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Contributed paper

Asymptotic confidence ellipses of parameters for the Birnbaum-Saunders distribution

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Abstract

The purpose of this study is to find the suitable covariance matrix for the construction of confidence regions of parameters in the Birnbaum-Saunders distribution and we need to calculate confidence ellipses and compare the coverage probabilities for asymptotic confidence ellipses of parameters in the Birnbaum-Saunders distribution. Monte Carlo simulation is used to compare the coverage probabilities of the asymptotic confidence ellipses. The result showed that the asymptotic confidence ellipses can work very well when the α values increase more than 2.0 and the sample sizes (n) increase. In the Birnbaum-Saunders distribution, we can use method of moment estimators instead of maximum likelihood estimators for confidence ellipses because of high efficiency of coverage probabilities.

Keywords: method of moment, method of maximum likelihood, confidence ellipse, monte carlo simulation.

1. Introduction

The Birnbaum–Saunders (BS) distribution is introduced by Birnbaum and Saunders [1]. It is also commonly known as the fatigue life distribution. Birnbaum–Saunders distribution is used extensively in reliability applications to model failure times. Desmond [2] provided a more general derivation based on a biological model and also strengthened the physical justification for the use of this distribution by relaxing the assumptions made by Birnbaum and Saunders [1]. Desmond [3] considered the relationship between the Birnbaum–Saunders and inverse Gaussian distributions.

A continuous random variable X has a Birnbaum-Saunders distribution if X has the following cumulative distribution function

$$F(x;\alpha,\beta)=\Phi\left[\frac{1}{\alpha}\left\{\left(\frac{x}{\beta}\right)^{\frac{1}{2}}+\left(\frac{\beta}{x}\right)^{\frac{1}{2}}\right\}\right] \;,\; x>0,\alpha>0,\beta>0\,, \qquad (1)$$

where $\Phi(.)$ is the cumulative distribution function of the standard normal distribution function. α and β are the shape and scale (the median) parameters, respectively. It is known that the probability density function of the Birnbaum–Saunders distribution is unimodal and although the hazard rate is not an increasing function of these, but the average hazard rate is nearly a non-decreasing function of x [4]. The maximum likelihood estimators (MLEs) were first discussed by Birnbaum and Saunders [5] and suggested some iterative schemes to solve the required non-linear equation. Ng et al. [6] considered the modified moment estimators for the parameters to overcome this problem. However, Wu and Wong [7] reported that those expressions for the intervals of estimators for β presented incorrectly by Ng et al. [6]. Moreover, there is no guarantee that the upper bounds of those intervals are always positive.

There are some popular distributions for failure data such as Lognormal, Weibull, Gamma, Inverse Gaussian, and BS distributions. All of these distributions may fit the failure data well within the central region of the distribution, but for a high reliability product, it is quite difficult to observe sufficient amount of failure data to distinguish among these possible distributions. For example, engineers are interested in predicting the lower percentile of the failure distribution. For those cases when the data fits well for several distributions, we can pick the distribution with theoretical support. In the fatigue model when failure mechanism follows the set of above conditions, the BS distribution could serve as a proper choice.

Some practical problems need more than one statistical interval to be computed from the same data and to be considered simultaneously. For this reason, we find simultaneous estimation of both parameters, which in this paper is called elliptical confidence regions of parameters for the Birnbaum-Saunders distribution. If we construct separate 99 percent confidence intervals for parameters. The difficulty is that the confidence of both parameters would not provide 99 percent confidence that the conclusions for both parameters are correct. The probability of both being correct would be 0.99^2 or only 0.98 (98 percent).

Therefore, in this paper, we focus only on the Birnbaum–Saunders distribution. We are going to construct asymptotic confidence ellipses at 98 percent confidence level which includes the investigation the accuracy of the confidence ellipses by the Monte-Carlo method. For a solution of this problem we compare the actual coverage probability with the nominal confidence coefficient.

2. Backgrounds

2.1 The Birnbaum-Saunders distribution

The general formula for the probability density function (PDF) of the Birnbaum-Saunders Distribution is

$$f(x) = \frac{1}{2\alpha\beta\sqrt{2\pi}}\left[\left(\frac{\beta}{x}\right)^{\frac{1}{2}} + \left(\frac{\beta}{x}\right)^{\frac{3}{2}}\right]\exp\left[-\frac{1}{2\alpha^2}\left(\frac{x}{\beta}+\frac{\beta}{x}-2\right)\right] \quad , \; x>0, \alpha>0, \beta>0 \quad (2)$$

where α and β are the shape and scale (the median) parameter. The expected value, variance, skewness and kurtosis are, respectively,

$$E(X) = \beta\left(1+\frac{1}{2}\alpha^2\right) \quad (3)$$

$$Var(X) = (\alpha\beta)^2\left(1+\frac{5}{4}\alpha^2\right) \quad (4)$$

$$\mu_3 = \frac{4\alpha\left(11\alpha^2+6\right)}{\left(5\alpha^2+4\right)^{3/2}} \tag{5}$$

$$\mu_4 = 3 + \frac{6\alpha^2\left(93\alpha^2+40\right)}{\left(5\alpha^2+4\right)^2}\,; \tag{6}$$

the expressions we give for the skewness and kurtosis correct those given by Johnson et al. [8]. As noted earlier, if $X \sim$ BS(α, β), then $X^{-1} \sim$ BS(α, β^{-1}); see Saunders [9]. It then follows that

$$E(X^{-1}) = \beta^{-1}\left(1+\frac{1}{2}\alpha^2\right) \tag{7}$$

and

$$Var(X^{-1}) = \alpha^2\beta^{-2}\left(1+\frac{5}{4}\alpha^2\right). \tag{8}$$

2.2 Maximum Likelihood Estimator

Let $x = \left(x_1, x_2, \ldots, x_n\right)$ denote a random sample of size n from the Birnbaum–Saunders distribution. The log-likelihood function, apart from an unimportant constant, is

$$\ln L(\alpha,\beta) = -n\ln(\alpha\beta) + \sum_{i=1}^{n}\ln\left[\left(\frac{\beta}{x_i}\right)^{\frac{1}{2}} + \left(\frac{\beta}{x_i}\right)^{\frac{3}{2}}\right] - \frac{1}{2\alpha^2}\sum_{i=1}^{n}\left(\frac{x_i}{\beta} - \frac{\beta}{x_i} - 2\right) \tag{9}$$

Then the maximum likelihood estimators (MLEs) $\hat{\alpha}^{(MLE)}$ and $\hat{\beta}^{(MLE)}$ of α and β, respectively, are obtained from the maximization of (9), as the solution to the following system of equations:

$$\frac{\partial}{\partial\alpha}\ln L(\alpha,\beta) = -\frac{n}{\alpha}\left(1+\frac{2}{\alpha^2}\right) + \frac{1}{\alpha^3\beta}\sum_{i=1}^{n}x_i + \frac{\beta}{\alpha^3}\sum_{i=1}^{n}\frac{1}{x_i} = 0 \tag{10}$$

$$\frac{\partial}{\partial \beta}\ln L(\alpha,\beta) = -\frac{n}{2\beta}+\sum_{i=1}^{n}\left(\frac{1}{x_i+\beta}\right)+\left(\frac{1}{2\alpha^2\beta^2}\right)\sum_{i=1}^{n}x_i-\left(\frac{1}{2\alpha^2}\right)\sum_{i=1}^{n}\frac{1}{x_i} = 0 \qquad (11)$$

From (10) and (11), Birnbaum and Saunders [5] showed that $\hat{\alpha}$ can be written as

$$\hat{\alpha}=\left[\frac{s}{\hat{\beta}}+\frac{\hat{\beta}}{r}-2\right]^{\frac{1}{2}} \qquad (12)$$

where $s=\frac{1}{n}\sum_{i=1}^{n}x_i$ and $r=\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\right)^{-1}$.

In order to find $\hat{\beta}$ it is necessary to solve a nonlinear equation in β ,that is, $\hat{\beta}$ is obtained as the positive root of

$$\beta^2-\beta[2r+K(\beta)]+r[s+K(\beta)]=0 \qquad (13)$$

where $K(\delta)$ is the harmonic mean function defined by

$$K(\delta)=\left[\frac{1}{n}\sum_{i=1}^{n}(\delta+x_i)^{-1}\right]^{-1} \quad ; \ \delta\ge 0$$

so that $r\equiv K(0)$

Since (13) is a non-linear equation in β, one needs to use an iterative procedure to solve for $\hat{\beta}$. Birnbaum and Saunders [5] proposed two iterative procedures (one simple and one complicated) to compute $\hat{\beta}$, but noted that the simple one works very well for small α $\left(<\frac{1}{2}\right)$ but may not work at all for large α (>2). The complicated one also does not work in certain range of the sample space.

Theoretically by using the Delta method for normal approximation it is possible to find the asymptotic covariance matrix of two parameter estimates by maximum likelihood $\hat{\alpha}^{(MLE)}$ and $\hat{\beta}^{(MLE)}$ and after to construct an asymptotic confidence ellipse.

However, the calculations are extremely cumbersome, and are not recommended for practical applications because we cannot find the maximum likelihood estimations for parameters β of the Birnbaum-Saunders distribution in the closed form.

2.3 Method of Moment Estimators

Let $x=(x_1,x_2,\ldots,x_n)$ denote a random sample of size n from the Birnbaum–Saunders distribution. Then the method of moments estimators (MMEs) $\hat{\alpha}^{(MME)}$ and $\hat{\beta}^{(MME)}$ of α and β are solution of these two equations

$$\hat{\alpha}^{(MME)}=\left\{2\left[\left(\frac{s}{r}\right)^{\frac{1}{2}}-1\right]\right\}^{\frac{1}{2}} \tag{14}$$

$$\hat{\beta}^{(MME)}=(rs)^{\frac{1}{2}} \tag{15}$$

where $s=\frac{1}{n}\sum_{i=1}^{n}x_i$ and $r=\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\right)^{-1}$.

Method of moments estimators of parameters α and β are found by equating the first two sample moments to the corresponding two population moments, and solving the resulting system of simultaneous equations. Defined as

$$m_1'=\frac{1}{n}\sum_{i=1}^{n}x_i\ ,\ \mu_1'=E(X)=\beta\left(1+\frac{1}{2}\alpha^2\right)$$

$$m_2'=\frac{1}{n}\sum_{i=1}^{n}x_i^2\ ,\ \mu_2'=E(X^2)=\beta^2\left(\frac{3}{2}\alpha^4+2\alpha^2+1\right)$$

To avoid solving the non-linear equation, moment type estimators of α and β have been proposed, and they can be obtained in explicit forms. It is basically obtained by equating $E(X)$ and $E(X^{-1})$ with the arithmetic mean and the harmonic mean of the data. They are as follows;

Defined as

$$m_1' = \frac{1}{n}\sum_{i=1}^{n} x_i = s\,,\quad \mu_1' = E\left(X\right) = \beta\left(1+\frac{1}{2}\alpha^2\right)$$

$$m_2' = \frac{1}{n}\sum_{i=1}^{n} \left(x_i\right)^{-1} = r^{-1}\,,\quad \mu_2' = E\left(X^2\right) = \beta^{-1}\left(1+\frac{1}{2}\alpha^2\right)$$

Ng et al. [6] showed that the asymptotic joint distribution of $\hat{\alpha}^{(MME)}$ and $\hat{\beta}^{(MME)}$ is bivariate normal and is given by

$$\begin{pmatrix} \hat{\alpha}^{(MME)} \\ \hat{\beta}^{(MME)} \end{pmatrix} \sim \mathrm{N}_2\left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \dfrac{\alpha^2}{2n} & 0 \\ 0 & \dfrac{(\alpha\beta)^2}{n}\left(\dfrac{1+\frac{3}{4}\alpha^2}{\left(1+\frac{1}{2}\alpha^2\right)^2}\right) \end{pmatrix}\right]. \tag{16}$$

From (16), the covariance matrix is denoted by

$$\mathbf{\Sigma} = \begin{pmatrix} \dfrac{\alpha^2}{2n} & 0 \\ 0 & \dfrac{(\alpha\beta)^2}{n}\left(\dfrac{1+\frac{3}{4}\alpha^2}{\left(1+\frac{1}{2}\alpha^2\right)^2}\right) \end{pmatrix}. \tag{17}$$

2.4 The Fisher Information of Parameters for the Birnbaum-Saunders distribution

Engelhardt et al. [10] showed that the Fisher information matrix of $\boldsymbol{\theta}$, where $\boldsymbol{\theta}$ = (α , β) is a two-dimensional vector of parameters, denoted by $\mathbf{I}\left(\boldsymbol{\theta}\right)$ defined as

$$\mathbf{I}(\boldsymbol{\theta}) = \mathbf{I}(\alpha,\beta) = \begin{pmatrix} \frac{2}{\alpha^2} & 0 \\ 0 & \frac{\left[\alpha(2\pi)^{-\frac{1}{2}}h(\alpha)+1\right]}{\alpha^2\beta^2} \end{pmatrix} \quad (18)$$

with $h(\alpha)=\alpha\sqrt{\frac{\pi}{2}}-\pi e^{\frac{2}{\alpha^2}}\left[1-\Phi\left(\frac{2}{\alpha}\right)\right]$.

Note that the expression we give for Fisher's information matrix only involves numerical integration through the evaluation of the standard normal distribution function $\Phi(t)$, where $\Phi(t)=\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$.

We need to compare the difference of confidence regions for two derivations of covariance matrix, so we assume that the method of moments estimators (MMEs) are under certain regularity conditions on $f(x;\theta)$, the MME $\hat{\theta}^{(MME)}$ of θ based on sample sizes n from $f(x;\theta)$ is asymptotic normal distribution (the same principle of MLEs). The vector of estimates $\hat{\boldsymbol{\theta}}^{(MME)'}=\left(\hat{\alpha}_n^{(MME)},\hat{\beta}_n^{(MME)}\right)$ has a two-dimensional normal distribution with the mean equal to the vector of true values of the parameters, that is $\boldsymbol{\theta}'=(\alpha,\beta)$ and the covariance matrix equal the inverse to the Fisher information matrix, denoted by $\boldsymbol{\Lambda}=\mathbf{I}_{\mathbf{n}}^{-1}(\boldsymbol{\theta})$. That is, as $n\to\infty$, $\hat{\boldsymbol{\theta}}_n^{(MME)}\sim\mathrm{N}_2(\boldsymbol{\theta},\boldsymbol{\Lambda})$.

And the inverse of the Fisher information matrix is computed as

$$\boldsymbol{\Lambda}=\mathbf{I}_{\mathbf{n}}^{-1}(\boldsymbol{\theta})=\frac{1}{n}\begin{pmatrix} \frac{\alpha^2}{2} & 0 \\ 0 & \frac{\alpha^2\beta^2}{\left[\alpha(2\pi)^{-\frac{1}{2}}h(\alpha)+1\right]} \end{pmatrix}. \quad (19)$$

3. Main Results

The $N_2\left(\boldsymbol{\theta},\boldsymbol{\Sigma}^{-1}\right)$ distribution assigns probability $1-\alpha$ to the ellipse $\left\{\hat{\boldsymbol{\theta}}_n^{(MME)}:\left(\hat{\boldsymbol{\theta}}_n^{(MME)}-\boldsymbol{\theta}\right)'\boldsymbol{\Sigma}^{-1}\left(\hat{\boldsymbol{\theta}}_n^{(MME)}-\boldsymbol{\theta}\right)\le\chi^2_{(2)}(\alpha)\right\}$, where $\chi^2_{(2)}(\alpha)$ denotes the upper 100α th percentile of the $\chi^2_{(2)}$ distribution.

For the first model, we take covariance matrix that comes from the asymptotic joint distribution that defined in (17) and construct a $100(1-\alpha)\%$ confidence region that we call "Model I".

A $100(1-\alpha)\%$ confidence region for parameters $\boldsymbol{\theta}'=(\alpha,\beta)$ of a two-dimensional normal distribution is the ellipse determined by all $\boldsymbol{\theta}$ such that

$$\left(\hat{\boldsymbol{\theta}}_n^{(MME)}-\boldsymbol{\theta}\right)'\boldsymbol{\Sigma}^{-1}\left(\hat{\boldsymbol{\theta}}_n^{(MME)}-\boldsymbol{\theta}\right)\le\chi^2_{(2)}(\alpha)$$

where

$$\left(\hat{\boldsymbol{\theta}}_n^{(MME)}-\boldsymbol{\theta}\right)'\boldsymbol{\Sigma}^{-1}\left(\hat{\boldsymbol{\theta}}_n^{(MME)}-\boldsymbol{\theta}\right) =$$

$$\begin{pmatrix}\hat{\alpha}_n^{(MME)}-\alpha & \hat{\beta}_n^{(MME)}-\beta\end{pmatrix}\ \boldsymbol{\Sigma}^{-1}\ \begin{pmatrix}\hat{\alpha}_n^{(MME)}-\alpha\\ \hat{\beta}_n^{(MME)}-\beta\end{pmatrix}$$

Model I: The $100(1-\alpha)\%$ confidence region for $\boldsymbol{\theta}$ consists of all value (α,β) satisfying

$$\left(\frac{2n}{\alpha^2}\left(\hat{\alpha}_n^{(MME)}-\alpha\right)^2+\frac{n}{(\alpha\beta)^2}\left(\frac{\left(1+\frac{1}{2}\alpha^2\right)^2}{1+\frac{3}{4}\alpha^2}\right)\left(\hat{\beta}_n^{(MME)}-\beta\right)^2\right)\le\chi^2_{(2)}(\alpha)\ . \qquad (20)$$

For the second model, we take the covariance matrix that comes from the Fisher information matrix that defined in (19) and construct a $100(1-\alpha)\%$ confidence region that we call "Model II".

A $100(1-\alpha)\%$ confidence region for parameters $\boldsymbol{\theta}' = (\alpha, \beta)$ of a two-dimensional normal distribution is the ellipse determined by all $\boldsymbol{\theta}$ such that

$$\left(\hat{\boldsymbol{\theta}}_n^{(MME)} - \boldsymbol{\theta}\right)' \boldsymbol{\Lambda}^{-1} \left(\hat{\boldsymbol{\theta}}_n^{(MME)} - \theta\right) \le \chi^2_{(2)}(\alpha)$$

From (19),

$$\left(\hat{\boldsymbol{\theta}}_n^{(MME)} - \boldsymbol{\theta}\right)' \mathbf{I_n}(\boldsymbol{\theta}) \left(\hat{\boldsymbol{\theta}}_n^{(MME)} - \boldsymbol{\theta}\right) \le \chi^2_{(2)}(\alpha)$$

which is equivalent

$$\left(\hat{\boldsymbol{\theta}}_n^{(MME)} - \boldsymbol{\theta}\right)' n\mathbf{I}(\boldsymbol{\theta}) \left(\hat{\boldsymbol{\theta}}_n^{(MME)} - \boldsymbol{\theta}\right) \le \chi^2_{(2)}(\alpha).$$

Model II: The $100(1-\alpha)\%$ confidence region for $\boldsymbol{\theta}$ consists of all value (α, β) satisfying

$$\frac{n}{\alpha^2}\left(2\left(\hat{\alpha}_n^{(MME)} - \alpha\right)^2 + \frac{\left[\alpha(2\pi)^{-\frac{1}{2}} h(\alpha) + 1\right]}{\beta^2}\left(\hat{\beta}_n^{(MME)} - \beta\right)^2 \right) \le \chi^2_{(2)}(\alpha). \qquad (21)$$

4. Monte Carlo simulation results

In order to compare the efficiency of all confidence regions, we performed a simulation study for different sample sizes and for different parameters values. We took the sample size as n = 10, 100 and 1,000, and the shape parameter as α = 0.1, 0.5, 1.0, 2.0 and 5.0. Since β is the scale parameter, β was kept fixed at 1.0, without loss of any generality. The experimental data are generated by the simulation technique using R program version 2.15.2. For each situation, the experiment is repeated 10,000 times to obtain the coverage probability. The results so obtained are reported in Table1.

The 98% confidence regions for α and β based on the method of moment estimators are given by

Model I: $$\left(\frac{2n}{\alpha^2}\left(\hat{\alpha}_n^{(MME)} - \alpha\right)^2 + \frac{n}{(\alpha\beta)^2}\left(\frac{\left(1+\frac{1}{2}\alpha^2\right)^2}{1+\frac{3}{4}\alpha^2} \right)\left(\hat{\beta}_n^{(MME)} - \beta\right)^2 \right) \leq \chi^2_{(2)}(\alpha)$$

Model II: $$\frac{n}{\alpha^2}\left(2\left(\hat{\alpha}_n^{(MME)} - \alpha\right)^2 + \frac{\left[\alpha(2\pi)^{-\frac{1}{2}} h(\alpha)+1\right]}{\beta^2}\left(\hat{\beta}_n^{(MME)} - \beta\right)^2 \right) \leq \chi^2_{(2)}(\alpha)$$

Table 1. Method of Moment estimates of α and β, their errors and coverage probabilities for confidence ellipses at 98% confidence level (Model I).

n	α	β	Method of Moment estimates		The percentages of absolute relative errors		Coverage probabilities
			$\overline{\hat{\alpha}}_n^{(MME)}$	$\overline{\hat{\beta}}_n^{(MME)}$	$\overline{\hat{\alpha}}_n^{(MME)}$	$\overline{\hat{\beta}}_n^{(MME)}$	
10	0.1	1.0	1.00480	1.00026	904.480	0.260	0
	0.5	1.0	1.10582	1.01146	121.164	2.292	0
	1.0	1.0	1.36450	1.04214	36.450	4.214	0.8854
	2.0	1.0	2.08118	1.10974	4.059	10.974	0.9470
	5.0	1.0	4.59170	1.24829	8.166	24.829	0.9080
100	0.1	1.0	1.00493	1.00005	904.930	0.005	0
	0.5	1.0	1.11647	1.00113	123.294	0.113	0
	1.0	1.0	1.40932	1.00316	40.932	0.316	0
	2.0	1.0	2.22008	1.00898	11.004	0.898	0.8622
	5.0	1.0	5.04636	1.01395	0.927	1.395	0.9735
1,000	0.1	1.0	1.00498	1.00000	904.480	0.003	0
	0.5	1.0	1.11793	1.00002	123.562	0.005	0
	1.0	1.0	1.41387	1.00006	41.380	0.039	0
	2.0	1.0	2.23500	1.00054	11.739	0.178	0.0019
	5.0	1.0	5.0936	1.00178	1.880	0.210	0.9739

Table 2. Method of Moment estimates of α and β, their errors and coverage probabilities for confidence ellipses at 98% confidence level (Model II).

n	α	β	Method of Moment estimates		The percentages of absolute relative errors		Coverage probabilities
			$\bar{\hat{\alpha}}_n^{(MME)}$	$\bar{\hat{\beta}}_n^{(MME)}$	$\bar{\hat{\alpha}}_n^{(MME)}$	$\bar{\hat{\beta}}_n^{(MME)}$	
10	0.1	1.0	1.00446	1.00050	904.460	0.050	0
	0.5	1.0	1.10564	1.00892	121.128	0.892	0
	1.0	1.0	1.10526	1.01253	60.526	1.253	0
	2.0	1.0	2.08218	1.10586	4.109	10.586	0.9575
	5.0	1.0	4.56994	1.23478	8.601	23.478	0.9604
100	0.1	1.0	1.00493	1.00003	904.93	0.003	0
	0.5	1.0	1.11669	1.00110	123.338	0.110	0
	1.0	1.0	1.40914	1.00365	40.914	3.650	0
	2.0	1.0	2.22306	1.00994	11.153	0.994	0.8774
	5.0	1.0	5.05031	1.01211	1.006	1.211	0.9731
1,000	0.1	1.0	1.00498	1.00003	904.980	0.003	0
	0.5	1.0	1.11781	1.00005	123.562	0.005	0
	1.0	1.0	1.41380	1.00039	41.380	0.039	0
	2.0	1.0	2.23439	1.00082	11.720	0.082	0.0016
	5.0	1.0	5.09402	1.00210	1.880	0.210	0.9739

From Table 1, the coverage probabilities of confidence ellipses for parameters of the Birnbaum-Saunders distribution increase when sample sizes (n) increase for the situation of $\alpha = 5.0$ when we fix $\beta = 1.0$. The confidence ellipses that we construct cannot work when alpha values are less than 2.0 but they are close to the confidence coefficient 0.98 when alpha values are greater than, or equal to 2.0 that except when sample sizes (n) is 1,000, they cannot work well when $\alpha = 2.0$.

From Table 2, the coverage probabilities of confidence ellipses for parameters of the Birnbaum-Saunders distribution of Model II increase when sample sizes (n) increase for the situation of $\alpha = 5.0$ when we fix $\beta = 1.0$. The confidence ellipses that we construct cannot work when alpha values are less than 2.0 as same as Model I but they are close to the confidence coefficient 0.98 when alpha values are greater than, or equal to 2.0 that except when sample sizes (n) is 1,000, they cannot work well when $\alpha = 2.0$ as same as Model I.

The difference between Model I and Model II is the covariance matrix that we obtained from the asymptotic joint distribution and the Fisher information matrix, respectively. From Table 1 and Table 2, the coverage probabilities in Model II is greater than Model I of all parameter values and sample sizes that it implies, the covariance matrix from the Fisher information matrix can be used in method of moment estimators for the Birnbaum-Saunders distribution well. Because of that we substitutes method of moment estimates instead of maximum likelihood estimates for asymptotic confidence ellipses of parameters for the Birnbaum-Saunders distribution.

We selected the highest coverage probabilities to construct graphs of 98% confidence ellipses of parameters for the Birnbaum-Saunders distribution when $\beta = 1$ and some cases of the confidence ellipses cannot work when alpha values are less than 2.0 that showed in Table 3

From Table 3, these graphs showed the confidence ellipse for parameters when $\alpha = 0.5, 5$ and $n = 10,\ 100$ and $1{,}000$ for both two models.

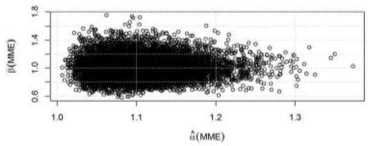
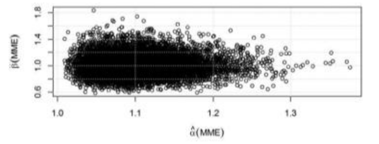
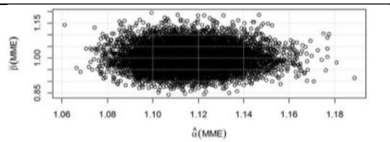
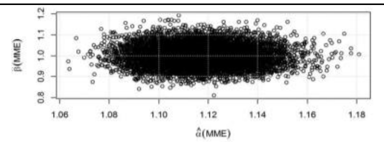
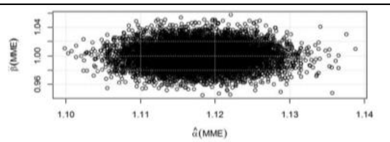
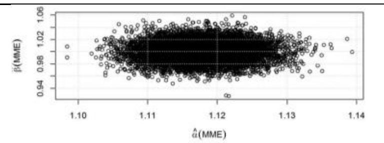
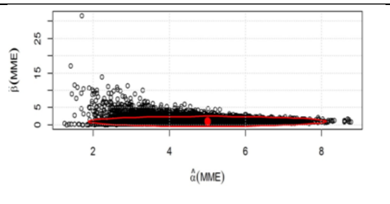
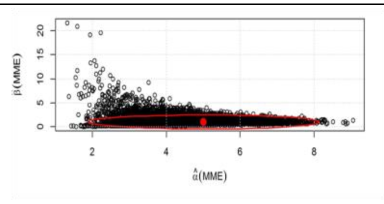
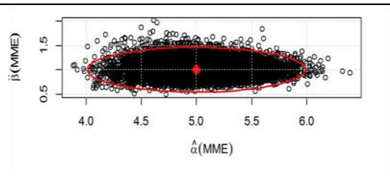
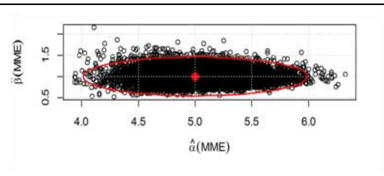
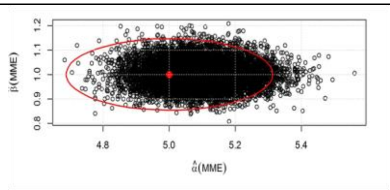
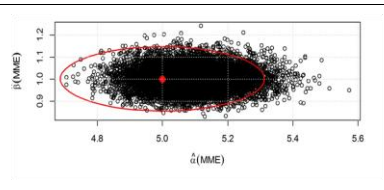
When $\alpha = 0.5$, all of point estimators that calculate from method of moment did not fall in the confidence ellipse. On the contrary, most of point estimators that calculate from method of moment fell in the confidence ellipse well when $\alpha = 5$. We can refer this result from the coverage probabilities values in Table 1.

5. Concluding Remarks

We have derived the two models (Model I and Model II) of confidence regions that can work very well when the α values increase more than 2.0 and the sample sizes (n) increase. The difference between Model I and Model II is the covariance matrix that we obtained from the asymptotic joint distribution and the Fisher information matrix, respectively.

In the Birnbaum-Saunders distribution, we can use method of moment estimators by using the covariance matrix from the Fisher information matrix instead of using the covariance matrix from the asymptotic joint distribution for construction confidence ellipses because of high efficiency of coverage probabilities.

Table 3. 98% Confidence Ellipse of Parameters for the Birnbaum-Saunders distribution when $\beta = 1$.

n	α	Confidence Ellipse for Parameters : Model I	Confidence Ellipses for Parameter : Model II
10	0.5		
100	0.5		
1,000	0.5		
10	5.0		
100	5.0		
1,000	5.0		

6. Future Research

The studies presented in this research suggest some directions for the future research as follows:

1. In the paper, we discussed only study of the Birnbaum-Saunders distribution which is useful in managing reliabilities for many parts of our lives. In future research we may suggest other distribution such as Weibull, Gamma, Lognormal etc.
2. We can use an alternative approach to construct confidence intervals and regions. An alternative approach to the problem is use of the bootstrap procedure. The bootstrap is a method for developing the sampling distribution of a statistic computed from a set of data by resampling the data and recomputing the statistic from the resampled data.
3. We can approve estimators by decreasing the bias. An alternative theorem to decrease the bias is Rao and Blackwell Theorem. The Rao and Blackwell Theorem is a theory for finding the Minimum Variance Unbiased Estimator from a sufficient statistic.

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