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# An Analytical Approach to EWMA Control Chart for AR(1) Process Observations with Exponential White Noise

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**Abstract**

We propose analytical solutions to evaluate the characteristic of the control chart - the average run length (ARL) when the process is in-control for EWMA control chart for AR(1) process observations with exponential white noise. We use the integral equation technique to derive an explicit formula for the average run length. In several examples, the analytical solutions are compared with the results obtained from the numerical approximations.

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**Keywords:** AR(1) process, average run length, exponential white noise, exponentially weighted moving average, integral equation.

## 1. Introduction

Control charts are usually used to detect aberrant observations in a process. Besides, they are often used for monitoring and improving quality in many real applications, for instance, in Computer Science, Engineering, Business and Economics, Medicine, Biomedical and Life Sciences, Chemistry and Materials Science, Humanities, Social Sciences and Law and in other areas of applications. Usually, the performance of the control chart is measured by the average run length (ARL). The $ARL_0$ is defined as the mean of false alarm time $(\tau)$ before an in-control process is taken to signal to be out of control. A sufficient large in-control $ARL_0$ is desired. When the process is out-of-control, the performance of a control chart is usually used as $ARL_1$ or the average of delay (AD) time. It is the expected number of observations taken from an out-of-control process until the control chart signals that the process is out-of-control. Ideally, the AD time should be small.

The methods for evaluating the characteristics of an Exponentially Weighted Moving Average (EWMA) control charts have been studied. Roberts [1] who first studied the properties of the EWMA chart in the case of normal distribution by using simulation. Next, the performance of EWMA, Cumulative Sum (CUSUM) and Shewhart control charts were compared by Roberts [2]. Brook and Evans [3] approximated the run length of EWMA by using a finite-state Markov Chain Approach (MCA) with discrete and continuous distributions. A numerically procedure for the tabulation of ARL of one- and two-sided EWMA charts were presented by Robinson and Ho [4]. A numerical procedure using integral equations for the tabulation of moments of run lengths of EWMA charts was presented by Crowder [5]. The two-sided EWMA chart assuming normal observations for both ARLs and standard deviations of run lengths were presented. Lucas and Saccucci [6] showed the properties of EWMA scheme used to monitor the mean of a normally distributed process by a Markov chain. A design procedure for EWMA schemes is given. The ARL of the Exponential EWMA chart were presented exactly based on the solution of a set of differential equations by Gan [7]. Borror [8] studied the ARL performance of EWMA chart for both skewed and heavy-tailed symmetric non-normal distributions using MCA. An integral equation for the ARL of the three-way control chart is analytically derived and numerical method for solving it has been studied for ARL by Calzada and Scariano [9]. They offered users software that reduce the computational burden and frees one to focus more fully on three-way design specifications appropriate for a particular application. Sukparungsee [10] introduced an analytical closed-form formula for determining the characteristics of EWMA charts for the cases of normal and some non-normal distributions by using a martingale-based technique. Wang [11] and Spliid [12] approximated the properties of the EWMA control chart by using the infinite state transition probability matrix similar to Lucas and Saccucci [6]. Explicit formulas for evaluating the characteristic of the control charts were introduced recently. Areepong [13] derived the explicit formulas of ARL and AD for exponential EWMA charts. Mititelu et al. [14] presented the explicit formula for ARL by Feldhom Integral Equation for one-sided EWMA control chart with Laplace distribution and CUSUM control chart with hyperexponential distribution.

In real applications, observations could be serially-correlated which often appear in finance and insurance. Autoregressive processes with non-Gaussian white noise are useful for modeling a wide range of observed phenomena which do not allow negative values or have a highly skewed distribution. Many problems such as daily flows of rivers, wind speeds, amount of dissolved oxygen in a river and etc. which are examples of real applications with serially-correlated processes. The full study of the case of the first-order autoregressive process AR(1) with marginal exponential distribution $\left(EAR(1)\ process\right)$ was introduced by Gaver and Lewis [15]. Bell and Smith [16] considered inference for non-negative autoregressive schemes $X_t = \rho X_{t-1} + Y_t$ where $\rho$ is the autoregressive coefficient is $\left(0 \le \rho < 1\right)$ and $Y_t$ are i.i.d. non - negative random variables. They assume that the initial observation is known and apply their results with problem in water quality analysis. Some references are Turkmann and Pereira [17], Andel [18], Andel and Zvara [19], Turkmann [20], Ibazizen and Fellag [21].

In this research we derived analytical formulas for ARL of the EWMA control chart when observations are AR(1) process with exponential white noise. The Integral Equations is used to derive this analytical formula. We compare our analytical results for ARL with results from numerical Integral Equation approximations. The paper is organized as follows: in section 2, we define the EWMA control chart for AR(1) process with exponential white noise. In sections 3, the explicit formula is derived for finding ARL of EWMA chart for AR(1) process with exponential white noise and Integral Equations for evaluating ARL for EWMA control chart for AR(1) process with exponential white noise will be discussed in section 4. In section 5, numerical comparisons with the analytical results are presented. The conclusions are discussed in section 6.

## 2. EWMA Control Chart for AR(1) Process with Exponential White Noise

The performance of the control chart is measured by the ARL. The $ARL_0$ is defined as the expected of false alarm time $\left(\tau\right)$ before an in-control process is taken to signal to be out of control. A sufficient large in-control $ARL_0$ is desired. When the process is out-of-control, the performance of a control chart is usually used as the $ARL_1$

or average of delay (AD) time. It is the expected number of observations taken from an out-of-control process until the control chart signals that the process is out-of-control. Ideally, the AD time should be small. The EWMA control chart is widely used to monitoring and detecting a small shift in the process mean. The EWMA control chart for the discrete time case is defined as a recursive form

$$X_t = (1-\lambda)X_{t-1} + \lambda\xi_t, \quad t = 1,2,\ldots, \tag{1}$$

where the smoothing parameter $\lambda \in (0,1)$, $X_t$ is the weighted average between current and previous observations, $X_0$ is initial value and the process $\{\xi_t, t = 1,2,...\}$ consists of is the independent observations with mean $\mu$ and variance $\sigma^2$. The control limits are as the form

$$UCL = \mu + L\sigma \qquad \text{and} \qquad LCL = \mu - L\sigma.$$

In this paper, we study the sequence $\{\xi_t\}$ consists of the first-order autoregressive (AR(1)) observations. The $AR(1)$ process is defined as a solution of equation

$$\xi_t = \phi\xi_{t-1} + y_t, \quad t = 1,2,\ldots, \tag{2}$$

where $y_t$ is white noise (see in Gaver and Lewis [15], Andel [18], Andel and Zvara [19], we assume that independent random variables $y_t$ is the random error term at time $t$ following $Exp(1)$ and $\phi$ is constant $(0 < \phi < 1)$ and $\xi_0$ is initial value. The target in-control parameter $\alpha_0$ is supposed to be steady at target value, $X_0$ and $\xi_0$ are usually chosen to be the process in-control parameter, i.e., $X_0 = \xi_0 = \alpha_0$. We mainly discuss the case an upper-sided EWMA control chart for AR(1) process with exponential white noise. The alarm times for this type of procedure are the following:

$$\tau_b = \inf\{t > 0 : X_t > b\}, \text{ where } b \text{ is the control limit.}$$

An exponential distribution with parameter 1 $(Exp(1))$ is defined by following function:

$$f(y) = e^{-y}, \, y \geq 0. \tag{3}$$

The ARL of EWMA control chart for AR(1) process with exponential white noise depends on the control limit $(b)$ and the smoothing parameter $(\lambda)$. The control limits are as the form

$$UCL = \mu + L\sigma = b \qquad \text{and} \qquad LCL = \mu - L\sigma = 0.$$

**3. Explicit Formula for Evaluating ARL for EWMA Control Chart for AR(1) Process with Exponential White Noise**

In this part, we use explicit formulas of the integral equation to find ARL for EWMA control chart for observations from the AR(1) process with exponential white noise. In real application, the AR(1) process often have a normal distribution. Nevertheless, a process may not follow a normal distribution as it may be right skewed. The AR(1) process with exponential white noise have been studied in the literature. Crowder's (1987) method for computing the ARL of EWMA control chart is applied below. Let $L(u)$ denote the ARL of one-sided EWMA control chart when the initial value is $u$. Since $y_t \geq 0$, we assume that the lower limit and the upper limit are $H_L = 0$ and $H_U = b$, respectively. $L(u)$ is given by the integral equation

$$L(u) = 1 + \int_{0 \leq (1-\lambda)u + \lambda(\phi X_0 + \eta_1) \leq b} L\left[(1-\lambda)u + \lambda(\phi u + y)\right] f(y)\,dy \,. \qquad (4)$$

After a change of variable,

$$L(u) = 1 + \frac{1}{\lambda}\int_0^b L(y) f\left(\frac{y-(1-\lambda)u}{\lambda} - \phi u\right) dy, \qquad (5)$$

where $L(u)$ is a Fredholm integral equation of the second kind.

From (5) if $y_t \sim Exp(1)$ then we have $f(y) = e^{-y}$, $y \geq 0$

$$f\left(\frac{y-(1-\lambda)u}{\lambda} - \phi u\right) = e^{-\frac{y}{\lambda}} e^{\frac{(1-\lambda)u}{\lambda}} e^{\phi u},$$

then we obtain,

$$L(u) = 1 + \frac{1}{\lambda}\int_0^b L(y) e^{-\frac{y}{\lambda}} e^{\frac{(1-\lambda)u}{\lambda} + \phi u} dy \,.$$

Let

$$C(u) = e^{\frac{(1-\lambda)u}{\lambda} + \phi u} \quad ; \quad 0 \leq u \leq b$$

so, we have

$$L(u) = 1 + \frac{1}{\lambda}\int_0^b L(y) e^{-\frac{y}{\lambda}} C(u)\,dy \quad ; \quad 0 \leq u \leq b$$

$$= 1 + \frac{C(u)}{\lambda}\int_0^b L(y) e^{-\frac{y}{\lambda}} dy.$$

We consider

$$d = \int_0^b L(y) e^{-\frac{y}{\lambda}} dy,$$

then we obtain

$$L(u)=1+\frac{C(u)}{\lambda}d. \tag{6}$$

Now we can express the constant $d$ as

$$\begin{aligned} d &= \int_0^b L(y)e^{-\frac{y}{\lambda}}dy \\ &= \int_0^b \left[1+\frac{C(y)}{\lambda}d\right]e^{-\frac{y}{\lambda}}dy \\ &= \frac{-\lambda\left(e^{-\frac{b}{\lambda}}-1\right)}{1+\frac{\left(e^{-b}-1\right)}{\lambda}e^{\phi u}}, \end{aligned} \tag{7}$$

where substitute (7) into (6) the solution for the integral equation (5) is

$$\begin{aligned} L(u) &= 1+\frac{1}{\lambda}e^{\frac{(1-\lambda)u}{\lambda}+\phi u}\frac{-\lambda\left(e^{-\frac{b}{\lambda}}-1\right)}{1+\frac{\left(e^{-b}-1\right)}{\lambda}e^{\phi u}} \\ &= 1-e^{\frac{(1-\lambda)u}{\lambda}}e^{\phi u}\frac{\lambda\left(e^{-\frac{b}{\lambda}}-1\right)}{\lambda+\left(e^{-b}-1\right)e^{\phi u}} \\ &= 1-\frac{\lambda e^{\frac{(1-\lambda)u}{\lambda}}\left(e^{-\frac{b}{\lambda}}-1\right)}{\lambda e^{-\phi u}+e^{-b}-1}. \end{aligned} \tag{8}$$

## 4. Integral Equations for Evaluating ARL for EWMA Control Chart for AR(1) Process with Exponential White Noise

In this section we present a numerical method to evaluate solution of the integral equations. Firstly, recall the equation (5)

$$L(u)=1+\frac{1}{\lambda}\int_0^b L(y)f\left(\frac{y-(1-\lambda)u}{\lambda}-\phi u\right)dy.$$

In general, we can approximate the Integral $\int_0^H f(z)dz$ by the sum of rectangles with bases $H/m$ with heights chosen as the values of $f$ at the midpoints of intervals of length $H/m$ beginning at zero, i.e. on the interval $[0,H]$ with the division points

$0 \le a_1 \le a_2 \le a_3 \le \ldots \le a_m \le H$ and weights $w_j = H/m$. We can rewrite the approximation for an integral is of the form:

$$\int_0^H f(z)dz \approx \sum_{j=1}^{m} w_j f(a_j),$$

where $a_j = \frac{H}{m}\left(j - \frac{1}{2}\right), \quad k = 1,2,\ldots,m.$

Then the Integral Equation (4) can be approximated as follows:

$$L(a_i) \approx 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j L(a_j)\, f\left(\frac{a_j - (1-\lambda)a_i}{\lambda} - \phi a_i\right), i = 1,2,3,\ldots,m.$$

That is

$$\begin{aligned}
L(a_1) &\approx 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j L(a_j)\, f\left(\frac{a_j - (1-\lambda)a_1}{\lambda} - \phi a_1\right)\\
L(a_2) &\approx 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j L(a_j)\, f\left(\frac{a_j - (1-\lambda)a_2}{\lambda} - \phi a_2\right)\\
L(a_3) &\approx 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j L(a_j)\, f\left(\frac{a_j - (1-\lambda)a_3}{\lambda} - \phi a_3\right)\\
&\vdots\\
L(a_m) &\approx 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j L(a_j)\, f\left(\frac{a_j - (1-\lambda)a_m}{\lambda} - \phi a_m\right)
\end{aligned}$$

or in matrix form as

$$L_{m\times 1} = 1_{m\times 1} + R_{m\times m} L_{m\times 1} \text{ or } (I_m - R_{m\times m}) L_{m\times 1} = 1_{m\times 1}, \tag{9}$$

where

$$L_{m\times 1} = \begin{pmatrix} L(a_1) \\ L(a_2) \\ \vdots \\ L(a_m) \end{pmatrix}, 1_{m\times 1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

and

$$R_{m\times m} = \begin{pmatrix}
\frac{1}{\lambda} w_1 f\left(\frac{a_1 - (1-\lambda)a_1}{\lambda} - \phi a_1\right) & \cdots & \frac{1}{\lambda} w_m f\left(\frac{a_m - (1-\lambda)a_1}{\lambda} - \phi a_1\right)\\
\frac{1}{\lambda} w_1 f\left(\frac{a_1 - (1-\lambda)a_2}{\lambda} - \phi a_2\right) & \cdots & \frac{1}{\lambda} w_m f\left(\frac{a_m - (1-\lambda)a_2}{\lambda} - \phi a_2\right)\\
\vdots & & \vdots\\
\frac{1}{\lambda} w_1 f\left(\frac{a_1 - (1-\lambda)a_m}{\lambda} - \phi a_m\right) & \cdots & \frac{1}{\lambda} w_m f\left(\frac{a_m - (1-\lambda)a_m}{\lambda} - \phi a_m\right)
\end{pmatrix}$$

and $I_m = diag(1,1,...,1)$ is the unit matrix of order m. If there exists $(I_m - R_{m\times m})^{-1}$, then the solution of the matrix equation (9)

$$L_{m\times 1} = (I_m - R_{m\times m})^{-1} 1_{m\times 1}.$$

We may approximate the function $L(u)$ as

$$\tilde{L}(u) \approx 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j L(a_j)\, f\left(\frac{a_j - (1-\lambda)u}{\lambda} - \phi u\right),$$

where $w_j = \frac{b}{m}$ and $a_j = \frac{b}{m}\left(j - \frac{1}{2}\right)$.

**5. Numerical Comparisons**

In this part, we compare the suggest formula results obtained with results by the numerical integral equations. In this work we assume that observations are from AR(1) processes with exponential white noise with parameter $\alpha = 1$ and will consider with smoothing constants $\lambda$ are 0.2, 0.3, 0.4 and $\phi$ are 0.1, 0.2, 0.3 for the explicit formula and numerical integral equation, as these values are recommended for EWMA control charts in the literature (i.e., Crowder [5] and Gan [7]). The results for ARL are computed for an initial value 0.1 for $Z_0$ and $X_0$.

In Table 1, we will consider the exponential distribution with mean 1 as this is representative of the situations of interest. We compare the numerical results obtained by explicit formulas with results by integral equations both for EWMA control chart with $\phi$ is 0.1 calculations with equations (5) and (8), respectively give ARL in control for the optimal set of parameters $\lambda$ and $b$. Obviously, the explicit formulas from the suggested formulas give results which are very closed to the numerical integral equations. Note that, the calculations with explicit formula (8) are much faster. For example, when given that $\phi$ is 0.1, calculations time based on our technique takes less than 1 second while the CPU time required for numerical integral equation for the EWMA run are inside the parentheses.

**Table 1.** Comparison of the ARL to the explicit formulas and the numerical integral equations when $\phi$ is 0.1. The entries inside the parentheses are CPU times in seconds.

| $\lambda$ | $b$ | ARL | |
|---|---|---|---|
| | | Explicit formulas | Integral Equations |
| 0.2 | 0.12 | 2.585 (0.0001) | 2.585 (42.1675) |
| | 0.17 | 5.099 (0.0034) | 5.099 (42.3767) |
| | 0.22 | 377.439 (0.0058) | 377.438 (42.2609) |
| 0.3 | 0.25 | 3.825 (0.0001) | 3.824 (42.3756) |
| | 0.30 | 7.331 (0.0033) | 7.330 (42.2582) |
| | 0.35 | 154.177 (0.0035) | 154.178(42.3941) |
| 0.4 | 0.40 | 5.428 (0.0001) | 5.429 (42.3789) |
| | 0.45 | 10.328 (0.0001) | 10.329 (43.7835) |
| | 0.50 | 131.003 (0.0001) | 131.002 (43.0657) |

In Tables 2 and 3, we present ARL for $\phi$ are 0.2 and 0.3 and smoothing constants $(\lambda)$ are 0.2, 0.3 and 0.4. It can be seen from ARL from that all of the explicit formulas give results which are close to the numerical integral equations with the same parameters. The entries inside the parentheses are CPU time.

**Table 2.** Comparison of the ARL by the explicit formulas and the numerical integral equations when $\phi$ is 0.2. The entries inside the parentheses are CPU times.

| $\lambda$ | $b$ | ARL | |
|---|---|---|---|
| | | Explicit formulas | Integral Equations |
| 0.2 | 0.11 | 2.374 (0.0034) | 2.375 (48.6926) |
| | 0.16 | 4.410 (0.0001) | 4.411 (53.2442) |
| | 0.21 | 30.281 (0.0001) | 30.280 (48.9279) |
| 0.3 | 0.24 | 3.585 (0.0052) | 3.584(49.6797) |
| | 0.29 | 6.547 (0.0001) | 6.546(48.7315) |
| | 0.34 | 45.060 (0.0001) | 45.061 (49.2111) |
| 0.4 | 0.39 | 5.188 (0.0001) | 5.187 (52.8878) |
| | 0.44 | 9.585 (0.0001) | 9.586 (55.5022) |
| | 0.49 | 70.746 (0.0001) | 70.747 (50.6929) |

**Table 3.** Comparison of the ARL by the explicit formulas and the numerical integral equations when $\phi$ is 0.3. The entries inside the parentheses are CPU times in seconds.

| $\lambda$ | $b$ | ARL | |
|---|---|---|---|
| | | Explicit formulas | Integral Equations |
| 0.2 | 0.13 | 2.976 (0.0033) | 2.975 (41.8028) |
| | 0.17 | 5.525 (0.0001) | 5.525 (42.5282) |
| | 0.21 | 42.503 (0.0001) | 42.501 (42.0791) |
| 0.3 | 0.26 | 4.530 (0.0040) | 4.531 (42.6632) |
| | 0.30 | 8.496 (0.0001) | 8.495 (42.3023) |
| | 0.34 | 89.455 (0.0038) | 89.454 (42.4041) |
| 0.4 | 0.41 | 6.749 (0.0001) | 6.748 (42.3786) |
| | 0.45 | 13.161 (0.0001) | 13.162 (42.2238) |
| | 0.49 | 408.918 (0.0032) | 408.919 (42.1813) |

**Table 4.** Comparison of the ARL obtained from the explicit formula and the numerical integral equations when $\phi$ is 0.4 and $\lambda$ is 0.2. The entries inside the parentheses are CPU times in seconds.

| $b$ | ARL | |
|---|---|---|
| | Explicit formulas | Integral Equations |
| 0.21 | 71.732 (0.0001) | 71.733 (42.2841) |
| 0.20 | 18.321 (0.0001) | 18.320 (42.1262) |
| 0.19 | 10.571 (0.0002) | 10.570 (42.2840) |
| 0.18 | 7.455 (0.0001) | 7.456 (42.5106) |
| 0.17 | 5.768 (0.0003) | 5.769 (44.3505) |
| 0.16 | 4.709 (0.0001) | 4.708 (42.9741) |
| 0.15 | 3.978 (0.0038) | 3.979 (43.5121) |

In Table 4 we calculate ARL by the explicit formulas and the numerical integral equations when given $\phi$ is 0.4 and $\lambda$ is 0.2, It can be seen from Table 4 that all of the explicit formulas give results close to ARL for AR(1) observations with exponential white noise with the same optimal values b.

**6. Conclusion**

We have presented that the explicit formulas for ARL of one-sided EWMA charts for the case of an AR(1) process with exponential white noise. We have shown that the proposed formulas are very accurate, easy to calculate and program. Using the formulas, we have been able to provide tables for the optimal weights, boundaries and approximations for ARL for one-sided EWMA charts for AR(1) process with exponential white noise. The performance comparison of the control charts has been based on the criteria of ARL.

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