Thailand Statistician
July 2011; 9(2) : 195-203
http://statassoc.or.th
Contributed paper

Some Properties of the Three-Parameter Crack Distribution

Phitchaphat Bowonrattanaset [a], Kamon Budsaba* [a,b]

[a] Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University, Phathum Thani, 12121, Thailand.

[b] Centre of Excellence in Mathematics, CHE, Si Ayutthaya Rd., Bangkok 10400, Thailand.

* Author for correspondence; e-mail: kamon@mathstat.sci.tu.ac.th

Received: 1 June 2011

Accepted: 21 June 2011

Abstract

The purpose of this paper is to investigate the main theoretical properties of a new lifetime three-parameter family of distributions. We will call this family as *Crack distribution* (CR) because it may be applied for modeling of some physical characteristics of fatigue cracks. CR-distribution relates to the following two-parameter distributions : the Birnbaum-Saunders distribution, the Inverse Gaussian distribution and the Length Biased Inverse Gaussian distribution. These are well-known fatigue-lifetime distributions. They are the special cases of CR-distribution based on non-classical parametrization. The main theoretical properties such as the characteristic function, the moment generating function and the cumulative distribution function on three-parameter CR-distribution are established in closed form.

Keywords: The Crack distribution, fatigue-lifetime distribution, parametrization.

1. Introduction

Reliability Theory achieved numerous applications in Physics, Engineering, Statistics, Environment Sciences, and Economics. One of important notions in the Reliability Theory is the notion of a lifetime distribution. In this paper, we will study the new three-parameter family of fatigue-lifetime distributions: the *Crack distribution*. It is related to three known two-parameter distributions: the Birnbaum-Saunders distribution, the Inverse Gaussian distribution, and the Length Biased Inverse Gaussian distribution. The value of random variables with these distributions could model the time before failure of an object due to a fatigue crack.

We refer to Birnbaum and Saunders [1,2] where the Birnbaum-Saunders distribution was introduced. Desmond [3] compared two fatigue-life models i.e. Birnbaum-Saunders distribution and Inverse Gaussian distribution. The Birnbaum-Saunders distribution is used in case of cyclic loading while the Inverse Gaussian distribution is used in case of non-cyclic loading. A practitioner could use the CR-distribution introduced in this paper as a general form that covers both cases.

Shuster [4] indicated a method to obtain the exact probabilities of the Inverse Gaussian distribution by using Standard Normal tables and Logs tables. Chhikara and Folks [5] gathered many properties of the Inverse Gaussian distribution. Khattree [6] studied about the Length Biased Inverse Gaussian distribution and Gamma distribution.

Ahmed, Budsaba, Lisawadi, and Volodin [7] proposed a new parametrization of the Birnbaum-Saunders distribution and provided various estimation strategies for its parameters. Their new parametrization is important since it fits the physical phenomena of fatigue cracks. The parameters (see the definition below) $\lambda > 0$ and $\theta > 0$ correspond to the thickness of the machine element and the nominal treatment pressure on the machine element, respectively.

In this paper, we will study the CR-distribution by adding the new *weight parameter* p. The engineering interpretation of Crack random variable is time after a machine element is started to be forced by a cyclic or non-cyclic loading until the crack achieves the critical value. After a machine element is forced, a slightly crack may happen but the element could still works. When it arrives the critical point, tolerance exceeds and the element does not properly work anymore.

The plan of the paper is as follows. First we introduce the probability density function (p.d.f.) based on the proposed parametrization of the Birnbaum-Saunders distribution, the Inverse Gaussian distribution, the Length Biased Inverse Gaussian distribution. Next we present an integral formula that we need for our calculations. After we will show how all four above-mentioned distributions are related. Next we will provide the characteristic function, the moment generating function and the cumulative distribution function of three-parameter CR-distribution in the closed form based on the proposed parameters of Ahmed, Budsaba, Lisawadi, and Volodin [7].

2. Preliminaries

2.1 The Birnbaum-Saunders Distribution

First we provide the density function of the Birnbaum-Saunders distribution in the *classical* parametrization.

A random variable X has the Birnbaum-Saunders distribution, if its p.d.f. is as follows

$$f_{BS}(x;\alpha,\beta)=\begin{cases}\dfrac{x+\beta}{2\alpha(2\pi\beta)^{1/2}x^{3/2}}\exp\left[-\dfrac{1}{2\alpha^2}\left(\dfrac{x}{\beta}+\dfrac{\beta}{x}-2\right)\right], & \text{if } x>0\\ 0 & , \textit{otherwise,}\end{cases}$$

where $\alpha>0$ is the shape parameter, and $\beta>0$ is the scale parameter and the median. The mean of X is $\beta\left(1+\frac{\alpha^2}{2}\right)$ and the variance is $\alpha^2\beta^2\left(1+\frac{5\alpha^2}{4}\right)$.

If a random variable Z has standard normal distribution, that is $Z\sim N(0,1)$ then the relations between X and Z are as follows

$$X=\beta\left[\frac{\alpha}{2}Z+\sqrt{\left(\frac{\alpha}{2}Z\right)^2+1}\right]^2$$

$$Z=\frac{1}{\alpha}\left[\sqrt{\frac{X}{\beta}}-\sqrt{\frac{\beta}{X}}\right].$$

The cumulative distribution function (c.d.f.) of X with the classical parametrization is given by

$$F_{BS}(x;\alpha,\beta)=\Phi\left(\frac{1}{\alpha}\left[\sqrt{\frac{x}{\beta}}-\sqrt{\frac{\beta}{x}}\right]\right)\ \text{for } x>0.$$

In this paper we consider the following new parametrization. A random variable X has the *Birnbaum-Saunders distribution*, denoted as BS(λ,θ) if its p.d.f. is

$$f_{BS}(x;\lambda,\theta)=\begin{cases}\dfrac{1}{2\theta\sqrt{2\pi}}\left[\lambda\left(\dfrac{\theta}{x}\right)^{3/2}+\left(\dfrac{\theta}{x}\right)^{1/2}\right]\exp\left[-\dfrac{1}{2}\left(\sqrt{\dfrac{x}{\theta}}-\lambda\sqrt{\dfrac{\theta}{x}}\right)^2\right] & , x>0\\ 0 & , \textit{otherwise.}\end{cases} \quad (1)$$

The relations between classical parameters α,β and proposed parameters λ,θ are as follows

$$\lambda=\frac{1}{\alpha^2}\ \text{and}\ \theta=\alpha^2\beta;$$

$$\alpha=\frac{1}{\sqrt{\lambda}}\ \text{and}\ \beta=\lambda\theta.$$

2.2 The Inverse Gaussian Distribution

Again, we start with the *classical parametrization*.

A random variable X has the Inverse Gaussian distribution, if its p.d.f. is as follows

$$f_{IG}(x;\mu,\beta)=\begin{cases}\sqrt{\dfrac{\beta}{2\pi}}x^{-3/2}\exp\left(-\dfrac{\beta(x-\mu)^2}{2\mu^2 x}\right) & ,x>0\\ 0 & ,otherwise,\end{cases}$$

where parameter $\mu>0$ is the mean of the distribution and $\beta>0$ is a scale parameter. The new parametrization of the Inverse Gaussian distribution, denoted as IG(λ,θ) is as follows. A random variable X has the Inverse Gaussian distribution if its p.d.f. is

$$f_{IG}(x;\lambda,\theta)=\begin{cases}\dfrac{\lambda}{\theta\sqrt{2\pi}}\left(\dfrac{\theta}{x}\right)^{3/2}\exp\left[-\dfrac{1}{2}\left(\sqrt{\dfrac{x}{\theta}}-\lambda\sqrt{\dfrac{\theta}{x}}\right)^2\right] & ,x>0\\ 0 & ,otherwise.\end{cases} \quad (2)$$

The relations between classical parameters α,β and proposed parameters λ,θ are as follows

$$\lambda=\frac{\beta}{\mu}\text{ and }\theta=\frac{\mu^2}{\beta};$$

$$\mu=\lambda\theta\text{ and }\beta=\lambda^2\theta.$$

2.3 The Length Biased Inverse Gaussian Distribution

Remind that the *length biased* pdf of its original p.d.f. is defined as follows.

Let X be a non-negative random variable having an absolutely continuous p.d.f. $f(\cdot)$ and a finite first moment $E[X]$. We say that a non-negative random variable Y has the length biased random variable associated with X, if its p.d.f. is given by the formula

$$h(x)=\frac{xf(x)}{E[X]},x>0.$$

We know that the first moment of the Inverse Gaussian distribution is $E(X)=\mu=\lambda\theta$. Hence, the p.d.f. of the Length Biased Inverse Gaussian distribution is given by the following formula

$$f_{LB}(x;\lambda,\theta)=\begin{cases}\dfrac{1}{\theta\sqrt{2\pi}}\left(\dfrac{\theta}{x}\right)^{1/2}\exp\left[-\dfrac{1}{2}\left(\sqrt{\dfrac{x}{\theta}}-\lambda\sqrt{\dfrac{\theta}{x}}\right)^2\right] & ,x>0\\ 0 & ,otherwise.\end{cases} \quad (3)$$

2.4 Definite Integrals of exponentials of Complicated Arguments and Powers

In this section we present some definite integrals which are used in calculations of the characteristic and moment generating functions for the CR-distribution. The formulae are taken from the famous *Table of integrals, series, and products* by Gradshteyn and Ryzhik [8].

Let ρ and q be complex numbers. Then

$$\int_0^{\infty} x^{-1/2} \exp(-\rho x - q/x)\,dx = \sqrt{\frac{\pi}{\rho}} \exp\left(-2\sqrt{\rho q}\right) \tag{4}$$

where $\operatorname{Re}(\rho) > 0, \operatorname{Re}(q) \geq 0$

and

$$\int_0^{\infty} x^{-n-1/2} \exp(-\rho x - q/x)\,dx = (-1)^n \sqrt{\frac{\pi}{\rho}} \frac{\partial^n}{\partial q^n} e^{-2\sqrt{\rho q}} \tag{5}$$

where $\operatorname{Re}(\rho) > 0, \operatorname{Re}(q) > 0$.

Note that from (5) for $n = 1$ we have

$$\int_0^{\infty} x^{-1-1/2} \exp(-\rho x - q/x)\,dx = -\sqrt{\frac{\pi}{\rho}} \frac{\partial}{\partial q} \exp\left(-2\rho^{1/2} q^{1/2}\right) = \sqrt{\frac{\pi}{q}} \exp\left(-2\sqrt{\rho q}\right).$$

Thus

$$\int_0^{\infty} x^{-3/2} \exp(-\rho x - q/x)\,dx = \sqrt{\frac{\pi}{q}} \exp\left(-2\sqrt{\rho q}\right), \tag{6}$$

where $\operatorname{Re}(\rho) > 0, \operatorname{Re}(q) > 0$.

3. The Probability Density Function of the Crack Distribution

We say that a continuous random variable X has the *Crack distribution* with parameters $\lambda > 0, \theta > 0$, and $0 \leq p \leq 1$ denoted as $CR(\lambda, \theta, p)$, if its p.d.f. is

$$f_{CR}(x; \lambda, \theta, p) = \begin{cases} \dfrac{1}{\theta\sqrt{2\pi}}\left[p\lambda\left(\dfrac{\theta}{x}\right)^{3/2} + (1-p)\left(\dfrac{\theta}{x}\right)^{1/2}\right] \exp\left[-\dfrac{1}{2}\left(\sqrt{\dfrac{x}{\theta}} - \lambda\sqrt{\dfrac{\theta}{x}}\right)^2\right] & , x > 0 \\ 0 & , otherwise. \end{cases} \tag{7}$$

Note the following relations of the p.d.f. for Birnbaum-Saunders, the Inverse Gaussian, and the Length Biased Inverse Gaussian distributions with the CR-distribution

The following are the special cases of the CR- distribution.

$$f_{CR}(x;\lambda,\theta,p)=\begin{cases} f_{IG}(x;\lambda,\theta) & ,p=1\\ f_{BS}(x;\lambda,\theta) & ,p=\frac{1}{2}\\ f_{LB}(x;\lambda,\theta) & ,p=0\end{cases}$$

Moreover,

$$f_{CR}(x;\lambda,\theta,p)=pf_{IG}(x;\lambda,\theta)+(1-p)f_{LB}(x;\lambda,\theta)$$

Note that

$$f_{BS}(x;\lambda,\theta)=\frac{1}{2}[f_{IG}(x;\lambda,\theta)+f_{LB}(x;\lambda,\theta)]$$

and it is mentioned in [7] that the new parameters of the Birnbaum-Saunders distribution have the following physical interpretation. Parameter $\lambda>0$ corresponds to the thickness of a machine element under consideration for a crack development and $\theta>0$ corresponds to nominal treatment pressure of the machine element.

4. The Characteristic Function of the Crack Distribution

Theorem. *The characteristic function of a random variable* $X\sim CR(\lambda,\theta,p)$ *is*

$$\varphi_X(t)=\frac{e^{\lambda(1-\sqrt{1-2\theta ti})}}{\sqrt{1-2\theta ti}}\left[1-p\left(1-\sqrt{1-2\theta ti}\right)\right].$$

Proof. By the definition

$$\begin{aligned}\varphi_X(t)&=E\left[e^{itX}\right]\\ &=\int_0^\infty \exp(itx)\frac{1}{\theta\sqrt{2\pi}}\left[p\lambda\left(\frac{\theta}{x}\right)^{3/2}+(1-p)\left(\frac{\theta}{x}\right)^{1/2}\right]\exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{\theta}{x}}\right)^2\right]dx\\ &=\frac{1}{\theta\sqrt{2\pi}}\int_0^\infty\left[p\lambda\left(\frac{\theta}{x}\right)^{3/2}+(1-p)\left(\frac{\theta}{x}\right)^{1/2}\right]\exp\left[itx-\frac{1}{2}\left(\frac{x}{\theta}-2\lambda+\frac{\lambda^2\theta}{x}\right)\right]dx\\ &=\frac{e^{\lambda}p\lambda\sqrt{\theta}}{\sqrt{2\pi}}\int_0^\infty x^{-3/2}\exp\left[-\left(\frac{1}{2\theta}-ti\right)x-\frac{\lambda^2\theta/2}{x}\right]dx\\ &\quad+\frac{e^{\lambda}(1-p)}{\sqrt{\theta}\sqrt{2\pi}}\int_0^\infty x^{-1/2}\exp\left[-\left(\frac{1}{2\theta}-ti\right)x-\frac{\lambda^2\theta/2}{x}\right]dx \end{aligned}\tag{8}$$

Let us consider each integral in (8) separately. From (6) with $\rho=\frac{1}{2\theta}-ti$ (obviously $\mathrm{Re}(\rho)=\frac{1}{2\theta}>0$) and $q=\frac{\lambda^2\theta}{2}$ (again, $\mathrm{Re}(q)=\frac{\lambda^2\theta}{2}>0$) we obtain

$$\int_0^\infty x^{-3/2}\exp\left[-\left(\frac{1}{2\theta}-ti\right)x-\frac{\lambda^2\theta/2}{x}\right]dx=\sqrt{\frac{2\pi}{\lambda^2\theta}}\exp\left(-\lambda\sqrt{1-2\theta ti}\right)\tag{9}$$

From (4) with the same ρ and q we obtain

$$\int_0^\infty x^{-1/2}\exp\left[-\left(\frac{1}{2\theta}-ti\right)x-\frac{\lambda^2\theta/2}{x}\right]dx=\sqrt{\frac{2\theta\pi}{1-2\theta ti}}\exp\left(-\lambda\sqrt{1-2\theta ti}\right) \tag{10}$$

Putting (9) and (10) into (8), we get that

$$\varphi_X(t)=\frac{e^{\lambda}p\lambda\sqrt{\theta}}{\sqrt{2\pi}}\sqrt{\frac{2\pi}{\lambda^2\theta}}\exp\left(-\lambda\sqrt{1-2\theta ti}\right)+\frac{e^{\lambda}(1-p)}{\sqrt{\theta}\sqrt{2\pi}}\sqrt{\frac{2\theta\pi}{1-2\theta ti}}\exp\left(-\lambda\sqrt{1-2\theta ti}\right)$$

$$=\frac{e^{\lambda\left(1-\sqrt{1-2\theta ti}\right)}}{\sqrt{1-2\theta ti}}\left[1-p\left(1-\sqrt{1-2\theta ti}\right)\right].$$

Note that as any characteristic function it is defined for all $-\infty<t<\infty$.

5. The Moment Generating Function

Theorem. *The moment generating function of* $X\sim CR(\lambda,\theta,p)$ *is*

$$\phi_X(t)=\frac{e^{\lambda\left(1-\sqrt{1-2\theta t}\right)}}{\sqrt{1-2\theta t}}\left[1-p\left(1-\sqrt{1-2\theta t}\right)\right]$$

and it is defined for $t<\frac{1}{2\theta}$.

Proof. Note that the proof of the formula for the moment generating function repeats the proof of the formula for the characteristic function if instead of it we consider just t and hence omitted. The only thing that we should mention is that the integrals exist if $t<\frac{1}{2\theta}$.

Not looking that the derivations of characteristic and moment generating functions are very similar, they present two different results. The characteristic function is defined for all real t, while the moment generating function is defined for $t<\frac{1}{2\theta}$.

6. The Cumulative Distribution Function

Theorem. *The cumulative distribution function of* $X\sim CR(\lambda,\theta,p)$ *is*

$$F_{CR}(x;\lambda,\theta,p)=\begin{cases}\Phi\left(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{\theta}{x}}\right)-(1-2p)e^{2\lambda}\left[1-\Phi\left(\sqrt{\frac{x}{\theta}}+\lambda\sqrt{\frac{\theta}{x}}\right)\right] & ,x>0\\ 0 & ,x\le 0,\end{cases}$$

where $\Phi(x)$ *is the standard normal distribution function.*

Proof. To prove the theorem, we show that

$$\frac{d}{dx}F_{CR}(x;\lambda,\theta,p)=f_{CR}(x;\lambda,\theta,p).$$

Note that

$$\frac{d}{dx}\Phi(x)=\phi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

Next, for $x > 0$

$$\frac{d}{dx}F_{CR}(x;\lambda,\theta,p)$$

$$=\frac{d}{dx}\left\{\Phi\left(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{\theta}{x}}\right)-(1-2p)e^{2\lambda}\left[1-\Phi\left(\sqrt{\frac{x}{\theta}}+\lambda\sqrt{\frac{\theta}{x}}\right)\right]\right\}$$

$$=\left[\phi\left(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{\theta}{x}}\right)\right]\frac{d}{dx}\left(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{\theta}{x}}\right)+(1-2p)e^{2\lambda}\left[\phi\left(\sqrt{\frac{x}{\theta}}+\lambda\sqrt{\frac{\theta}{x}}\right)\right]\frac{d}{dx}\left(\sqrt{\frac{x}{\theta}}+\lambda\sqrt{\frac{\theta}{x}}\right)$$

$$=\frac{1}{2x\sqrt{2\pi}}\cdot\left(\sqrt{\frac{x}{\theta}}+\lambda\sqrt{\frac{\theta}{x}}\right)\exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{\theta}{x}}\right)^2\right]$$

$$+\frac{1-2p}{2x\sqrt{2\pi}}\left(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{\theta}{x}}\right)\exp\left[2\lambda-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}}+\lambda\sqrt{\frac{\theta}{x}}\right)^2\right]$$

$$=\frac{1}{\theta\sqrt{2\pi}}\left[p\lambda\left(\frac{\theta}{x}\right)^{3/2}+(1-p)\left(\frac{\theta}{x}\right)^{1/2}\right]\exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}}-\lambda\sqrt{\frac{\theta}{x}}\right)^2\right].$$

It is clear that $\frac{d}{dx}F_{CR}(x;\lambda,\theta,p)=f_{CR}(x;\lambda,\theta,p)=0$ for $x\le 0$.

It is obvious that $F(-\infty)=0$ and $F(\infty)=1$.

7. Conclusion

This paper can be considered as a starting point for study on the new three-parameter lifetime distribution, the CR- distribution, which relates to three known two-parameter distributions. We provided the closed forms of the characteristic function, moment generating function (including their existing conditions), and the cumulative distribution function of CR- distribution based on the new parametrization. The existing condition $t<\frac{1}{2\theta}$ of the moment generating function depends on parameter θ only.

There are still many topics on CR- distribution to investigate, such as statistical inference and the survival analysis. It will give us more results if we can join the statistical knowledge and the engineering knowledge on this subject.

Acknowledgements

This research is partially supported by the Center of Excellence in Mathematics, the Commission on Higher Education, Thailand.

References

[1] Birnbaum, Z. W., & Saunders, S. C. A new family of life distributions, *J. Applied Probability*, 1969; 6: 319-327.

[2] Birnbaum, Z. W., & Saunders, S. C. Estimation for a family of life distributions with applications to fatigue, *J. Applied Probability*, 1969; 6: 328-347.

[3] Desmond, A. F. On the relationship between two fatigue-life models, *IEEE Reliability*, 1986; 35: 167-169.

[4] Shuster, J. On the Inverse Gaussian distribution function, *J.American Statistical Association*, 1968; 63: 1514-1516.

[5] Chhikara, R. S., & Folks, J. L. *The inverse Gaussian distribution-theory, methodology, and applications,* Now York: Marcel Dekker, 2008.

[6] Khattree, R. Characterization of inverse-Gaussian and gamma distributions through their length-biased distributions, *IEEE Reliability*, 1989; 38: 610-611.

[7] Ahmed S. E., Budsaba K., Lisawadi S., & Volodin, A. I. Parametric estimation for the Birnbaum-Saunders lifetime distribution based on a new parametrization, *Thailand Statistician*, 2008; 6: 213-240.

[8] Gradshteyn, I. S., & Ryzhik, I. M. *Table of integrals, series, and products* (7ed.), MA, Elsevier Academic Press, 2007.