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# Complete Convergence for Weighted Sums of Arrays of Rowwise $\tilde{\rho}$-mixing Random Variables

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**Abstract**

In this paper we obtain some new results on complete convergence for weighted sums of arrays of rowwise $\tilde{\rho}$-mixing random variables. Our results improve and extend the some results established for sequences of independent random variables.

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**Keywords:** complete convergence, $\tilde{\rho}$-mixing random variables, slowly varying function, weighted sums.

## 1. Introduction

The concept of complete convergence was introduced by Hsu and Robbins in [1] as follows: A sequence of random variables $\{X_n, n \geq 1\}$ are said to converge completely to a constant $C$ if $\sum_{n=1}^{\infty} P(|X_n - C| > \varepsilon) < \infty$ for all $\varepsilon > 0$. From then on, many authors have devoted their study to complete convergence.

Recently, Sung [2] proved the following two results. In Theorems A and B we assume that $\{X_n, n \geq 1\}$ is a sequence of zero-mean independent random variables stochastically dominated by a random variable $X$, that is, $P(|X_n| > x) \leq CP(|X| > x)$ for all $x > 0$ and all $n \geq 1$ and some positive constant $C$. Moreover, $\{a_{ni}, i \geq 1, n \geq 1\}$ is an array of real numbers satisfying such that $\sup_{n \geq 1, i \geq 1} |a_{ni}| < \infty$ and $\sum_{i=1}^{\infty} a_{ni} X_i$ is finite almost surely for all $n \geq 1$. Finally, let $t \geq -1, -\infty < \beta < \infty, p > 0$ be constants such that $\gamma = p(t + \beta + 1) > 0$.

**Theorem A.** *Assume that* $E|X|^{\gamma} < \infty$ *and*

$$\sum_{i=1}^{\infty} |a_{ni}|^{\alpha} = O(n^{\beta}) \quad \text{for some } \alpha < \gamma. \tag{1}$$

(i) If $1 \leq \gamma < 2$, or

(ii) if $\gamma \geq 2$, and

$$\sum_{i=1}^{\infty} a_{ni}^2 = O(n^q) \quad \text{for some } q < 2/p, \tag{2}$$

then

$$\sum_{n=1}^{\infty} n^t P(n^{-1/p} |\sum_{i=1}^{\infty} a_{ni} X_i| > \varepsilon) < \infty \quad \text{for all } \varepsilon > 0. \tag{3}$$

**Theorem B.** *Assume that* $E|X|^{\gamma} \log |X| < \infty$ *and*

$$\sum_{i=1}^{\infty} |a_{ni}|^{\gamma} = O(n^{\beta}) \quad \text{for all } n \geq 1. \tag{4}$$

*(i) If* $1 \leq \gamma < 2$ *or*

*(ii) If* $\gamma \geq 2$ *and* $\sum_{i=1}^{\infty} a_{ni}^{2} = O(n^{q})$ for some $q < 2/p$,

*then (3) holds.*

Let $Z$ be the set of integers and $\{a_{nk}, n \geq 1, k \in Z\}$ be an array of constants. Denote

$$N(n, m+1) = \#\{k \in Z : |a_{nk}| \geq (m+1)^{-1/p}\}, p \geq 2, n \geq 1, m \geq 1,$$

where the symbol $\#A$ stands for the number of elements in the set $A$. For two sequences of real numbers $\{a_m, m \geq 1\}$ and $\{b_m, m \geq 1\}$, we write $a_m \approx b_m$ as $m \to \infty$, if $a_m = O(b_m)$ and vise versa $b_m = O(a_m)$ as $m \to \infty$.

Wang et al. [3] proved the following result:

**Theorem C.** *Let* $r > 1$ *and* $\{X_i, i \in Z\}$ *be a sequence of i.i.d. random variables and let* $\{a_{ni}, n \geq 1, i \in Z\}$ *for be an array of constants.*

(I) *If* $p > 2$ *and*

$$N(n, m+1) \approx m^{q(r-1)/p}, n \geq 1, \text{ as } m \to \infty, \text{ when } 2 \leq q < p, \quad (5)$$

$$EX = 0, \text{ when } 1 \leq q(r-1), \quad (6)$$

$$\sum_{i \in Z} a_{ni}^{2} = O(n^{\delta}), n \to \infty, \text{ when } 2 \leq q(r-1), \text{ for some } 0 < \delta < 2/p, \quad (7)$$

*then the following statements are equivalent:*

$$(i) E|X|^{p(r-1)} < \infty;$$

$$(ii) \sum_{i=1}^{\infty} 2^{i(r-1)} \max_{2^{i-1} \leq n < 2^{i}} P\left( n^{-1/p} \left| \sum_{k \in Z} a_{nk} X_k \right| > \varepsilon \right) < \infty \text{ for all } \varepsilon > 0.$$

(II) *If* $p = q = 2$ *and*

$$N(n, m+1) \approx m^{(r-1)}, n \geq 1, \text{ as } m \to \infty,$$

$$EX = 0, \text{ when } 1 \leq 2(r-1),$$

$$\sum_{i\in Z} |a_{ni}|^{2(r-1)} = O(1), n \to \infty,$$

*then the following statements are equivalent:*

$$(i) E|X|^{2(r-1)} \log(1+|X|) < \infty;$$

$$(ii) \sum_{i=1}^{\infty} 2^{i(r-1)} \max_{2^{i-1} \le n < 2^i} P\left( n^{-1/2} | \sum_{k\in Z} a_{nk} X_k | > \varepsilon \right) < \infty, \text{ for all } \varepsilon > 0.$$

The main purpose of this paper is to generalize the above mentioned results for $\tilde{\rho}$-mixing random variables (see the definition below). Theorem A and Theorem B and the sufficient part of Theorem C are extended and improved for $\tilde{\rho}$-mixing case.

Let $\{\Omega, \Im, P\}$ be a probability space. In the following, all random variables are assumed to be defined on $\{\Omega, \Im, P\}$. For a sequence of random variables $\{X_n, n \ge 1\}$ we denote $\Im_S = \sigma(X_n : n \in S \subset N)$. Given two $\sigma$-subalgebras $\Im_1, \Im_2 \subset \Im$, denote

$$\rho(\Im_1, \Im_2) = \sup\{|\operatorname{corr}(\zeta, \eta)|, \zeta \in L_2(\Im_1), \eta \in L_2(\Im_2)\},$$

where the correlation coefficient is defined in usual way

$$\operatorname{corr}(\zeta, \eta) = \frac{E(\zeta\eta) - E\zeta E\eta}{\sqrt{\operatorname{Var}(\zeta)\operatorname{Var}(\eta)}}$$

and by $L_2(\Im)$ we denote the space of all $\Im$-measurable random variables $\zeta$ such that $E(\zeta^2) < \infty$.

Stein [4] introduced the following coefficients of dependence (with slightly different notations):

$$\tilde{\rho}(k) = \sup\{\rho(\Im_S, \Im_T) : \text{all finite subsets } S, T \subset N \text{ such that } \operatorname{dist}(S,T) \ge k\},$$

$k \ge 0$. Obviously, $0 \le \tilde{\rho}(k+1) \le \tilde{\rho}(k) \le 1, k \ge 0,$ and $\tilde{\rho}(0) = 1.$

**Definition.** A sequence of random variables $\{X_n, n \geq 1\}$ are said to be a $\tilde{\rho}$ *-mixing sequence* if there exists $k \in N$ such that $\tilde{\rho}(k) < 1$. An array of random variables $\{X_{nk}, k \geq 1, n \geq 1\}$ are said to be an array of rowwise $\tilde{\rho}$ -mixing random variables, if, for every positive integer $n$ the sequence of random variables $\{X_{nk}, k \geq 1\}$ is a $\tilde{\rho}$ -mixing sequence.

For fixed $n$ -th row of an array of rowwise $\tilde{\rho}$ -mixing random variables $\{X_{nk}, n \geq 1, k \geq 1\}$ we denote the coefficients of dependence of the sequence $\{X_{nk}, k \geq 1\}$ as $\tilde{\rho}_n(\cdot)$ for every $n \geq 1$.

The notion of $\tilde{\rho}$ -mixing assumption is similar to $\rho$ -mixing, but they are quite different from each other. A number of publications are devoted to $\tilde{\rho}$ -mixing sequence. We refer to Bradley [5,6] for the central limit theorem, Bryc and Smolenski [7] for moment inequalities and almost sure convergence, Shanchao [8] for
moment inequalities and strong law of large numbers, Gut and Peligrad [9], Wu [10,11], and Shixin [12] for almost sure convergence, Utev and Peligrad [13] for maximal inequalities and the invariance principle, Dehua and Shixin [14,15] for complete convergence, Dehua and Shixin [16] for Hájeck-Rènyi inequality and strong law of large numbers among many others.

Recall that a measurable function $h$ is said to be *slowly varying* if for each $\lambda > 0$

$$\lim_{x \to \infty} \frac{h(\lambda x)}{h(x)} = 1.$$

We refer to Seneta [17] for other equivalent definitions and for detailed and comprehensive study of properties of such functions.

Throughout this paper, we assume that $\sum_{i=1}^{\infty} a_{ni} X_{ni}$ is finite almost surely, $C$ is a positive constant which may vary from one place to another, the symbol $[x]$ denotes the greatest integer less than $x$, and the symbol $\lfloor x \rfloor$ denotes the least integer more than $x$.

**2. Lemmata**

In order to prove our main result, we need the following lemmas. The proof of the first lemma could be found in Utev and Peligrad [13].

**Lemma 1.** *For a positive integer* $J$ *and* $0 \le r < 1$ *and* $u \ge 2$*, there exists a positive constant* $C = C(u,J,r)$ *such that if* $\{X_n, n \ge 1\}$ *is a sequence of random variables with* $\tilde{\rho}(J) \le r, EX_k = 0$*, and* $E|X_k|^u < \infty$ *for every* $k \ge 1$*, then for all* $n \ge 1$,

$$E \max_{1 \le i \le n} \left| \sum_{k=1}^{i} X_k \right|^u \le C \left\{ \sum_{k=1}^{n} E|X_k|^u + \left( \sum_{k=1}^{n} EX_k^2 \right)^{u/2} \right\}.$$

The second lemma is well known and we do not present the proof.

**Lemma 2.** *Let* $\{X_{nk}, n \ge 1, k \ge 1\}$ *be an array of random variables stochastically dominated by a random variable X, then there exists a constant* $D$ *such that for all* $u > 0$ *and* $x > 0$,

(i) $E|X_{nk}|^u I(|X_{nk}| \le x) \le D\{E|X|^u I(|X| \le x) + x^u P(|X| > x)\}$,

(ii) $E|X_{nk}|^u I(|X_{nk}| > x) \le DE|X|^u I(|X| > x)$.

The proof of the last lemma could be found in Bai and Su [18]

**Lemma 3.** *Let* $h(x) > 0$ *be a slowly varying function as* $x \to +\infty$*, then*

(i) $$\lim_{k \to +\infty} \sup_{2^k \le x \le 2^{k+1}} \frac{h(x)}{h(2^k)} = 1,$$

$$\lim_{x \to +\infty} x^{\delta} h(x) = +\infty, \quad \lim_{x \to \infty} x^{-\delta} h(x) = 0, \text{ for all } \delta > 0.$$

(ii) For all $\delta > 0, \eta > 0$, and all positive integers $k$

$$C \cdot 2^{k\delta} h(2^k \cdot \eta) \le \sum_{j=1}^{k} 2^{j\delta} h(2^j \cdot \eta) \le C \cdot 2^{k\delta} h(2^k \cdot \eta).$$

(iii) For all $\delta < 0, \eta > 0$ all positive integers $k$

$$C \cdot 2^{k\delta} h(2^k \cdot \eta) \le \sum_{j=k}^{\infty} 2^{j\delta} h(2^j \cdot \eta) \le C \cdot 2^{k\delta} h(2^k \cdot \eta).$$

**3. Main Results and Proofs**

With the preliminaries accounted for, we can now formulate and prove main results of this paper.

**Theorem 1.** *Let* $p > 0, t, \beta$ *be constants such that* $t + \beta > -1$, $h(x) > 0$ *be a slowly varying function,* $\{X_{nk}, n \ge 1, k \ge 1\}$ *be an array of zero-mean rowwise* $\tilde{\rho}$ *-mixing random variables stochastically dominated by a random variable* $X$ *, and* $\{a_{ni}, i \ge 1, n \ge 1\}$ *be an array of constants satisfying (1). Assume that*

$$\overline{\lim_{k \to \infty}} \sup_n \tilde{\rho}_n(k) < 1 \text{ and } E|X|^{\gamma} h(|X|^p) < \infty,$$

*where* $\gamma = p(t + \beta + 1) > 0$.

*If* $t = -1$ *we additionally assume that* $E|X|^{\gamma} < \infty$.

*(i) If* $\gamma = 1$, *and* $E|X| < \infty$, *then*

$$\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} P\left( n^{-1/p} \left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| > \varepsilon \right) < \infty \text{ for all } \varepsilon > 0, \quad (8)$$

*moreover*

$$\sum_{n=1}^{\infty} n^t h(n) P\left( n^{-1/p} \left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| > \varepsilon \right) < \infty \text{ for all } \varepsilon > 0. \quad (9)$$

*(ii) If* $1 < \gamma < 2$, *then (8) and (9) hold.*

*(iii) If* $\gamma = 2, \{a_{ni}, n \ge 1, i \ge 1\}$ *satisfies (2), and* $E|X|^2 < \infty$, *then (8) and (9) hold.*

*(iv) If* $\gamma \ge 2$ *and* $\{a_{ni}, n \ge 1, i \ge 1\}$ *satisfies (2), then (8) and (9) hold.*

**Proof.** First of all we note that it is enough to show that (8) holds. Really, by Lemma 3 we have

$$\sum_{n=1}^{\infty} n^{t} h(n) P\left( n^{-1/p} \left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| > \varepsilon \right)$$

$$= \sum_{j=0}^{\infty} \sum_{2^j \le n < 2^{j+1}} n^{t} h(n) P\left( n^{-1/p} \left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| > \varepsilon \right)$$

$$\le C + C \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^{j}) \max_{2^j \le n < 2^{j+1}} P\left( n^{-1/p} \left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| > \varepsilon \right) < \infty,$$

therefore, (9) holds.

If $t < -1$, then by Lemma 3 (i) we obtain that (8) holds. Thus, we assume that $t \ge -1$. Since $\sum_{i=1}^{\infty} a_{ni} X_{ni}$ is finite almost surely for each $n \ge 1$, there exists positive integer $k_n$ such that

$$P(n^{-1/p} \left| \sum_{i=k_n+1}^{\infty} a_{ni} X_{ni} \right| > \varepsilon/2) < n^{-t-2}, \text{ for all } n \ge 1.$$

By Lemma 3 (iii), in order to prove (8), it is enough to show that

$$\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^{j}) \max_{2^j \le n < 2^{j+1}} P(n^{-1/p} \left| \sum_{i=1}^{k_n} a_{ni} X_{ni} \right| > \varepsilon / 2) < \infty. \qquad (10)$$

Without loss of generality, we assume that $a_{ni} > 0$ for all $n \ge 1, i \ge 1, \sup_{i \ge 1, n \ge 1} a_{ni} = 1$, and $\sum_{i=1}^{\infty} a_{ni}^{\alpha} \le n^{\beta}$. Thus, for any $\theta \ge 0$, we have

$$\sum_{i=1}^{\infty} a_{ni}^{\alpha+\theta} \le n^{\beta}. \qquad (11)$$

For $i \ge 1, n \ge 1$ we define

$$U_{ni} = X_{ni} I(|a_{ni} X_{ni}| \le n^{1/p}), V_{ni} = X_{ni} I(|a_{ni} X_{ni}| > n^{1/p}).$$

Since $EX_{ni} = 0$, we obtain

$$\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} P(n^{-1/p} |\sum_{i=1}^{k_n} a_{ni} X_{ni}| > \varepsilon/2)$$

$$\le \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} P(n^{-1/p} |\sum_{i=1}^{k_n} a_{ni} (U_{ni} - EU_{ni})| > \varepsilon/4)$$

$$+ \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} P(n^{-1/p} |\sum_{i=1}^{k_n} a_{ni} (V_{ni} - EV_{ni})| > \varepsilon/4)$$

$$\stackrel{Def}{=} J_1 + J_2. \tag{12}$$

We estimate each term $J_1$ and $J_2$ separately.

For $J_2$, we first prove that

$$n^{-1/p} |\sum_{i=1}^{k_n} a_{ni} EV_{ni}| \to 0 \ as \ n \to \infty. \tag{13}$$

If $\gamma = 1$, since $E|X| < \infty$, by Lemma 2 and (11), we have

$$n^{-1/p} |\sum_{i=1}^{k_n} a_{ni} EV_{ni}| \le n^{-1/p+\beta} E|X| I(|X| > n^{1/p})$$

$$\le n^{-(t+1)} E|X| I(|X| > n^{1/p}) \to 0 \text{ as } n \to \infty.$$

If $\gamma > 1, t > -1$, select $\eta$ such that $\max\{p\beta, \alpha, 1\} < \eta < \gamma$. Since $E|X|^{\gamma} h(|X|^p) < \infty$, then by Lemma 3 (i), we have $E|X|^{\eta} < \infty$. Therefore, by Lemma 2 and (11), we obtain

$$n^{-1/p} |\sum_{i=1}^{k_n} a_{ni} EV_{ni}| \le n^{-1/p} \sum_{i=1}^{\infty} n^{-(\eta-1)/p} E|a_{ni} X_{ni}|^{\eta} I(|a_{ni} X_{ni}| > n^{1/p})$$

$$\le n^{-\eta/p+\beta} E|X|^{\eta} I(|X| > n^{1/p})$$

$$= n^{(p\beta-\eta)/p} E|X|^{\eta} I(|X|> n^{1/p}) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

If $\gamma > 1, t = -1$, since $E|X|^{\gamma} < \infty$ we obtain

$$n^{-1/p} |\sum_{i=1}^{k_n} a_{ni} EV_{ni}| \leq n^{-1/p} \sum_{i=1}^{\infty} n^{-(\gamma-1)/p} E|a_{ni}X_{ni}|^{\gamma} I(|a_{ni}X_{ni}|> n^{1/p})$$

$$\leq n^{-\gamma/p+\beta} E|X|^{\gamma} I(X|> n^{1/p})$$

$$= E|X|^{\gamma} I(X|> n^{1/p}) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Thus, (13) holds. Hence, there exists $n$ large enough such that

$$n^{-1/p} |\sum_{i=1}^{k_n} a_{ni} EV_{ni}| < \varepsilon/8. \qquad (14)$$

Select $\delta > 0$ such that $\gamma - \delta > 0$ and $\gamma - \delta > \alpha$, by (14), (11), Lemma 2 (ii) and Lemma 3 (ii),we have

$$J_2 \leq C + \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} P(n^{-1/p} |\sum_{i=1}^{k_n} a_{ni} V_{ni}| > \varepsilon/8)$$

$$\leq C + \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} \sum_{i=1}^{k_n} P(|a_{ni}X_{ni}| > n^{1/p})$$

$$\leq C + \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} \sum_{i=1}^{\infty} n^{-(\gamma-\delta)/p} E|a_{ni}X_{ni}|^{\gamma-\delta} I(|a_{ni}X_{ni}|> n^{1/p})$$

$$\leq C + \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) 2^{-j(\gamma-\delta)/p + j\beta} E|X|^{\gamma-\delta} I(|X|> 2^{j/p})$$

$$= C + \sum_{j=0}^{\infty} 2^{j\delta/p} h(2^j) E|X|^{\gamma-\delta} I(|X|> 2^{j/p})$$

$$= C + \sum_{j=0}^{\infty} 2^{j\delta/p} h(2^j) \sum_{i=j}^{\infty} E|X|^{\gamma-\delta} I(2^{i/p} < |X| \le 2^{(i+1)/p})$$

$$= C + \sum_{i=0}^{\infty} E|X|^{\gamma-\delta} I(2^{i/p} < |X| \le 2^{(i+1)/p}) \sum_{j=0}^{i} 2^{j\delta/p} h(2^j)$$

$$\le C + C \sum_{i=0}^{\infty} h(2^i) E|X|^{\gamma} I(2^{i/p} < |X| \le 2^{(i+1)/p})$$

$$\le C + CE|X|^{\gamma} h(|X|^p) < \infty. \qquad (15)$$

In order to estimate $J_1$, we first note that obviously for every positive integer $n$, $\{U_{ni} - EU_{ni}, 1 \le i \le k_n\}$ is a sequence of zero-mean $\tilde{\rho}$-mixing random variables with the mixing coefficient not greater than $\tilde{\rho}_n(\cdot)$.

Fix any $v \ge 2$ and $v > \gamma$ (the value of $v$ will be specified later). By Markov's inequality, Lemma 1, and $C_r$-inequality, , we have

$$J_1 \le C \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} n^{-v/p} \left\{ \sum_{i=1}^{k_n} E|a_{ni} U_{ni}|^v + \left( \sum_{i=1}^{k_n} E|a_{ni} U_{ni}|^2 \right)^{v/2} \right\}$$

$$\le C \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} n^{-v/p} \left\{ \sum_{i=1}^{\infty} E|a_{ni} U_{ni}|^v + \left( \sum_{i=1}^{\infty} E|a_{ni} U_{ni}|^2 \right)^{v/2} \right\}$$

$$\stackrel{Def}{=} J_3 + J_4. \qquad (16)$$

Let $I_{nk} = \{i : (k+1)^{-1/p} < |a_{ni}| \le k^{-1/p}\}, k \ge 1, n \ge 1$, then $\cup_{k=1}^{\infty} I_{nk} = N$ for all $n \ge 1$. Since $v > \gamma > \alpha$, we have $k^{(v-\alpha)/p} > j^{(v-\alpha)/p}$ for all $k > j, j, k \ge 1$. For $\alpha > 0$

$$n^{\beta} \geq \sum_{i=1}^{\infty} |a_{ni}|^{\alpha} = \sum_{k=1}^{\infty} \sum_{i \in I_{nk}} |a_{ni}|^{\alpha} \geq \sum_{k=1}^{\infty} (\#I_{nk})(k+1)^{-\alpha/p}$$

$$\geq \sum_{k=j}^{\infty} (\#I_{nk})(k+1)^{-v/p} (j+1)^{(v-\alpha)/p}$$

$$> 2^{-\alpha/p} \sum_{k=j}^{\infty} (\#I_{nk}) k^{-v/p} j^{(v-\alpha)/p}.$$

For $\alpha < 0$, we also have

$$n^{\beta} \geq \sum_{i=1}^{\infty} |a_{ni}|^{\alpha} = \sum_{k=1}^{\infty} \sum_{i \in I_{nk}} |a_{ni}|^{\alpha} \geq \sum_{k=1}^{\infty} (\#I_{nk}) k^{-\alpha/p}$$

$$> \sum_{k=j}^{\infty} (\#I_{nk}) k^{-v/p} j^{(v-\alpha)/p}.$$

Therefore,

$$\sum_{k=j}^{\infty} (\#I_{nk}) k^{-v/p} \leq C n^{\beta} j^{-(v-\alpha)/p} \quad \text{for all } j \geq 1. \tag{17}$$

By the same way as we proved (15) and by Lemma 2(i), we have

$$J_3 \leq C \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} n^{-\frac{v}{p}} \sum_{i=1}^{\infty} \left( n^{\frac{v}{p}} P(|a_{ni} X| > n^{1/p}) + E|a_{ni} X|^{v} I(|a_{ni} X| \leq n^{1/p}) \right)$$

$$\leq C \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} \sum_{i=1}^{\infty} P(|a_{ni} X| > n^{1/p})$$

$$+ C \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} \sum_{k=1}^{\infty} (\#I_{nk})(nk)^{-v/p} E|X|^{v} I(|X| < (n(k+1))^{1/p})$$

$$= C + C\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} \sum_{k=1}^{\infty} (\# I_{nk})(nk)^{\frac{v}{p}} \sum_{i=1}^{n(k+1)} E\,|X|^v \; I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$= C + C\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} \sum_{k=1}^{\infty} (\# I_{nk})(nk)^{\frac{v}{p}} \sum_{i=1}^{2n} E\,|X|^v \; I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$+ C\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} \sum_{k=2}^{\infty} (\# I_{nk})(nk)^{\frac{v}{p}} \sum_{i=2n+1}^{n(k+1)} E\,|X|^v \; I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$\overset{Def}{=} C + J_5 + J_6. \tag{18}$$

Since $v > \gamma$, we have that $(\gamma - v)/p < 0$. Then by (17) and Lemma 3

$$J_5 \le C\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} n^{-v/p+\beta} \sum_{i=1}^{2n} E\,|X|^v \; I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$\le C\sum_{j=0}^{\infty} 2^{j(\gamma - v)/p} h(2^j) \sum_{i=1}^{4} E\,|X|^v \; I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$+ C\sum_{j=1}^{\infty} 2^{j(\gamma - v)/p} h(2^j) \sum_{i=5}^{2^{j+2}} E\,|X|^v \; I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$\le C + C\sum_{i=5}^{\infty} E\,|X|^v \; I((i-1)^{1/p} \le |X| < i^{1/p}) \sum_{j=\lfloor \log_2 i \rfloor - 2}^{\infty} 2^{j(\gamma - v)/p} h(2^j)$$

$$\le C + C\sum_{i=5}^{\infty} i^{(\gamma - v)/p} h(i) E\,|X|^v \; I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$\le C + CE\,|X|^{\gamma} \; h(|X|^p) < \infty. \tag{19}$$

Next,

$$J_6 \le C\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} \sum_{i=2n+1}^{\infty} \sum_{k=[\frac{i}{n}-1]}^{\infty} (\#I_{nk})(nk)^{-v/p} E|X|^v \, I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$\le C\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} \sum_{i=2n+1}^{\infty} n^{-v/p+\beta} (\frac{i}{n})^{-(v-\alpha)/p} E|X|^v \, I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$\le C\sum_{i=3}^{\infty} i^{-(v-\alpha)/p} E|X|^v \, I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$+ C\sum_{j=1}^{\infty} 2^{j(t+1)} h(2^j) \sum_{i=2^j}^{\infty} 2^{j(\beta-\alpha/p)} i^{-(v-\alpha)/p} E|X|^v \, I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$\le CE|X|^{\alpha} + C\sum_{i=2}^{\infty} i^{-(v-\alpha)/p} E|X|^v \, I((i-1)^{1/p} \le |X| < i^{1/p}) \sum_{j=1}^{[\log_2 i]} 2^{j(\gamma-\alpha)/p} h(2^j)$$

$$\le C + C\sum_{i=2}^{\infty} i^{-(v-\gamma)/p} h(i) E|X|^v \, I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$\le C + CE|X|^{\gamma} \, h(|X|^p) < \infty. \qquad (20)$$

Therefore, from (18), (19) , and (20) we have that $J_3 < \infty$ for $\gamma \ge 1$.

For $J_4$, if $\gamma \ge 2$ by (2) we have

$$\sum_{i=1}^{\infty} E|a_{ni}U_{ni}|^2 \le C\sum_{i=1}^{\infty} E|a_{ni}X|^2 \le Cn^q. \qquad (21)$$

Since $q < 2/p$, we can chose $v$ large enough such that $(t+1) + v(q/2 - 1/p) < 0$.

By Lemma 3 (iii) we obtain

$$J_4 \le C\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \le n < 2^{j+1}} n^{-v/p} n^{vq/2} \le C\sum_{j=0}^{\infty} 2^{j\{(t+1)+v(q/2-1/p)\}} h(2^j) < \infty. \quad (22)$$

If $1 \le \gamma < 2$, let $v = 2$, then $J_4 = J_3 < \infty$. Therefore $J_1 < \infty$ for $\gamma \ge 1$. By (12), (10) holds.

**Remark 1.** (i) If there exists a positive constant $M>0$ such that $h(x)\geq M$ for sufficiently large $x$, then the assumption $E|X|^{p(t+\beta+1)}\ h(|X|^p)<\infty$ implies that $E|X|^{p(t+\beta+1)}<\infty$.

(ii) Let $h(x)=1, X_{ni}=X_i$, for all $i\geq 1, n\geq 1$, and $\{X_i, i\geq 1\}$ be a sequence of independent random variables. Then Theorem A follows from Theorem 1, since independent random variables are a special case of $\tilde{\rho}$-mixing random variables.

(iii) Let $\beta=0, t=r-2$, and $h(x)=1$. If condition (5) holds, then (1) holds according to (2.11) of Wang et al. [2], with $\alpha=\tilde{q}\ (r-1), \gamma=p(r-1), 2\leq q<\tilde{q}<p$. When $0<q(r-1)<2$, by (2.11) of Wang et al. [2], we have that $\sum_{i\in Z} a_{ni}^2=O(1)$. Therefore, if (5) and (7) hold, we have $\sum_{i\in Z} a_{ni}^2=O(n^{\delta})$, for $0<\delta<2/p$. Thus Theorem 1 extends and improves the sufficient part of Theorem C (I) for the case of $\tilde{\rho}$-mixing random variables.

If condition (1) on the weights is replaced by a weaker condition (4), we obtain the following theorem.

**Theorem 2.** *Let* $\{X_{nk}, k\geq 1, n\geq 1\}$ *be an array of zero-mean rowwise* $\tilde{\rho}$*-mixing random variables stochastically dominated by a random variable* $X$*. Assume that* $\overline{\lim}_{k\to\infty}\sup_n \tilde{\rho}_n(k)<1$ *and* $E|X|^{\gamma}\log|X|<\infty$*, where* $\gamma=p(t+\beta+1)>0$ *and* $p>0$*. Let* $\{a_{ni}, i\geq 1, n\geq 1\}$ *be an array of real numbers satisfying (4).*

(i) *If* $1\leq\gamma<2$*, then*

$$\sum_{j=0}^{\infty}2^{j(t+1)}\max_{2^j\leq n<2^{j+1}}P\left(n^{-1/p}\left|\sum_{i=1}^{\infty}a_{ni}X_{ni}\right|>\varepsilon\right)<\infty \ \textit{for all}\ \varepsilon>0, \tag{23}$$

*moreover*

$$\sum_{n=1}^{\infty}n^{t}P\left(n^{-1/p}\left|\sum_{i=1}^{\infty}a_{ni}X_{ni}\right|>\varepsilon\right)<\infty \ \textit{for all}\ \varepsilon>0. \tag{24}$$

*(ii) If* $\gamma \geq 2$ *and* $\{a_{ni}, i \geq 1, n \geq 1\}$ *satisfies (2), then (23) and (24) hold.*

**Proof.** Let $U_{nk}, V_{nk}, I_{nk}, J_k$ be as in the proof of Theorem 1. From this proof, it is sufficient to show $J_2 < \infty$ and $J_j < \infty, j = 4,5,6$ with $h(x) = 1$.

For $J_2$, we first prove that

$$n^{-1/p} \left| \sum_{i=1}^{k_n} a_{ni} EV_{ni} \right| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Since $E|X|^{\gamma} \log|X| < \infty$, we have $E|X|^{\gamma} < \infty$ and hence

$$n^{-1/p} \left| \sum_{i=1}^{k_n} a_{ni} EV_{ni} \right| \leq n^{-1/p} \sum_{i=1}^{\infty} n^{-(\gamma-1)/p} E|a_{ni} X_{ni}|^{\gamma} I(|a_{ni} X_{ni}| > n^{1/p})$$

$$\leq n^{-\gamma/p+\beta} E|X|^{\gamma} I(|X| > n^{1/p})$$

$$= n^{-(t+1))} E|X|^{\gamma} I(|X| > n^{1/p}) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Therefore, there exists $n$ large enough such that

$$n^{-1/p} \left| \sum_{i=1}^{k_n} a_{ni} EV_{ni} \right| < \varepsilon/8.$$

Thus, similar to the proof of (15)

$$J_2 \leq C + \sum_{j=0}^{\infty} 2^{j(t+1)} \max_{2^j \leq n < 2^{j+1}} \sum_{i=1}^{k_n} P(|a_{ni} X_{ni}| > n^{1/p})$$

$$\leq C + \sum_{j=0}^{\infty} 2^{j(t+1)} \max_{2^j \leq n < 2^{j+1}} \sum_{i=1}^{\infty} n^{-\gamma/p} E|a_{ni} X_{ni}|^{\gamma} I(|a_{ni} X_{ni}| > n^{1/p})$$

$$\leq C + C \sum_{j=0}^{\infty} 2^{j(t+1)} 2^{-j\gamma/p + j\beta} E|X|^{\gamma} I(|X| > 2^{j/p})$$

$$= C + C\sum_{j=0}^{\infty} E|X|^{\gamma} I(|X| > 2^{j/p})$$

$$= C + C\sum_{j=0}^{\infty}\sum_{i=j}^{\infty} E|X|^{\gamma} I(2^{i/p} < |X| \le 2^{(i+1)/p})$$

$$= C + C\sum_{i=0}^{\infty} iE|X|^{\gamma} I(2^{i/p} < |X| \le 2^{(i+1)/p})$$

$$\le C + CE|X|^{\gamma} \log|X| < \infty.$$

Since $v > \gamma$, we have

$$n^{\beta} = \sum_{i=1}^{\infty} |a_{ni}|^{\gamma} = \sum_{k=1}^{\infty}\sum_{i\in I_{nk}} |a_{ni}|^{\gamma} \ge \sum_{k=1}^{\infty} (\#I_{nk})(k+1)^{-\gamma/p}$$

$$\ge \sum_{k=j}^{\infty} (\#I_{nk})(k+1)^{-v/p}(j+1)^{(v-\gamma)/p}$$

$$> 2^{-v/p}\sum_{k=j}^{\infty} (\#I_{nk})k^{-v/p} j^{(v-\gamma)/p}.$$

Hence

$$\sum_{k=j}^{\infty} (\#I_{nk})k^{-v/p} \le Cn^{\beta} j^{-(v-\gamma)/p} \quad \text{for all} \quad j \ge 1. \tag{25}$$

By (25), similar to the proof of (19), we obtain

$$J_5 \le C\sum_{j=0}^{\infty} 2^{j(t+1)} \max_{2^j \le n < 2^{j+1}} n^{-v/p+\beta} \sum_{i=1}^{2n} E|X|^{v} I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$\le C + C\sum_{i=5}^{\infty} i^{(\gamma-v)/p} E|X|^{v} I((i-1)^{1/p} \le |X| < i^{1/p})$$

$$\leq C + CE|X|^{\gamma} < \infty.$$

By (25), similar to the proof of (20), we obtain

$$J_6 \leq C\sum_{i=1}^{\infty} i^{-(v-\gamma)/p} E|X|^{v} I((i-1)^{1/p} \leq |X| < i^{1/p})$$

$$+ C\sum_{j=1}^{\infty} 2^{j(t+1)} \sum_{i=2^j}^{\infty} 2^{j(\beta-\gamma/p)} i^{-(v-\alpha)/p} E|X|^{v} I((i-1)^{1/p} \leq |X| < i^{1/p})$$

$$\leq CE|X|^{\gamma} + C\sum_{i=1}^{\infty} i^{-(v-\gamma)/p} E|X|^{v} I((i-1)^{1/p} \leq |X| < i^{1/p})$$

$$\leq C + C\sum_{i=2}^{\infty} i^{-(v-\gamma)/p} E|X|^{v} I((i-1)^{1/p} \leq |X| < i^{1/p})$$

$$\leq C + CE|X|^{\gamma} < \infty.$$

Similar to the proof of Theorem 1, we have $J_4 < \infty$.

**Remark 2.** Obviously, Theorem B follows from Theorem 2 by let $h(x) = 1, X_{ni} = X_i$, for all $i \geq 1, n \geq 1$, and $\{X_i, i \geq 1\}$ be a sequence of independent random variables. Furthermore, Theorem 2 extends and improves the sufficiency part of Theorem C (II) for the case of $\tilde{\rho}$-mixing random variables.

**Corollary 1.** *Let* $\{X_{nk}, k \geq 1, n \geq 1\}$ *be an array of zero-mean rowwise* $\tilde{\rho}$ *-mixing random variables stochastically dominated by a random variable* $X$*. Assume that* $\overline{\lim}_{k\to\infty} \sup_n \tilde{\rho}_n(k) < 1$ *and* $E|X|^{p} < \infty$ *for some* $p > 2$*. Let* $\{a_{ni}, i \geq 1, n \geq 1\}$ *be an array of real numbers satisfying (2) and*

$$\sum_{i=1}^{\infty} |a_{ni}|^{\alpha} = O(1) \text{ for some } 2 \leq \alpha < p.$$

*Then*

$$\sum_{j=0}^{\infty} 2^{j} \max_{2^{j}\le n<2^{j+1}} P\left( n^{-1/p} \left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| > \varepsilon \right) < \infty \text{ for all } \varepsilon > 0$$

and

$$\sum_{n=1}^{\infty} P(n^{-1/p} \left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| > \varepsilon) < \infty \text{ for all } \varepsilon > 0.$$

**Proof.** Let $t = 0$ and $\beta = 0$ and $h(x) = 1$. Clearly $|a_{ni}| = O(1)$. Thus the result follows from Theorem 1 (iii).

**Corollary 2.** *Let* $\{X_{nk}, k \ge 1, n \ge 1\}$ *be an array of zero-mean rowwise* $\tilde{\rho}$ *-mixing random variables stochastically dominated by a random variable* $X$. *Assume that* $\overline{\lim}_{k\to\infty} \sup_n \tilde{\rho}_n(k) < 1$ *and* $E|X|^2 \log|X| < \infty$. *Let* $\{a_{ni}, i \ge 1, n \ge 1\}$ *be an array of real numbers satisfying*

$$\sum_{i=1}^{\infty} |a_{ni}|^2 = O(1)$$

*Then*

$$\sum_{j=0}^{\infty} 2^{j} \max_{2^{j}\le n<2^{j+1}} P\left( n^{-1/p} \left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| > \varepsilon \right) < \infty \text{ for all } \varepsilon > 0$$

*and*

$$\sum_{n=1}^{\infty} P(n^{-1/2} \left| \sum_{i=1}^{\infty} a_{ni} X_{ni} \right| > \varepsilon) < \infty \text{ for all } \varepsilon > 0.$$

**Proof.** Let $t = 0, \beta = 0$, and $p = 2$. Clearly $|a_{ni}| = O(1)$. Thus the result follows from Theorem 2 (ii).

**Remark 3.** Set $X_{ni} = X_i$ for all $n \geq 1$ and $i \geq 1$, let $\{X_i, i \geq 1\}$ be a sequence of i.i.d. random variables. In this particular case Corollaries 1 and 2 were proved by Li et al. [19]. Hence Corollaries 1 and 2 extend the results of Li et al. [19].

As a corollary of Theorem 1, we can obtain the following result on the rate of convergence for moving average processes.

**Corollary 3.** *Let* $\{X_{nk}, k \in Z, n \in Z\}$ *be an array of zero-mean rowwise* $\tilde{\rho}$*-mixing random variables stochastically dominated by a random variable* $X$*. Assume that* $\overline{\lim}_{k\to\infty} \sup_n \tilde{\rho}_n(k) < 1$ *and* $E|X|^{p(t+2)} < \infty$ *for some* $0 < p < 2$ *and* $p(t+2) > 1$*. Let* $\{a_n, -\infty < n < \infty\}$ *be a sequence of real numbers such that* $\sum_{n=-\infty}^{\infty} |a_n| < \infty$*. Set* $a_{ni} = \sum_{j=i+1}^{i+n} a_j$ *for each* $i$ *and* $n$*. Then*

$$\sum_{j=0}^{\infty} 2^{j(t+1)} \max_{2^j \leq n < 2^{j+1}} P\left( n^{-1/p} \left| \sum_{i=-\infty}^{\infty} a_{ni} X_{ni} \right| > \varepsilon \right) < \infty \text{ for all } \varepsilon > 0,$$

*and*

$$\sum_{n=1}^{\infty} n^t P\left( \left| \sum_{i=-\infty}^{\infty} a_{ni} X_{ni} \right| / n^{1/p} > \varepsilon \right) < \infty \text{ for all } \varepsilon > 0.$$

**Proof.** Repeats the proof of Sung [2] and hence omitted.

**Remark 4.** Corollary 3 extends Corollary 3 of Sung [2] for arrays of rowwise $\tilde{\rho}$-mixing random variables.

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