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Information Measures for Generalized Beta Distributions as an Income Distribution

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Abstract

In this paper, we study the information properties of generalized beta of the second kind $(GB2)$ distribution. We derived discrimination information function between two $GB2$ distributions with different parameters and between the $GB2$ and its subfamilies, including the generalized gamma (GG), Beta of the second kind, Singh-Maddala, Dagum, Fisk, Lomax and Inverse Lomax distributions. For large values of q, we have shown through using Stirling's formula the discrimination information function between $GB2$ and $GG_{\circ}$ as a special case, where $GG_{\circ}$ is a generalized gamma distribution with specified parameters. So, our results are the extended version of what was obtained by Dadpay, et al. [1]. Our achievement for a big class of distributions such as the one mentioned above is held. Finally, we discussed the minimum discrimination information model for $GB2$ family.

Keywords: Entropy, Kullback-Liebler Information, Minimum Discrimination Information, Generalized Beta of Second Kind, Generalized Gamma.

1. Introduction

The generalized beta of the second kind ($GB2$) is particularly a useful family of distributions. This distribution has been widely utilized in many branches of science, especially in economics. The GB2 family is flexible in that it includes several well-known models as subfamilies. the GB2 family, which encompasses Singh-Maddala, Beta distribution of second kind, Dagum, Fisk, Lomax, Inverse Lomax and generalized gamma distribution as a limiting distribution, has been used in economics. Also, members of this family and their inverse distributions have significant potential for improving the distributional fit in many applications involving thin or heavy-tailed distributions. Members of the GB2 family can be generated as mixtures of well-known distributions and provide a model for heterogeneity in claims distributions. Some authors studied properties of GB2 and its subfamilies, for example, McDonald [2], McDonald and Xu [3], Cummins, et al. [4], Kleiber [5], Soofi, et al. [6], Nadarajah and Kotz [7] and so on. Also, Dadpay, et al. [1] studied information properties of the GG distribution which is a special case of $GB2$.

In this paper, we studied information properties of the generalized beta of second kind ($GB2$) distribution and derive discrimination information function between $GB2$ and its subfamilies. In special cases, we computed discrimination information function between $GB2$ and generalized gamma distribution with specified parameters $(GG_{\circ})$ and it was concluded that for large values of q, the result is equal to discrimination information function between GG and $GG_{\circ}$, by using Stirling's formula. Also, minimum discrimination information model in $GB2$ family is discussed.

2. Preliminaries

In this section, we studied generalized beta of the second kind distribution and its subfamilies. Then, we interpreted their relationships. Additionally, we studied some properties of $GB2$ family needed in the next section.

2.1 Generalized Beta distribution of the second kind (GB2)

A random variable X is said to be a standard beta distribution with parameter α and β if its probability density function is:

$$f(x)=\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \qquad 0<x<1\ ,\ \alpha>0\ ,\ \beta>0, \tag{1}$$

where

$B(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx,$

denotes the beta function.

Many generalizations of standard beta distributions have been proposed in by many scholars, some of which were mentioned in introduction. The generalized beta distribution of the second kind ($GB2$) is one of them that has an important role in income and loss distributions. The probability density function of the generalized beta distribution of the second kind $GB2(a,b,p,q)$ is:

$$f(y\mid a,b,p,q)=\frac{ay^{ap-1}}{b^{ap}B(p,q)\left[1+(\frac{y}{b})^{a}\right]^{p+q}}, \qquad y>0, \tag{2}$$

where b is scale and $a,\ p,\ q$ are shape parameters and all parameters are positive. The cumulative distribution function of $GB2(a,b,p,q)$ can also be expressed as follows:

$$F(y)=\frac{(\frac{y}{b})^{a}}{pB(p,q)[1+(\frac{y}{b})^{a}]^{p}}\cdot{}_2F_1[p,1-q;\ p+1;\ \frac{(\frac{y}{b})^{a}}{1+(\frac{y}{b})^{a}}], \quad y>0, \tag{3}$$

where

$${}_2F_1(a,b;\ c;\ x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\cdot\frac{x^n}{n!}, \qquad |x|<1,$$

it shows the Gauss's hypergeometric function (see Nadarajah and Kotz [7]).

The $GB2$ family contains a large number of income and loss distributions. For example, by putting $(p=1),\ (a=1),\ (q=1),\ (p=q=1),\ (a=p=1)$ and $(a=q=1)$ in $GB2(a,b,p,q)$ form, we obtain Singh-Maddala (SM)[1], Beta distribution of second kind (B_2), Dagum (D)[2], Fisk[3], Lomax (L) and Inverse Lomax (IL)

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[1] - Singh – Madala is called Burr (XII) also.
[2] - Dagum is called Burr (III) also.
[3] - Fisk is called Log Logistic also.

distributions respectively as special cases of $GB2(a,b,p,q)$. The generalized gamma distribution (GG) is obtained from $GB2$, when $b=\beta q^{\frac{1}{a}}$ and letting $q\to\infty$. Its density is

$$f(y\mid a,\beta,p)=\frac{ay^{ap-1}e^{-(\frac{y}{\beta})^{a}}}{\beta^{ap}\Gamma(p)},\quad y\geq 0. \tag{4}$$

The exponential (EXP), gamma (GA), weibull (W), and generalized normal (GN) distributions that are special cases of GG can be derived from $GB2$, as well. Links of $GB2(a,b,p,q)$ with other distributions are shown in figure 1.

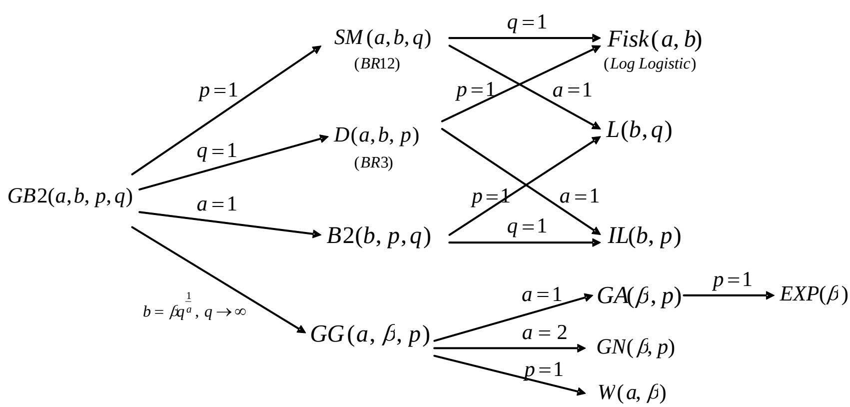

Figure 1. GB2 distribution tree.

Some properties of the $GB2$ distribution have been studied by Kleiber [5]. We concentrate on three important properties used in this paper.

$$Y\sim GB2(a,b,p,q)\Rightarrow\begin{cases} Y^{r}\sim GB2(\frac{a}{r},b^{r},p,q),\quad r>0 & (5)\\ \frac{1}{Y}\sim GB2(a,\frac{1}{b},p,q) & (6)\\ E(Y^{k})=\dfrac{b^{k}\Gamma(p+\frac{k}{a}).\Gamma(q-\frac{k}{a})}{\Gamma(p).\Gamma(q)},\quad -ap<k<aq. & (7)\end{cases}$$

3. Main Results

Our aim in this section is to further investigate the information properties of $GB2$ family. Therefore, we consider a class of distribution functions namely Ω_θ, then we show that the maximum entropy model in this class is $GB2(a,b,p,q)$. We derive Shannon entropy of $GB2$ and its subfamilies; also, we discuss entropy ordering in $GB2$ distribution. At last, we derive discrimination information function between $GB2$ and its subfamilies. In special cases, we compute discrimination information function between $GB2$ and $GG_\circ$ by using Stirling's formula and we illustrate that for large values of q, the result is equal to discrimination information function between GG and $GG_\circ$. So, our result is an extended version of what is obtained by Dadpay, et al. [1]. Our finding is correct for a big class of distributions such as those mentioned above. Finally, we discuss minimum discrimination information model in $GB2$ family.

3.1 Information Properties of GB2 Family

To study the information properties of $GB2$ family, we consider the class of distribution functions

$$\Omega_\theta = \{F(y \mid \theta) : E_f[T_j(Y) \mid \theta] = \theta_j, \quad j = 0, 1, 2\}, \tag{8}$$

where $\theta = (\theta_\circ, \theta_1, \theta_2)$, such that $\theta_\circ = T_\circ(y) = 1$ are the normalizing of the density.

$$T_1(y) = \ln y \quad \Rightarrow \quad \theta_1 = \mu(a,b,p,q) = E_{GB2}(\ln Y) = \ln b + \frac{\psi(p) - \psi(q)}{a}, \tag{9}$$

where $\psi(x) = \dfrac{d \ln \Gamma(x)}{dx}$ is the digamma function.

and

$$T_2(y) = \ln[1 + (\frac{y}{b})^a] \;\Rightarrow\; \theta_2 = \gamma(p,q) = E_{GB2}[\ln(1 + (\frac{Y}{b})^a] = \psi(p+q) - \psi(q). \tag{10}$$

The entropy of a random variable X with distribution F in Ω_θ is given by

$$H(f) = -\int_{-\infty}^{+\infty} f(y \mid a,b,p,q) \ln f(y \mid a,b,p,q) dy. \tag{11}$$

Via maximum entropy theorem (an easy proof of it can be found in Kagan, *et* al. [8]) we obtain that the maximum entropy model is $F^{*}=GB2^{*}=GB2(a,b,p,q)$ when the constraints are as in (8), and its entropy is:

$$H(GB2^{*})=\max[H(f)]=\ln B(p,q)+\ln\frac{b}{a}+(1-ap)\{\frac{\psi(p)-\psi(q)}{a}\}+(p+q)\psi(p+q)-(p+q)\psi(q). \tag{12}$$

(More details in appendix, A).

- For specific values of the parameters, (12) implies entropy expressions for Singh-Maddala, beta distribution of second kind, Dagum, log logistic, Lomax and Inverse Lomax distribution when $(p=1)$, $(a=1)$, $(q=1)$, $(p=q=1)$, $(a=p=1)$ and $(a=q=1)$, respectively.
- By setting $b=\beta q^{\frac{1}{a}}$ and $q\to\infty$, we obtained maximum entropy for $GB2(a,b,p,q)$ that is equal to entropy of $GG\,(a,\beta,p)$ with the following form:

$$H(GG)=\ln\beta+\ln\Gamma(p)+p-\ln a+(\frac{1}{a}-p)\psi(p), \tag{13}$$

which is achieved by Nadarajah and Zografos [9]. (More details in appendix, B).

Entropy ordering of GG was studied in Dadpay, et al. [1], also entropy ordering of distributions within many parametric families were studied in Ebrahimi, et al. [10], in view of this method, the entropy ordering of $GB2$ is achieved. It is clear that the entropy of $GB2$ family is ordered by scale parameter b. For the entropy orderings in terms of the shape parameters, we have

$$\frac{\partial H(GB2)}{\partial a}>0\Leftrightarrow a>\psi(p)-\psi(q),$$

$H(GB2)$ can be increasing in $a,$ if $(a,p,q)\in D$ where

$$D=\{(a,p,q):a>\psi(p)-\psi(q)\}.$$

$\frac{\partial H(GB2)}{\partial p} > 0 \Rightarrow H(GB2)$ can increase in $p,$ if $(a, p, q) \in S$ where

$$S = \{(a, p, q) : p < \frac{aq\psi_p(p+q) + \psi_p(p)}{a\psi_p(p) - a\psi_p(p+q)}\},$$

where $\psi_p(\circ)$ mean derivation of ψ with respect to $p.$

Finally, $\frac{\partial H(GB2)}{\partial q} > 0 \Rightarrow H(GB2)$ can increase in $q,$ if $(a, p, q) \in T$ where

$$T = \{(a, p, q) : q > \frac{\psi_q(q) - ap\psi_q(p+q)}{a\psi_q(p+q) - a\psi_q(q)}\},$$

where $\psi_q(\circ)$ mean derivation of ψ with respect to $q.$

3.2 Discrimination Information Properties

Information, in a technical sense, in many seemingly diverse statistical problems is quantified in a unified manner by using suitably chosen discrimination information (Kullback-Liebler) function,

$$K(F : F_\circ) = \int f(y) \log \frac{f(y)}{f_\circ(y)} dy, \tag{14}$$

where $F_\circ$ refers to as the reference distribution (Soofi [11]). It is well-known that $K(F : F_\circ) \geq 0$, its equality holds if and only if $f(y) = f_\circ(y)$ for all y in the support of the distributions.

Let $F_\circ = GB2_\circ = GB2(a_\circ, b_\circ, p_\circ, q_\circ)$ be a given $GB2$ distribution. After some mathematical computations, it can be concluded that the discrimination information functions between $F = GB2(a, b, p, q)$ and $F_\circ$ is given by:

$$K(GB2:GB2_\circ) = -\ln\frac{B(p,q)}{B(p_\circ,q_\circ)} + \ln\frac{\varphi_a}{\varphi_b^{\,p\varphi_a}} + (\varphi_a p - p_\circ)\mu(\varphi_a,\varphi_b,p,q) - (p+q)\gamma(p,q) +$$

$$\left\{ \frac{1}{\varphi_b^{\,p\varphi_a}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(p+\frac{n+1}{\varphi_a})} {}_2F_1(p+\frac{n+1}{\varphi_a}, p+q; p+\frac{n+1}{\varphi_a}+1; -\frac{1}{\varphi_b^{\,\varphi_a}}) \right.$$

$$+\varphi_b^{\,q\varphi_a} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(q+\frac{n+1}{\varphi_a})} {}_2F_1(q+\frac{n+1}{\varphi_a}, p+q; q+\frac{n+1}{\varphi_a}+1; -\frac{1}{\varphi_b^{\,\varphi_a}}) \qquad (15)$$

$$\left. -\frac{1}{\varphi_a \varphi_b^{\,q\varphi_a}} \sum_{n=0}^{\infty} \binom{-p-q}{n} \frac{1}{(q+n)^2 \varphi_b^{\,n\varphi_a}} \right\} \frac{p_\circ + q_\circ}{B(p,q)},$$

where $\varphi_a = \dfrac{a}{a_\circ}$, $\varphi_b = (\dfrac{b}{b_\circ})^{a_\circ}$.

(More details in appendix, C).

Although $K(GB2:GB2_\circ)$ is a complicated function of the parameters, (15) is a general representation of the discrimination information function between the $GB2$ and its subfamilies, between distributions within each subfamily, and between distributions of different subfamilies.

Let, we put in (15), $(p_\circ = 1)$, $(p_\circ = a_\circ = 1)$, $(p_\circ = q_\circ = 1)$, $(a_\circ = 1)$, $(q_\circ = 1)$ and $(a_\circ = q_\circ = 1)$, then, $K[GB2(a,b,p,q):SM(a_\circ,b_\circ,q_\circ)]$, $K[GB2(a,b,p,q):L(b_\circ,q_\circ)]$, $K[GB2(a,b,p,q):Fisk(a_\circ,b_\circ)]$, $K[GB2(a,b,p,q):B2(b_\circ,p_\circ,q_\circ)]$, $K[GB2(a,b,p,q):D(a_\circ,b_\circ,p_\circ)]$ and $K[GB2(a,b,p,q):IL(b_\circ,p_\circ)]$ show the discrimination information between the two types of distributions, respectively.

In special cases, the discrimination information between $GB2(a,b,p,q)$ and $GB2(a_\circ,b_\circ,p_\circ,q_\circ)$ where $a = a_\circ$, $b = b_\circ$ and $p \neq p_\circ$, $q \neq q_\circ$ is given by:

$$K(GB2,GB2_\circ) = \ln\frac{B(p_\circ,q_\circ)}{B(p,q)} + a_\circ(p-p_\circ)\mu(a_\circ,b_\circ,p,q) + [(p-p_\circ)+(q-q_\circ)]\gamma(p,q). \qquad (16)$$

Let $F_\circ = GG_\circ(a_\circ, \beta_\circ, p_\circ)$ be a given GG distribution, we derive discrimination information function between $F = GB2(a,b,p,q)$ and $F_\circ$ as follows:

$$K(GB2, GG_\circ) = \ln\frac{a}{a_\circ} - \ln\frac{B(p,q)}{\Gamma(p_\circ)} - ap\ln b + a_\circ p_\circ \ln\beta_\circ + (\frac{b}{\beta_\circ})^{a_\circ} \frac{\Gamma(p+\frac{a_\circ}{a}).\Gamma(q-\frac{q_\circ}{a})}{\Gamma(p).\Gamma(q)}$$
$$+ (ap - a_\circ p_\circ)\mu(a,b,p,q) - (p+q)\gamma(a,b).$$

(17)

- For large values of $q,$ the gamma function and digamma function are

  $$\Gamma(x) \cong e^{-x} x^{x-\frac{1}{2}}\sqrt{2\pi},$$

  $$\psi(x) \cong \ln x - \frac{1}{2x},$$

  it is approximated by Stirling's formula.

Then, we have:

$$\lim_{\substack{b=\beta q^{\frac{1}{a}} \\ q\to+\infty}} K(GB2:GG_\circ) = \ln\frac{a}{a_\circ} - \ln\frac{\Gamma(p)}{\Gamma(p_\circ)} - p + (p - \frac{ap_\circ}{a})\psi(p) - a_\circ p_\circ \ln\frac{\beta}{\beta_\circ} + (\frac{\beta}{\beta_\circ})^{a_\circ}\frac{\Gamma(p+\frac{a}{a_\circ})}{\Gamma(p)}$$
$$= K(GG:GG_\circ).$$

Now, $K(GG:GG_\circ)$ what has been achieved by Dadpay, et al. [1] can be a special case of $K(GB2:GG_\circ)$. If we put in (17), $(a_\circ = 1)$, $(p_\circ = 1)$, $(a_\circ = p_\circ = 1)$ and $(a_\circ = 2, p_\circ = 2p)$, then (17) is $K[GB2(a,b,p,q):G(p_\circ,\beta_\circ)]$, $K[GB2(a,b,p,q):W(a_\circ,\beta_\circ)]$, $K[GB2\ (a,b,p,q):EXP\ (\beta_\circ)]$ and $K[GB2(a,b,p,q):GN(p_\circ,\beta_\circ)]$ respectively. These results lead to Dadpay et al. [1], if $b = \beta q^{\frac{1}{a}}$ and $q \to +\infty$.

Kullback [12] developed the theoretical grounds for various applications of the minimum discrimination information (MDI) statistics. The MDI model, relative to a reference distribution $F_\circ$, was obtained by minimizing $K(F:F_\circ)$ subject to the constraint with the form $E_f[T_j(X)]=\theta_j,\ \ j=0,1,2.$ The MDI theorem of Kullback [12] states the MDI model $F^*,$ if it exists, then it has the following form

$$f^*(y\mid\theta)=\eta_\circ f_\circ(y)y^{\eta_1}[1+(\frac{y}{b})^a]^{\eta_2}, \tag{18}$$

where $\eta_\circ=\eta_\circ(\theta)$ is the normalizing factor, $\eta_1=\eta_1(\theta)$, $\eta_2=\eta_2(\theta)$ are Lagrange multipliers for the moment constraints $E[\ln X]=\theta_1$ and $E[\ln\{1+(\frac{X}{b})^a\}]=\theta_2$, respectively. From (18), we note the MDI distributions in reference to the GG and all its subfamilies contain: exponential, gamma, Weibull, generalized normal … which are not members of the $GB2$ family. The following MDI properties of $GB2$ distribution are obtained from (18).

- The MDI distribution in Ω_θ relative to the reference distribution

  $F_\circ=GB2_\circ=GB2(a_\circ,b_\circ,p_\circ,q_\circ),\ a_\circ=a,\ b_\circ=b,\ p_\circ\neq p$ and $q_\circ\neq q$

  is $F^*=GB2^*(a_\circ,b_\circ,p,q)$, and MDI function is given by

  $$K(GB2^*:GB2_\circ)=\min_{F\in\Omega_\theta}K(F:GB2_\circ)=K(B2:B2_\circ)$$

  $$=\ln\frac{B(p_\circ,q_\circ)}{B(p,q)}+a_\circ(p-p_\circ)\mu(a,\frac{b}{b_\circ},p,q)+[(p-p_\circ)+(q-q_\circ)]\gamma(p,q).$$

- The MDI distribution in Ω_θ relative to the reference distribution

  $F_\circ=SM(a_\circ,b_\circ,q_\circ),\ \ a_\circ=a,\ \ b_\circ=b,\ \ q_\circ\neq q$ is

  $F^*=GB2^*(a_\circ,b_\circ,p,q)$ and the MDI function

  $$K(GB2:SM_\circ)=\min_{F\in\Omega_\theta}K(F:SM_\circ)$$

  $$=-\ln q_\circ-\ln B(p,q)+(p-1)[\psi(p+q)-\psi(p)]+(q-q_\circ)[\psi(p+q)-\psi(q)].$$

- The MDI distribution in Ω_θ relative to the reference distribution

  $F_\circ=D(a_\circ,b_\circ,p_\circ),\ a=a_\circ,\ b_\circ=b,\ p_\circ\neq p$ is $F^*=GB2^*(a_\circ,b_\circ,p,q)$

and the MDI function $K(GB2^* : D_\circ) = \min_{F \in \Omega_\theta} K(F : D)$

$$= \ln \frac{B(p_\circ, 1)}{B(p,q)} + (p - p_\circ)[\psi(p+q) - \psi(p)] + (q-1)[\psi(p+q) + \psi(q)].$$

- The MDI distribution in Ω_θ relative to the reference distribution

  $F_\circ = Fisk(a_\circ, b_\circ),\ a = a_\circ,\ b_\circ = b$ is $F^* = GB2(a_\circ, b_\circ, p, q)$ and the MDI function $K(GB2^* : Fisk_\circ) = \min_{F \in \Omega_\theta} K(F : Fisk)$

  $$= -\ln B(p,q) + (p-1)[\psi(p+q) - \psi(p)] + (q-1)[\psi(p+q) - \psi(q)].$$

3.3 Data Transformation

Let $Y \sim GB2(a, b, p, q)$, from (5) we have $Y^s \sim GB2(\frac{a}{s}, b^s, p, q)$ (because $GB2$ family is closed under power transformation). Thus, the effect of power transformation by $K(Y : Y^s)$, can be estimated, this effect may be interpreted as the loss of information due to transformation.

In this case, considering similar arguments that were introduced in (15), $\varphi_a = s$ and $\varphi_b = b^{\frac{a}{s} - a}$.

We derive

$$K(Y : Y^s) = -\ln \frac{B(p,q)}{B(p_\circ, q_\circ)} + \ln s b^{ap(s-1)} + (ps - p_\circ)\mu(s, b^{\frac{a}{s} - a}, p, q) - (p+q)\gamma(p,q) +$$
$$\left\{ b^{ap(s-1)} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(p + \frac{n+1}{s})} {}_2F_1(p + \frac{n+1}{s}, p+q; p + \frac{n+1}{s} + 1; -b^{a(s-1)}) \right.$$
$$+ b^{aq(1-s)} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(q + \frac{n+1}{s})} {}_2F_1(q + \frac{n+1}{s}, p+q; q + \frac{n+1}{s} + 1; -b^{a(s-1)}) \qquad (19)$$
$$\left. - \frac{b^{aq(s-1)}}{s} \sum_{n=0}^{\infty} \binom{-p-q}{n} \frac{b^{an(s-1)}}{(q+n)^2} \right\} \frac{p_\circ + q_\circ}{B(p,q)}.$$

Particularly, we derive $K(Y:Y^{a})$ by putting $s=a$ in (19), that is discrepancy between $GB2(a,b,p,q)$ and $B2(b^{a},p,q)$.

4. Conclusions

In this paper, we derived discrimination information function between $GB2$ distribution and its subfamily. In special cases, we computed discrimination information function between $GB2$ and GG family (GG distribution and its subfamily) and established that for large values of q the results are equal to discrimination information function between GG and $GG_{\circ}$. So, our achievements are the extended version of the some results in Dadpay, et al. [1]. Also, we discussed minimum discrimination information model in $GB2$ family.

Appendix

A)

if $\quad Y \mapsto GB2 \Rightarrow f(y,a,b,p,q) = \frac{ay^{ap-1}}{b^{ap}B(p,q)(1+\left(\frac{y}{b}\right)^{a})^{p+q}}, \qquad y \geq 0$

then

$$H(f) = -\int \frac{ay^{ap-1}}{b^{ap}B(p,q)(1+\left(\frac{y}{b}\right)^{a})^{p+q}} ln \frac{ay^{ap-1}}{b^{ap}B(p,q)(1+\left(\frac{y}{b}\right)^{a})^{p+q}} dy$$

let $\quad A = \frac{a}{b^{ap}B(p,q)}$

$$H(f) = -\left[\int \frac{y^{ap-1}}{\left(1+\left(\frac{y}{b}\right)^{a}\right)^{p+q}} lnAdy + \int A \frac{y^{ap-1}}{\left(1+\left(\frac{y}{b}\right)^{a}\right)^{p+q}} lny^{ap-1}dy - \int A \frac{y^{ap-1}}{\left(1+\left(\frac{y}{b}\right)^{a}\right)^{p+q}} ln\left(1+\left(\frac{y}{b}\right)^{a}\right)^{p+q} dy\right]$$

where

$$\int \frac{y^{ap-1}}{\left(1+\left(\frac{y}{b}\right)^{a}\right)^{p+q}} lnAdy = lnA = -lna - aplnb - lnB(p,q)$$

and

$$\int A \frac{y^{ap-1}}{\left(1+\left(\frac{y}{b}\right)^{a}\right)^{p+q}} lny^{ap-1}dy = A(ap-1)\int \frac{y^{ap-1}}{\left(1+\left(\frac{y}{b}\right)^{a}\right)^{p+q}} lnydy$$

we know that : $\int_0^{+\infty} \frac{y^{ap-1}}{\left(1+\left(\frac{y}{b}\right)^a\right)^{p+q}} dy = \frac{b^{ap} B(p,q)}{a}$

on the other hand: $E\{Y^{ra}\} = A\int_0^{+\infty} \frac{y^{a(r+p)-1}}{\left(1+\left(\frac{y}{b}\right)^a\right)^{p+q}} dy = \frac{a}{b^{ap} B(p,q)} \times \frac{b^{a(r+p)} B(p+r,q-r)}{\mathrm{a}} =$

$\frac{b^{ar} B(p+r,q-r)}{\mathrm{B(p,q)}}$

$$\Rightarrow \frac{d}{dr} E\{Y^{ra}\} = A\int_0^{+\infty} \frac{a y^{a(r+p)-1}}{\left(1+\left(\frac{y}{b}\right)^a\right)^{p+q}} lnydy$$

$$= \frac{ab^{ar} lnbB(p+r,q-r) + b^{ar} \frac{\acute{\Gamma}(p+r)\Gamma(q-r) - b^{ar}\acute{\Gamma}(q-r)\Gamma(p+r)}{\Gamma(p+q)}}{B(p,q)}$$

When r=0 $\Rightarrow \frac{d}{dr} Er\,\{Y^{ra}\} = \frac{alnb + \acute{\Gamma}(p)\Gamma(q) - \acute{\Gamma}(q)\Gamma(p)}{\Gamma(\mathrm{p})\Gamma(\mathrm{q})}$

$$\Rightarrow A\int_0^{+\infty} \frac{y^{ap-1} lny}{\left(1+\left(\frac{y}{b}\right)^a\right)^{p+q}} dy = \left[lnb + \frac{\psi(p) - \psi(q)}{a}\right]$$

$$\Rightarrow \int A \frac{y^{ap-1}}{\left(1+\left(\frac{y}{b}\right)^a\right)^{p+q}} lny^{ap-1} = (ap-1)\left\{lnb + \frac{\psi(p) - \psi(q)}{a}\right\}$$

with similar process: $\int A \frac{y^{ap-1}}{\left(1+\left(\frac{y}{b}\right)^a\right)^{p+q}} ln\left(1+\left(\frac{y}{b}\right)^a\right)^{p+q} dy = (\mathrm{p}+\mathrm{q})\{\psi(p+q) - \psi(q)\}$

$$\Rightarrow H(f) = aplnb - lna + lnB(p,q) - (ap-1)\frac{\mathrm{lnb} + \psi(p) - \psi(q)}{a} + (p+q)\{\psi(p+q) - \psi(q)\}$$

$$= lnB(p,q) - lna + lnb + (1-ap)\left\{\frac{\psi(p) - \psi(q)}{a}\right\} + (p+q)\psi(p+q) - (p+q)\psi(q).$$

B)

Note: $\Gamma(\alpha+1) = e^{-\alpha}\alpha^{\alpha-\frac{1}{2}}\sqrt{2}\pi$ & $\Gamma(\alpha+1) \triangleq \alpha\Gamma(\alpha) \Rightarrow \Gamma(\alpha) \triangleq e^{-\alpha}\alpha^{\alpha-\frac{3}{2}}\sqrt{2}\pi$

(Stirling's formula)

$$\Rightarrow ln\Gamma(\alpha) = -\alpha + \left(\alpha - \frac{3}{2}\right) ln\alpha + ln\sqrt{2\pi}$$

$$\psi(\alpha) = \frac{d}{dy} ln\Gamma(\alpha) \Rightarrow \psi(\alpha) = ln\alpha - \frac{3}{2\alpha}$$

let $b = \beta q^{\frac{1}{a}}$ & $q \to +\infty$

then $H(f) = -lna + ln\beta + \left(\frac{1}{a} - p\right)\psi(p) + ln\Gamma(p) + p.$

C)

if $Y \mapsto GB2(a,b,p,q) \Rightarrow f_Y(y) = \frac{y^{ap-1}}{b^{ap} B(p,q)\left(1+\left(\frac{y}{b}\right)^a\right)^{p+q}}$ $\quad y \geq 0$

let : $A = \frac{a}{b^{ap}B(p,q)}$ & $A_0 = \frac{a_0}{{b_0}^{a_0 p_0}B(p_0,q_0)}$

$$K(GB2, GB2_0) = \int f(y) ln \frac{f(y)}{f_0(y)} dy$$

$$= \int_0^{+\infty} \frac{y^{ap-1}}{\left(1+\left(\frac{y}{b}\right)^a\right)^{p+q}} ln \frac{A \frac{y^{ap-1}}{\left(1+\left(\frac{y}{b}\right)^a\right)^{p+q}}}{A_0 \frac{y^{a_0 p_0 -1}}{\left(1+\left(\frac{y}{b_0}\right)^{a_0}\right)^{p_0+q_0}}} dy$$

$$= -H(f) - \int_0^{+\infty} A \frac{y^{ap-1}}{\left(1+\left(\frac{y}{b}\right)^a\right)^{p+q}} lnA_0 dy - \int_0^{+\infty} A \frac{y^{ap-1}}{\left(1+\left(\frac{y}{b}\right)^a\right)^{p+q}} ln y^{a_0 p_0 -1} dy$$

$$+ \int_0^{+\infty} A \frac{y^{ap-1}}{\left(1+\left(\frac{y}{b}\right)^a\right)^{p+q}} ln\left(1+\left(\frac{y}{b_0}\right)^{a_0}\right)^{p_0+q_0} dy$$

$$-H(f) = -lnB(p,q) + lna - lnb - (1-ap)\left\{\frac{\psi(p) - \Psi(q)}{a}\right\} \psi - (\mathrm{p}+\mathrm{q})\{\psi(p+q) - \psi(q)\}$$

and

$$\int_0^{+\infty} A \frac{y^{ap-1}}{\left(1+\left(\frac{y}{b}\right)^a\right)^{p+q}} lnA_0 dy = -lnA_0 = -lna_0 + a_0 p_0 lnb_0 + lnB(p_0, q_0)$$

and

$$\int_0^{+\infty} A \frac{y^{ap-1}}{\left(1+\left(\frac{y}{b}\right)^a\right)^{p+q}} ln y^{a_0 p_0 -1} dy = (1 - a_0 p_0)\left\{lnb + \frac{\psi(p) - \psi(q)}{a}\right\}$$

then

$$\int_0^{+\infty} A \frac{y^{ap-1}}{\left(1+\left(\frac{y}{b}\right)^a\right)^{p+q}} ln\left(1+\left(\frac{y}{b_0}\right)^{a_0}\right)^{p_0+q_0} dy$$

$$= \left\{\frac{1}{b^{ap}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1){b_0}^{a_0(n+1)}} \; {}_2F_1 \left[p + \frac{a_0}{a}n + \frac{a_0}{a}, (p+q), p + \frac{a_0}{a}n + \frac{a_0}{a} + 1, -(\frac{b_0}{b})^a\right]\right.$$

$$+ (\frac{b}{b_0})^{aq} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \; {}_2F_1 \left[q + \frac{a_0}{a}n + \frac{a_0}{a}, (p+q), q + \frac{a_0}{a}n + \frac{a_0}{a}\right.$$

$$\left. + 1, -(\frac{b_0}{b})^a\right]$$

$$\left. - a a_0 (\frac{b_0}{b})^{aq} \sum_{n=0}^{\infty} \frac{\Gamma[1-(p+q)]}{\Gamma[1-(p+q)-n]\Gamma(n+1)} (\frac{b_0}{b})^{an} \frac{1}{(aq+an)^2}\right\} \frac{p_0+q_0}{B(p,q)}.$$

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