Thailand Statistician
January 2010; 8(1) : 47-61
www.statassoc.or.th
Contributed paper

An Integral Equation Approach to EWMA Chart for Detecting a Change in Lognormal Distribution

Yupaporn Areepong and Saowanit Sukparungsee*
Department of Applied Statistics, Faculty of Applied Science,
King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand.
* **Author for correspondence**; e- mail: swns@kmutnb.ac.th

Received: 29 September 2009
Accepted: 7 January 2010

Abstract

The Exponentially Weighted Moving Average (EWMA) procedure is a popular chart in quality control and surveillance for detecting small changes in parameters of distribution. We use the Integral Equation approach to compute properties of the EWMA chart when observations are lognormally distributed. We compute the Average Run Length (ARL) of false alarm times and Average Delay (AD) of true alarm times by numerical solution of the Integral Equations and then compare the results with Monte Carlo simulations. A discussion of the efficiency of the different quadrature rules that can be used for numerical integration is given. We use the Integral Equation approach to obtain tables of optimal parameter values for one-sided lognormal EWMA designs. For given ARL values of 300 and 500 and out-of-control-parameter values of 0.1 - 0.9, tables are given of optimal values for weights and alarm boundaries and for optimal AD. The Gauss-Legendre rule for numerical result of the Integral Equations is superior to other rules. Also, the Integral Equations approach is much better than MC because its computational time is much less than the latter especially, for finding an optimal parameter of EWMA designs.

Keywords: average delay, average run length, exponentially weighted moving average, integral equations, quadrature rules.

1. Introduction

Statistical Process Control (SPC) charts play a vital role in quality control in many applications. They have been used in monitoring, measuring, and quality control in areas such as manufacturing, computer sciences and telecommunications, finance and economics, epidemiology, environmental statistics and in other fields. In this paper, we discuss the Exponentially Weighted Moving Average (EWMA) chart. This chart is used for detecting small changes of parameters [1-3].

Two common characteristics of control charts are the Average Run Length (ARL) and the Average Delay (AD). The ARL is a measure of the number of observations that are made before an in-control process is falsely signaled to be out of control. Ideally, an acceptable ARL of in-control processes should be sufficiently large. The AD is a measure of the number of observations that are made before an out-of-control process is correctly signaled to be out of control. Ideally, an acceptable AD of out-of-control processes should be small. Many methods for evaluating ARL and AD for the EWMA procedure have been studied in the literature, e.g. Markov Chain Approach (MCA) [3, 4], Integral Equation approach (IE) [2] and Monte Carlo simulations (MC). Roberts [1] was the first researcher to introduce the ARL of EWMA. He used simulations to derive nomograms for the case of a Gaussian distribution. Crowder [2] used Integral Equations (IE) to find both ARL and AD. Later, Lucas and Saccucci [3] evaluated ARL by using a finite-state Markov Chain Approach (MCA). Borror [5] examined the ARL performance of EWMA charts for both skewed and heavy-tailed symmetric non-normal distributions using MCA and also showed that a EWMA chart is more robust to the assumption of normality than Cumulative Sum (CUSUM) charts.

MCA, IE and MC are the most popular methods for evaluating the characteristics of EWMA charts. These methods have the following features. The IE method is the most advanced method but it has only been used before for the special case of Gaussian distribution. The MCA requires a discretisation of the continuity of the process into many steps and a large number of calculations of matrix inverses. MC is simple to program and is good for checking accuracy but it is usually very time consuming as it requires a large number of sample trajectories. In addition, it is also difficult to find optimal designs by using MC.

In this paper we use the Integral Equation method to evaluate the ARL and AD of EWMA charts for observations from a lognormal distribution. The Integral Equations are solved numerically by using Gauss-Legendre integration rules to approximate the integrals. The results for ARL are compared with results from Monte Carlo simulations using numerical algorithm written in C++ and Mathematica programming. Optimal parameter values for lognormal EWMA designs are obtained.

2. Chart Characteristics

Let ξ_t, $t = 1, 2, \ldots$ be a sequence of independent identically distributed (i.i.d.) random variables with a distribution $F(x,\alpha)$, where α is a parameter. It is usually assumed that under some "standard" (or "in-control") conditions the parameter has a known in-control value $\alpha = \alpha_0$. Then at some unknown time ν, which is called the change-point time $(\nu < \infty)$, the parameter α could be changed to an "out-of-control" value $\alpha \neq \alpha_0$.

As stated above, two measures that are commonly used to analyse chart characteristics are the Average Run Length (ARL) and the Average Delay (AD) time. The ARL is the average number of observations that will occur before an in-control process falsely gives an out-of-control signal. To reduce the number of false out-of-control signals a sufficiently large ARL is required. The AD is a measure of the average number of observations that will occur before an out-of-control process correctly gives an out-of-control signal and to reduce the time that the process is out-of-control, a small AD is required. Therefore the ARL and AD are two conflicting criteria that must be balanced to give an optimal control chart.

All popular charts such as Shewhart, CUSUM, EWMA and Shiryayev-Roberts charts (see e.g. Page [6]; Woodall and Adams [7]; Hawkins and Olwell [8]) are based on the use of a stopping time τ. The typical condition on choice of the stopping time τ is that

$$ARL = E_\infty(\tau) = T, \tag{1}$$

where T is given (usually large) and $E_\infty(.)$ is the expectation under distribution $F(x,\alpha_0)$ for the in-control state. The quantity $E_\infty(\tau)$ is usually called the Average Run Length (ARL).

Another typical characteristic of a control chart is obtained by minimizing the quantity

$$AD = E_\nu(\tau - \nu + 1 \mid \tau \geq \nu), \tag{2}$$

where $E_\nu(.)$ is the expectation under the assumption that change-point occurs at time $\nu < \infty$ and α is the value of the parameter after change-point. In practice, the condition in (2) is usually calculated when $\nu = 1$. The quantity $E_1(\tau)$ is usually called the Average Delay (AD) time and a sequential chart has a near optimal performance if its AD is close to a minimal value.

3. EWMA Procedure

The Exponentially Weighted Moving Average (EWMA) chart was introduced by Roberts [1]. It is usually used to monitor and detect a small change in a process mean. Crowder [2], Gan [9], Lucas and Saccucci [3] and Knoth [10] have given detailed discussions and numerical comparisons for characteristics of EWMA procedure. The EWMA for the discrete time case is defined by the following recursion:

$$X_t = (1-\lambda)X_{t-1} + \lambda\xi_t, \quad t = 1, 2, ..., \tag{3}$$

where ξ_t is a sequence of independent identically distributed random variables, $\lambda \in (0,1)$ is a weighting constant and X_t is the weighted average between current and previous observations. The target in-control parameter α_0 is supposed to be steady and the initial value X_0 is usually chosen to be the process in-control parameter, i.e. $X_0 = \alpha_0$.

If the observations ξ_t are independent random variables with variance σ^2, then the variance of EWMA statistics X_t is

$$\sigma_{X_t}^2 = \sigma^2 (\frac{\lambda}{2-\lambda})[1-(1-\lambda)^{2t}], \text{ t} = 1, 2, ...$$

Since $0 < 1-\lambda < 1$, we have that $(1-\lambda)^{2t} \to 0$ as $t \to \infty$, and therefore the asymptotic value of the variance is

$$\sigma_{X_t}^2 = \sigma^2 (\frac{\lambda}{2-\lambda}). \tag{4}$$

When constant upper and lower control limits are preferred for detecting change-point, the standard deviation used in the limits is usually the asymptotic value. Using the expression in Equation (4), the upper control limit of the EWMA chart is the following:

$$UCL = H_U = \alpha_0 + L\sigma\sqrt{\frac{\lambda}{2-\lambda}}$$

and the lower control limit of the EWMA chart is

$$LCL = H_L = \alpha_0 - L\sigma\sqrt{\frac{\lambda}{2-\lambda}},$$

where L is a constant to be chosen. The process will be declared to be in an out-of-control state when $X_t > H_U$ or $X_t < H_L$.

The first passage time of an EWMA chart is:

$$\tau_{L,U} = \inf\{t > 0 : X_t < H_L \quad or \quad X_t > H_U\}. \tag{5}$$

In this paper, we mainly discuss the case of a positive change in distribution where crossing the upper limit raises an alarm. This is called an “upper-sided EWMA” procedure. We then use the notation $H_U = H.$

The first passage time of an upper-sided EWMA chart is:

$$\tau_H = \inf\{t > 0 : X_t > H\},$$

where H is the control limit.

The ARL of the EWMA chart depends on the control limit (H) and the smoothing parameter $(\lambda).$ The optimal design parameters (λ, H) are given by minimizing AD after change in a process mean.

4. Integral Equations for Evaluating ARL and AD for lognormal EWMA Chart.

In this part, we use the Integral Equation approach to find numerical values for ARL and AD for EWMA charts for observations from lognormal distribution. In real applications, particularly in industry, the observations often have normal distribution. Nevertheless, a process may not follow the normal distribution as it may be positive or right skewed. Shewhart control charts for a lognormal distribution have been developed by Morrison [11] and Kotz and Lovelace [12].

Integral Equations were first used by Crowder [2] for approximating ARL and AD for Gaussian distribution. Later, Champ and Rigdon [13] intensively studied them by comparing the run length distributions obtained from the Integral Equation technique and MCA for the case of Gaussian distribution (for detail, see Gan [9], Srivastava [14]).

Let $\xi_t, t = 1, 2, \ldots$ be sequentially observed independent random variables. The change point models are the following:

$$\xi_t \sim \begin{cases} lognormal\left(\alpha_0, \sigma^2\right) & ;t = 1,\ 2,\ ...,\ \nu - 1 \\ lognormal\left(\alpha, \sigma^2\right) & ;t = \nu,\ \nu + 1,\ ..., \quad \text{where} \quad \alpha \neq \alpha_0. \end{cases}$$

A lognormal distribution is defined by the following function:

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln(x)-\alpha)^2}{2\sigma^2}}.$$

We assume that the EWMA statistic X_t is defined as in Equation (3). We first consider the EWMA procedure when $\xi_1, \xi_2, ...$ are i.i.d. with a probability density function $f(x)$. We assume that the system is in-control at time t if the EWMA statistic X_t is in the range $H_L \leq X_t \leq H_U$ and out-of-control if $X_t > H_U$ or $X_t < H_L$, where H_L is a constant lower bound and H_U is a constant upper bound. We also assume that the system is initially in an in-control state u, i.e. $X_0 = u$ and $H_L \leq u \leq H_U$.

We now define a function $G(u)$ as follows:

$$G(u) = E_\infty\left(\tau_{L,U}\right), \quad X_0 = u, \tag{6}$$

where $\tau_{L,U}$ is the stopping time defined in Equation (5). Then $G(u)$ is the ARL for initial value u. There are two possibilities for X_1 after the first observation, ξ_1, is made. If ξ_1 gives an out-of-control value for X_1, then

$$X_1 = (1-\lambda)X_0 + \lambda\xi_1 > H_U \quad \text{or} \quad \mathrm{X}_1 = (1-\lambda)X_0 + \lambda\xi_1 < H_L.$$

In this case the run length will be 1 because there will be an immediate out-of-control signal. If ξ_1 gives X_1 in an in-control state, then

$$H_L < (1-\lambda)X_0 + \lambda\xi_1 < H_U. \tag{7}$$

Then one observation will have been made and on average $G(X_1) = G((1-\lambda)X_0 + \lambda\xi_1)$ more observations will be made before an out-of-control signal occurs. The inequality (7) can be rewritten in the form

$$\frac{H_L - (1-\lambda)X_0}{\lambda} < \xi_1 < \frac{H_U - (1-\lambda)X_0}{\lambda}. \tag{8}$$

The probability that ξ_1 satisfies the bounds in Equation (8) for a probability distribution function $f(\xi_1)$ is given by:

$$P\left(\frac{H_L-(1-\lambda)X_0}{\lambda}<\xi_1<\frac{H_U-(1-\lambda)X_0}{\lambda}\right) = \int_{\frac{H_L-(1-\lambda)X_0}{\lambda}}^{\frac{H_U-(1-\lambda)X_0}{\lambda}} f(x)dx,$$

where $f(x)$ is the probability density function.

Then, following the method given in Champ and Rigdon [13], if we let initial $X_0 = u$ and make the substitution $x=\xi_t$, we can rewrite the formula for the $G(u)$ defined in Equation (6) as follows:

$$G(u)=\left(1-P\left(\frac{H_L-(1-\lambda)u}{\lambda}<\xi_1<\frac{H_U-(1-\lambda)u}{\lambda}\right)\right)$$

$$+\int_{\frac{H_L-(1-\lambda)u}{\lambda}}^{\frac{H_U-(1-\lambda)u}{\lambda}}\left(1+G\left(\left(1-\lambda\right)u+x\right)\right)f(x)dx$$

$$=1+\int_{\frac{H_L-(1-\lambda)u}{\lambda}}^{\frac{H_U-(1-\lambda)u}{\lambda}}G\left(\left(1-\lambda\right)u+\lambda x\right)f(x)dx,$$

or finally, on changing the integration variable, we obtain

$$G(u)=1+\frac{1}{\lambda}\int_{H_L}^{H_U}G(x)\,f\left(\frac{x-(1-\lambda)u}{\lambda}\right)dx.$$

In this paper we consider mainly the upper-sided case for nonnegative random variable ξ_t. Since $X_t \geq 0$ for the case of nonnegative ξ_t we can assume $H_L=0$ and $H_U=H$. Then we get

$$G(u)=1+\frac{1}{\lambda}\int_{0}^{H}G(x)\,f\left(\frac{x-(1-\lambda)u}{\lambda}\right)dx. \tag{9}$$

Note that the AD also can be found from Equation (9) where $f(x)$ is the probability density function with an out-of-control parameter α.

In general, the Integral Equations (9) cannot be solved analytically for $G(u)$ and it is necessary to use numerical methods to solve them. We shall use a quadrature rule to approximate the integral by a finite sum. Any quadrature rule is defined by a set of points $\{a_j,\ j=1,\ 2,\ ...,\ m\}$ on the interval $[0,H]$ and a set of constant weights $\{w_j,\ j=1,\ 2,\ ...,\ m\}$. The approximation for an integral is of the form:

$$\int_0^H W(x)g(x)dx \approx \sum_{j=1}^{m} w_j g(a_j), \tag{10}$$

where $W(x)$ and $g(x)$ are given functions. Different choices for the function $W(x)$ which are the sets of points and weights define different quadrature rules. The quadrature rules that we use to obtain numerical solution of Integral Equations for ARL of EWMA are midpoint rule, trapezoidal rule, Simpson's rule and Gauss-Legendre rule. In next section, we will compare the numerical results from these techniques. We give a summary of quadrature rules and their application to Integral Equations.

The main criteria used in selecting the function $W(x)$, the set of points $\{a_j, j=1,2,...,m\}$ and the weights $\{w_j, j=1,2,...,m\}$ to integrate $\int_0^H W(x)g(x)dx$ are as follows. First, the function $W(x)$ is chosen so that a set of polynomials will give a good approximation to the function $g(x)$ to be integrated. The sets of points and weights are then chosen so that the quadrature rule is exact if $g(x)$ is replaced by the highest possible degree polynomial for the given choice of points.

The choices for the weights and sets of points for the four different rules can be found in Atkinson [15, 16]. Using the quadrature rule (10), we obtain a numerical approximation $\tilde{G}(u)$ for Integral Equations as a system of linear algebraic equations:

$$\tilde{G}(a_i) = 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j \tilde{G}(a_j)\, f\left(\frac{a_j-(1-\lambda)a_i}{\lambda}\right), \quad i=1,2,...,m. \tag{11}$$

That is

$$\tilde{G}(a_1) = 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j \tilde{G}(a_j)\, f\left(\frac{a_j-(1-\lambda)a_1}{\lambda}\right)$$

$$\tilde{G}(a_2) = 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j \tilde{G}(a_j)\, f\left(\frac{a_j - (1-\lambda)a_2}{\lambda}\right)$$
$$\vdots$$
$$\tilde{G}(a_m) = 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j \tilde{G}(a_j)\, f\left(\frac{a_j - (1-\lambda)a_m}{\lambda}\right)$$

The above set of m equations in m unknowns can be rewritten in a matrix form as follows. We define a column vector $\mathbf{G}$ with components:

$$G_j = \tilde{G}(a_j), \quad j = 1, 2, ..., m$$

and an $m \times m$ matrix R with matrix entries given by:

$$[\mathbf{R}]_{ij} = \frac{1}{\lambda} w_j\, f\left(\frac{a_j - (1-\lambda)a_i}{\lambda}\right).$$

The system of algebraic linear equations can then be written as the matrix equation:

$$\mathbf{G} = \mathbf{1} + \mathbf{RG},$$

where vector $\mathbf{1_m}$ is $m \times 1$ vector of ones. Therefore

$$\mathbf{G} - \mathbf{RG} = \mathbf{1}$$

or, equivalently,

$$(\mathbf{I} - \mathbf{R})\mathbf{G} = \mathbf{1}.$$

This equation can be easily solved numerically to obtain the vector $\mathbf{G}$ and therefore the components $\tilde{G}(a_j) = G_j$. After substituting the values for $\tilde{G}(a_j)$ into Equation (11) and replacing the a_i by u, we obtain an approximation for the function $G(u)$ as

$$\tilde{G}(u) = 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j \tilde{G}(a_j)\, f\left(\frac{a_j - (1-\lambda)u}{\lambda}\right). \quad (12)$$

5. Numerical Comparisons

In this part, we compare the numerical results obtained by Integral Equations with results by Monte Carlo simulations. We have solved the Integration Equations using the four quadrature rules, composite midpoint rule, composite trapezoidal rule, composite Simpson's rule and Gauss-Legendre. In the numerical tests we assume that observations are from a lognormal distribution with parameter $\sigma^2 = 1$ and that a typical set of EWMA chart parameters is $\lambda = 0.05$ and $H = 2.253.$ Numerical values for ARL are computed for an in-control parameter value $\alpha = 0$ and numerical values for AD are computed for a range of out-of-control parameter values, $\alpha = 0.2, 0.4, 0.6, 0.8$ and $1.$ The CPU times have been obtained for each computation. The results of the tests are shown in Table 1.

It can be seen from Table 1 that all of the Integral Equation method give results of similar accuracy for ARL and AD for the lognormal distribution and that the ARL and AD values are in good agreement with values from the MC simulations. The CPU times for the different quadrature rules are similar with the CPU times in increasing order being Gauss-Legendre, Simpson's, Midpoint, and Trapezoidal rule. Because the Gauss-Legendre rule is expected to give higher accuracy for a given number of nodes than the other rules (see detail; Atkinson [15, 16]), we will use it in all further computations.

Table 1. Comparison of ARL and AD from Integral Equations with Monte Carlo simulations

α	Integral Equations				Simulations
	Midpoint	Trapezoidal	Simpson	Gauss-Legendre	
0	201.778 (36.34)[a]	201.804 (36.803)	201.786 (34.047)	201.743 (32.484)	201.247 ± 0.016
0.2	80.856 (36.094)	80.861 (36.867)	80.855 (35.766)	80.858 (32.906)	80.928 ± 0.017
0.4	44.437 (37.344)	44.438 (38.078)	44.435 (35.860)	44.437 (33.063)	44.428 ± 0.023
0.6	29.037 (38.063)	29.037 (38.297)	29.036 (35.344)	29.037 (33.984)	29.014 ± 0.013
0.8	20.774 (36.984)	20.774 (37.203)	20.773 (36.063)	20.774 (33.329)	20.757 ± 0.009
1.0	15.653 (37.125)	15.653 (37.609)	15.653 (35.376)	15.653 (33.562)	15.645 ± 0.006

[a] The numbers in parentheses are CPU times in seconds.

In Table 2, we compare the numerical results for ARL and AD obtained from the Integral Equation method in Equation (12) with results obtained from Monte Carlo simulations (10^5 sample trajectories). The results are shown in Table 2 for a value of $T = 500,$ for an in-control parameter value $\alpha = 0$ and for out-of-control parameter values α from 0.1 to 3. For the lognormal EWMA we obtain a pair of optimal parameter values of $\lambda = 0.01$ and $H = 1.774.$ The table shows that the outputs obtained by Integral Equation method are accurate compared with MC results.

Table 2. Comparison of ARL and AD from Integral Equations with Monte Carlo simulations

α	ARL and AD	
	Integral Equations	Monte Carlo simulations
0	500.322	499.431 ± 0.232[b]
0.1	292.745	292.952 ± 0.355
0.2	207.352	207.228 ± 0.195
0.3	160.636	159.785 ± 0.129
0.4	129.562	129.234 ± 0.096
0.5	107.649	107.510 ± 0.075
0.6	91.259	91.192 ± 0.063
0.7	78.284	78.195 ± 0.054
0.8	67.838	67.861 ± 0.047
0.9	59.241	59.243 ± 0.041
1.0	52.049	52.054 ± 0.037
2.0	17.151	17.176 ± 0.016
3.0	6.808	6.803 ± 0.008

[b] standard deviation.

We next describe a procedure for obtaining optimal designs for the lognormal EWMA chart. The criterion used for choosing optimal values for the EWMA weight parameter λ and alarm boundary parameter H is minimisation of AD for a given in-control parameter value $\alpha_0 = 0,$ $ARL = T$ and a given out-of-control parameter value $\alpha.$ We use the procedure to compute optimal $\left(\lambda^*, H^*\right)$ values for $ARL = 300$ and 500 and for out-of-control parameter values of 0.1, 0.3, 0.5, 0.7, 0.9. Plots of the optimal parameter values are shown in Figure 1 and a table of the values is given in Table 3.

The numerical procedure for obtaining optimal parameters for EWMA designs

1. Select an acceptable in-control value of ARL and decide on the change parameter value α for an out-of-control state.

2. For given α and $T,$ find graphically optimal parameter values of λ^* and H^* to minimise the AD values given by Equation (12) subject to the constraint that $ARL = T$ in Equation (12).

In Table 3 we present AD as a function of λ for $ARL = 300$ and $500.$ For example, given $ARL = 300$ and $\alpha = 0.5,$ calculations with formula (12) give $AD^* = 38.395$ for the optimal set of parameters $\lambda^* = 0.08$ and $H^* = 2.8459.$ This optimal AD value agrees with the value of $AD^* = 38.419$ that we computed from MC for the same parameter values. Plots of the optimal parameter values are shown in Figure 1.

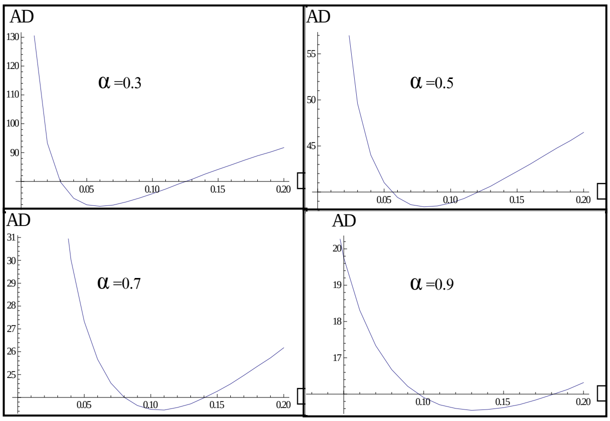
Table 3. Optimal parameters and AD of one-sided lognormal EWMA by Integral Equations.

ARL	α	λ^*	H^*	AD^*	AD by simulations
ARL=300	0.1	0.04	2.2520	167.318	167.249 [0.040][c]
	0.3	0.06	2.5580	71.392	71.594 [0.016]
	0.5	0.08	2.8459	38.395	38.419 [0.008]
	0.7	0.11	3.2600	23.450	23.402 [0.005]
	0.9	0.13	3.5308	15.554	15.562 [0.003]

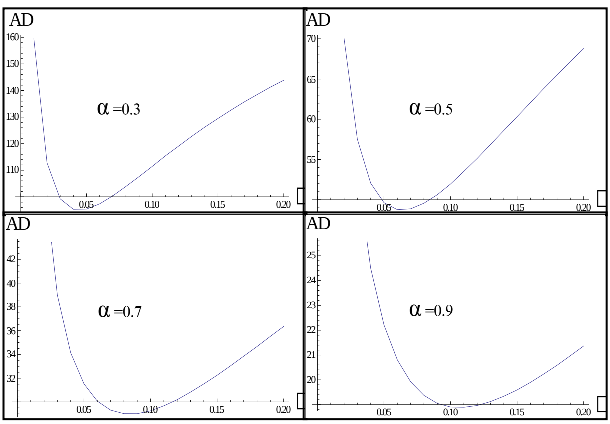
ARL=500	0.1	0.03	2.2352	247.897	247.16 [0.058]
	0.3	0.04	2.4220	95.311	95.858 [0.019]
	0.5	0.06	2.7719	48.774	48.859 [0.009]
	0.7	0.09	3.2703	29.018	28.941 [0.006]
	0.9	0.11	3.5957	18.895	18.873 [0.004]

[c] standard deviation.

Figure1. Curves of AD for optimal lognormal EWMA designs



a. ARL = 300



b. ARL = 500

6. Conclusion

We have used the Integral Equation method to calculate the ARL and AD for the EWMA chart for the lognormal distribution. This method is based on a numerical solution of Integral Equations based on Gauss-Legendre integration rules to approximate the integrals. The numerical results for this distribution have been found to be in good agreement with results obtained from the Monte Carlo simulation (MC). The values for ARL and AD were obtained with CPU times approximately 30 seconds whereas the MC values required very time consuming. In addition, we have found optimal designs for EWMA charts for fixed values of ARL and given parameter change α which this technique can be used with arbitrary values of ARL, in-control parameter (α_0), out-of-control parameter (α), and initial values (X_0).

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