^{\lambda}(\log w_j)^2}{(1-w_j^{\lambda})^2}\\
&+\beta\sum_{j=1}^{m}(1-w_j^{\lambda})^{\beta-2}w_j^{\lambda}(\log w_j)^2(2-\beta-w_j^{\lambda})+\beta(\delta-1)\sum_{i=1}^{n}\frac{(\log z_i)^2e^{1-(1-z_i^{\lambda})^{\beta}}(1-z_i^{\lambda})^{\beta-1}z_i^{\lambda}}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)^2}\\
&\Big[\frac{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)(1-\beta z_i^{\lambda})}{(1-z_i^{\lambda})}+\beta z_i^{\lambda}(1-z_i^{\lambda})^{\beta-1}e^{1-(1-z_i^{\lambda})^{\beta}}(1-\log z_i)\Big]\\
&+\beta(\delta-1)\sum_{j=1}^{m}\frac{(\log w_j)^2e^{1-(1-w_j^{\lambda})^{\beta}}(1-w_j^{\lambda})^{\beta-1}w_j^{\lambda}}{(e^{1-(1-w_j)^{\beta}}-1)^2}\\
&\Big[\frac{(e^{1-(1-w_j^{\lambda})^{\beta}}-1)(1-\beta w_j^{\lambda})}{(1-w_j^{\lambda})}+\beta w_j^{\lambda}(1-w_j^{\lambda})^{\beta-1}e^{1-(1-w_j^{\lambda})^{\beta}}(1-\log w_j)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 logL}{\partial\lambda\partial\beta} =&\sum_{i=1}^{n}(1-z_i^{\lambda})^{\beta-1}z_i^{\lambda}\log z_i(1+\beta\log(1-z_i^{\lambda}))\sum_{j=1}^{m}(1-w_j^{\lambda})^{\beta-1}w_j^{\lambda}\log w_j(1+\beta\log(1-w_j^{\lambda}))-\sum_{i=1}^{n}\frac{z_i^{\lambda}\log z_i}{1-z_i^{\lambda}}\\
&-\sum_{j=1}^{m}\frac{w_j^{\lambda}\log w_j}{1-w_j^{\lambda}}-(\delta-1)\sum_{i=1}^{n}\frac{e^{1-(1-z_i^{\lambda})^{\beta}}z_i^{\lambda}\log z_i(1-z_i^{\lambda})^{\beta-1}}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)^2}((e^{1-(1-z_i^{\lambda})^{\beta}}-1)(\beta-\beta\log(1-z_i^{\lambda})-1)\\
&-\beta e^{1-(1-z_i^{\lambda})^{\beta}}(1-z_i^{\lambda})^{\beta}\log(1-z_i^{\lambda}))+(\delta-1)\sum_{j=1}^{m}\frac{e^{1-(1-w_j^{\lambda})^{\beta}}w_j^{\lambda}\log w_j(1-w_j^{\lambda})^{\beta-1}}{(e^{1-(1-w_j^{\lambda})^{\beta}}-1)^2}\\
&((e^{1-(1-w_j^{\lambda})^{\beta}}-1)(\beta-\beta\log(1-w_j^{\lambda})-1)-\beta e^{1-(1-w_j^{\lambda})^{\beta}}(1-w_j^{\lambda})^{\beta}\log(1-w_j^{\lambda}))
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 logL}{\partial\beta^2} =&\frac{-(n+m)}{\beta^2}+\beta(\delta-1)\sum_{i=1}^{n}\frac{\log(1-z_i^{\lambda})e^{1-(1-z_i^{\lambda})^{\beta}}(1-z_i^{\lambda})^{\beta-1}(e^{1-(1-z_i^{\lambda})^{\beta}}-1-(1-z_i^{\lambda})^{\beta})}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)^2}\\
&+\beta(\delta-1)\sum_{j=1}^{m}\frac{\log(1-w_j^{\lambda})e^{1-(1-w_j^{\lambda})^{\beta}}(1-w_j^{\lambda})^{\beta-1}(e^{1-(1-w_j^{\lambda})^{\beta}}-1-(1-w_j^{\lambda})^{\beta})}{(e^{1-(1-w_j^{\lambda})^{\beta}}-1)^2}\\
&-\sum_{i=1}^{n}\beta(1-z_i^{\lambda})^{\beta-1}\log(1-z_i^{\lambda})-\sum_{j=1}^{m}\beta(1-w_j^{\lambda})^{\beta-1}\log(1-w_j^{\lambda})
\end{aligned}
$$

$$\frac{\partial^2 logL}{\partial\lambda\partial\delta_1} = \beta\sum_{i=1}^{n}\frac{e^{1-(1-z_i^{\lambda})^{\beta}}(1-z_i^{\lambda})^{\beta-1}z_i^{\lambda}\log z_i}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)}$$

$$\frac{\partial^2 logL}{\partial\lambda\partial\delta_2} = \beta\sum_{j=1}^{m}\frac{e^{1-(1-w_j^{\lambda})^{\beta}}(1-w_j^{\lambda})^{\beta-1}w_j^{\lambda}\log w_j}{(e^{1-(1-w_j^{\lambda})^{\beta}}-1)}$$

$$\frac{\partial^2 logL}{\partial\beta\partial\delta_1} = -\sum_{i=1}^{n}\frac{e^{1-(1-z_i^{\lambda})^{\beta}}(1-z_i^{\lambda})^{\beta}\log(1-z_i^{\lambda})}{(e^{1-(1-z_i^{\lambda})^{\beta}}-1)}$$

$$\frac{\partial^2 logL}{\partial\beta\partial\delta_2} = \sum_{j=1}^{m}-\frac{e^{1-(1-w_j^{\lambda})^{\beta}}(1-w_j^{\lambda})^{\beta}\log(1-w_j^{\lambda})}{(e^{1-(1-w_j^{\lambda})^{\beta}}-1)}$$

$$\frac{\partial^2 logL}{\partial\delta_1^2} = \frac{-n}{\delta_1^2};\quad \frac{\partial^2 logL}{\partial\delta_2^2} = \frac{-m}{\delta_2^2};\quad \frac{\partial^2 logL}{\partial\delta_1\partial\delta_2} = 0$$