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Effect of Preliminary Unit Root Tests on Predictors for an Unknown Mean Gaussian AR(1) Process

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Abstract

It is known from Diebold and Kilian [1] and Niwitpong [2] that a preliminary Dickey-Fuller unit root test [3] is useful for a one-step-ahead forecast of the trend of an AR(1) process and an unknown mean Gaussian AR(1) process respectively. In this paper, the more powerful preliminary unit root tests, based on the weighted symmetric estimator described by Fuller [4], Pantula et al. [6] and Shin and So [5], are compared with Dickey-Fuller unit root test. Monte Carlo simulation results are given to compare the relative efficiencies of predictors using the scaled prediction mean squares error for an unknown mean AR(1) process after preliminary unit root tests.

Keywords: AR(1), preliminary unit root test, scaled prediction mean square error.

1. Introduction

Applications in econometrics of an unknown mean Gaussian AR(1) process have been described by Hamilton [6]. In addition, Hamilton described the need to use the unit root test to find the correct model for the series of the nominal interest rate of the United States from 1947-1949 and the real GNP for the United States from 1947-1989, see Figures 17.2-17.3 of Hamilton ([6], pp. 503). Hamilton described that there is no guarantee in economic theory suggesting that the nominal interest rate series should be a deterministic time trend model, although Figure 17.2 shows an upward trend over the sample data. The model for these data might be a random walk without trend or a stationary process model with a constant term. Therefore, the interesting question arises whether a one-step-ahead prediction of the true model of the nominal interest rate series should be computed from a random walk model or a stationary model. To answer this question, the unit root testing of Dickey and Fuller [3] will be used to choose between these models. We further construct a one-step-ahead prediction for an unknown mean Gaussian AR(1) process when ρ is close to one from the result of unit root test.

Diebold and Kilian [1] and Niwitpong [2] used a simple Dickey-Fuller unit root test as a preliminary test to improve the forecast from the trend of the Gaussian AR(1) process and an unknown mean Gaussian AR(1) process, when the autoregressive parameter is near one. These authors used prediction mean squares error (PMSE) and scaled PMSE respectively, as the criteria to assess a forecast from an AR(1) process. Diebold and Kilian also pointed out that the more powerful unit root tests might help to improve a forecast from a near non-stationary process. It is, therefore, of interest to use preliminary unit root tests that are more powerful than the Dickey-Fuller unit root test when computing a one-step-ahead forecast from an unknown mean AR(1) process. In this paper we consider the weighted symmetric unit root tests which are more powerful tests than Dickey-Fuller tests, see e.g. Pantula et al. [7] and Shin and So [5] for the preliminary unit root tests compared to Dickey-Fuller unit root tests. As in Niwitpong [2], we use the scaled PMSE as a criterion to assess a one-step-ahead forecast from an AR(1) process. This scaled PMSE was proved by Niwitpong [2] to be functionally independent of (μ, σ). This important result allows us to set $(\mu, \sigma) = (0, 1)$ in Monte Carlo simulation and the results are valid for all values of parameters (μ, σ). This leads to a great reduction in computational effort. In this paper we also report the relative efficiencies of the scaled PMSE and scaled PMSE after preliminary unit root tests as in Niwitpong [2].

In section 2, preliminary unit root tests are reviewed. Section 3 describes the scaled PMSE after the preliminary unit root test. For the estimators of $(\hat{\rho}, \hat{\mu})$, we prove that the scaled PMSE after the preliminary unit root test is functionally independent of (μ, σ). This important result implies that this scaled PMSE is a function of ρ. This quantity is evaluated by Monte Carlo simulations that are reported in Section 4. The conclusion is presented in Section 5.

2. Review of Unit Root Tests

The unknown mean Gaussian AR(1) process Y_t satisfies

$$Y_t - \mu = \rho(Y_{t-1} - \mu) + e_t \qquad (1)$$

where $\rho \in (-1, 1)$ and the $e_t, t = 2, 3, ..., T$ are independent and identically $\mathrm{N}(0, \sigma^2)$ distributed.

Niwitpong [2] considered the hypothesis testing case 2 of Hamilton ([6], pp. 490 - 495). The null hypothesis H_0 and the alternative hypothesis H_a are as follows, $H_0 : \rho = 1$ and $H_a : \rho < 1.$ Dickey and Fuller [3] proposed two unit root tests based on the ordinary least squares (OLS) estimator for the null hypothesis H_0. The unit root tests based on the OLS estimator, denoted by $\hat{\rho}_0$, are

$$\hat{\kappa}_0 = \mathrm{T}\left(\hat{\rho}_0 - 1\right) \text{ and } \hat{\tau}_0 = \left(\mathrm{se}\left(\hat{\rho}_0\right)\right)^{-1}\left(\hat{\rho}_0 - 1\right) = \left(\hat{\rho}_0 - 1\right)\left[\sum_{t=2}^{T}\left(\mathrm{Y}_{t-1} - \bar{Y}\right)^2\right]^{1/2}\sigma_0^{-1}$$

where

$$\hat{\rho}_0 = \frac{\sum_{t=2}^{T}\left(\mathrm{Y}_t - \bar{Y}\right)\left(Y_{t-1} - \bar{Y}\right)}{\sum_{t=2}^{T}\left(Y_{t-1} - \bar{Y}\right)^2}, \bar{Y} = T^{-1}\sum_{t=1}^{T} Y_t \text{ and}$$

$$\sigma_0^2 = (\mathrm{T}\text{-}2)^{-1}\sum_{t=2}^{T}\left(\mathrm{Y}_t - \bar{Y} - \hat{\rho}_0\left(Y_{t-1} - \bar{Y}\right)\right)^2$$

The weighted symmetric unit root tests described by Pantula et al. [7] and Shin and So [5] are

$$\hat{\kappa}_W = \mathrm{T}\left(\hat{\rho}_W - 1\right) \text{ and}$$

$$\hat{\tau}_W = \left(\mathrm{se}\left(\hat{\rho}_W\right)\right)^{-1}\left(\hat{\rho}_W - 1\right) = \left(\hat{\rho}_W - 1\right)\left[\sum_{t=3}^{T}\left(\mathrm{Y}_{t-1} - \bar{Y}\right)^2 + T^{-1}\sum_{t=1}^{T}\left(\mathrm{Y}_t - \bar{Y}\right)^2\right]^{1/2}\sigma_W^{-1}$$

where

$$\hat{\rho}_\mathrm{W} = \frac{\sum_{t=2}^{T}\left(\mathrm{Y}_t - \bar{Y}\right)\left(Y_{t-1} - \bar{Y}\right)}{\sum_{t=3}^{T}\left(Y_{t-1} - \bar{Y}\right)^2 + T^{-1}\sum_{t=1}^{T}(Y_t - \bar{Y})^2},$$

$$\sigma_W^2 = (\text{T-2})^{-1}\left[\sum_{t=2}^{T} W_t\left(Y_t - \bar{Y} - \hat{\rho}_W\left(Y_{t-1} - \bar{Y}\right)\right)^2 + \sum_{t=1}^{T-1}(1 - W_{t+1})\left(Y_t - \bar{Y} - \hat{\rho}_W\left(Y_{t+1} - \bar{Y}\right)\right)^2\right]$$

and $W_t = (t-1)/T$.

The quantiles of these statistics; $\hat{\kappa}_0, \hat{\tau}_0, \hat{\kappa}_W$ and $\hat{\tau}_W$ estimated by simulation, for $H_0 : \rho = 1$ are provided in Fuller [4]. The hypothesis $H_0 : \rho = 1$ is rejected if these test statistics are less than the corresponding critical values in Fuller [4]. Shin and So [5] and Pantula et al. [7] pointed out that the statistic $\hat{\tau}_W$ is shown to be the most powerful unit root tests. We will therefore use these statistics; $\hat{\kappa}_0, \hat{\tau}_0, \hat{\kappa}_W$ and $\hat{\tau}_W$ for prior analyzing the data, then we construct the prediction mean square error suited to these data based on the results of unit root tests.

3. The scaled prediction mean square error after a preliminary unit root test

In this section, we review the scaled PMSE and the scaled PMSE after preliminary unit root tests as noted by Niwitpong [2].

Suppose Y_t is an unknown mean AR(1) process specified by (1). Our aim for model (1), whether the hypothesis H_a is satisfied or not, is to predict Y_{T+1}, based on data $Y_1, Y_2, ..., Y_T$. If H_0 is satisfied then $Y_t = Y_{t-1} + e_t$ and the predictor of Y_{T+1}, based on data $Y_1, Y_2, ..., Y_T$ is Y_T. If H_a is satisfied then $Y_t - \mu = \rho(Y_{t-1} - \mu) + e_t$ and the predictor of Y_{T+1}, based on data $Y_1, Y_2, ..., Y_T$ is $\hat{\mu} + \hat{\rho}(Y_T - \hat{\mu})$. For the predictor Y_T of Y_{T+1}, the error is

$$\beta_0(Y_1, Y_2, ..., Y_T, Y_{T+1}) = e_{T+1} - (Y_T - \mu) + \rho(Y_T - \mu).$$

Also, for the predictor $\hat{\mu} + \hat{\rho}(Y_T - \hat{\mu})$ of Y_{T+1}, the error is

$$\beta_1(Y_1, Y_2, ..., Y_T, Y_{T+1}) = e_{T+1} + (\mu - \hat{\mu}) - \hat{\rho}(Y_T - \hat{\mu}) + \rho(Y_T - \mu).$$

Now consider a preliminary unit root test based on the statistic $\hat{\kappa}_0$.

We may write the sample space $\Omega = A_1 \cup A_2$ where A_1 and A_2 are the events that the hypothesis H_0 is accepted and rejected respectively.

Let

$$\phi(Y_1, Y_2, ..., Y_T, Y_{T+1}) = \beta_0(Y_1, Y_2, ..., Y_T, Y_{T+1}) : if\ \omega \in A_1$$

$$\phi(Y_1, Y_2, ..., Y_T, Y_{T+1}) = \beta_1(Y_1, Y_2, ..., Y_T, Y_{T+1}) : if\ \omega \in A_2.$$

Let $PMSE_P$ denote the PMSE after a preliminary unit root test. In other words,

$$PMSE_P = E(\phi(Y_1, Y_2\ , ... Y_T, Y_{T+1})^2)$$

The scaled $PMSE_P$ is defined to be $PMSE_P\ /\ \sigma^2$ .In other words, the scaled $PMSE_P$ is

$$E(\frac{1}{\sigma^2}\phi(Y_1, Y_2, ..., Y_T, Y_{T+1})^2).$$

The following argument shows that this quantity does not depend on (μ, σ^2).

$$\frac{PMSE_P}{\sigma^2} = E(\frac{1}{\sigma^2}\phi(Y_1, Y_2, ..., Y_T, Y_{T+1})^2)$$

$$= E\left(\frac{1}{\sigma^2}\phi(Y_1, Y_2, ..., Y_T, Y_{T+1})^2 (I(A_1, \omega) + I(A_2, \omega))\right)$$

$$= E\left(\frac{1}{\sigma^2}\beta_0(Y_1, Y_2, ... Y_T, Y_{T+1})^2 I(A_1, \omega)\right) + E\left(\frac{1}{\sigma^2}\beta_1(Y_1, Y_2, ... Y_T, Y_{T+1})^2 I(A_2, \omega)\right)$$

where

$$I(A_1, \omega) = 1:\ \text{if}\, \Lambda_i < c_i \ \text{ and 0 oherwise}$$

and

$$I(A_2, \omega) = 1:\ \text{if}\, \Lambda_i < c_i \ \text{ and 0 oherwise}$$

where $\Lambda_i, i = 1, 2, 3, 4$ are, respectively, the statistics $\hat{\tau}_0, \hat{\kappa}_0, \hat{\tau}_W$ and $\hat{\kappa}_W$ and $c_i, i = 1, 2, 3, 4$ is a corresponding critical values in Fuller [4]. Niwitpong [2] proved that, $\hat{\rho}_0$ is a function of $(X_1, X_2, ..., X_T)$ when we define $X_T = (Y_T - \mu)/\sigma, \upsilon = (\mu - \hat{\mu})/\sigma$ and $\eta_T = e_T\ /\ \sigma.$ For the estimator $\hat{\rho} = \hat{\rho}_W$, the weighted symmetric estimator of ρ , and the estimator $\hat{\mu} = \hat{\mu}_0$,the sample mean of $(Y_1, Y_2, Y_3, ..., Y_T)$, we may show that $\hat{\rho}_W$ is a function of $(X_1, X_2, ...X_T)$ and

$$\frac{1}{\sigma^2}\beta_0(Y_1, Y_2, ..., Y_T, Y_{T+1})^2 = \left(\frac{e_{T+1}}{\sigma} + \frac{(\rho-1)(Y_T-\mu)}{\sigma}\right)^2 = \left(\eta_{T+1} + (\rho-1)X_T\right)^2$$

and

$$\frac{1}{\sigma^2}\beta_1(Y_1, Y_2, ..., Y_T, Y_{T+1})^2 = \left(\frac{e_{T+1}}{\sigma} + \frac{(\mu-\hat{\mu}) - \hat{\rho}(Y_T-\hat{\mu}) + \rho(Y_T-\mu)}{\sigma}\right)^2$$

$$= \left(\frac{e_{T+1}}{\sigma} + \frac{(1-\hat{\rho})(\mu-\hat{\mu}) + (\rho-\hat{\rho})(Y_T-\mu)}{\sigma}\right)^2 = \left(\eta_{T+1} + (1-\hat{\rho})v + (\rho-\hat{\rho})X_T\right)^2$$

which are functions of $(X_1, X_2, ..., X_T, \eta_{T+1})$ and ρ. Thus,

$$\frac{1}{\sigma^2}\beta_0(Y_1, Y_2, ...Y_T Y_{T+1})^2 I(A_1, \omega) \; and \; \frac{1}{\sigma^2}\beta_1(Y_1, Y_2, ...Y_T Y_{T+1})^2 I(A_2, \omega)$$

are functions of $(X_1, X_2, ..., X_T, \eta_{T+1})$ and ρ. As pointed out in Niwitpong [2], the distribution of $(X_1, X_2, ..., X_T, \eta_{T+1})$ does not depend on (μ, σ^2). Hence

$$E\left(\frac{1}{\sigma^2}\beta_0(Y_1, Y_2, ...Y_T Y_{T+1})^2 I(A_1, \omega)\right) \quad and \quad E\left(\frac{1}{\sigma^2}\beta_1(Y_1, Y_2, ...Y_T Y_{T+1})^2 I(A_2, \omega)\right)$$

do not depend on (μ, σ^2) i.e. they are functions of ρ.

It is also straightforward to show that the statistics $\hat{\tau}_0, \hat{\kappa}_0, \hat{\tau}_W$ and $\hat{\kappa}_W$ have probability distributions which do not depend on (μ, σ^2).

4. Monte Carlo simulation estimation of scaled PMSE

The scaled PMSE and the scaled $PMSE_p$ are estimated using Monte Carlo simulation. As shown in the previous section, these quantities do not depend on (μ, σ^2), we therefore set $(\mu, \sigma^2) = (0, 1)$ in these simulations. Suppose that each Monte Carlo simulation consists of M independent runs. Let the observed values of $X_t, v \; and \; \hat{\rho}$ be denoted by $x_t^{(k)}, v^{(k)}$ and $\hat{\rho}^{(k)}$ respectively for the k th run. We estimate the scaled PMSE by (see, Niwitpong [2])

$$1+\frac{\sum_{k=1}^{M}\left((1-\hat{\rho}^{(k)})v^{(k)}+(\rho-\hat{\rho}^{(k)})x_T^{(k)}\right)^2}{M}.$$

From the previous section, the scaled $PMSE_p$ is equal to

$$E\left(\left(\eta_{T+1}+(\rho-1)X_T\right)^2 I(A_1,\omega)\right)+E\left(\left(\eta_{T+1}+(1-\hat{\rho})v+(\rho-\hat{\rho})X_T\right)^2 I(A_2,\omega)\right).$$

Niwitpong [5] has shown that this scaled $PMSE_p$ is estimated by

$$1+\sum_{k\in M_0}\left((\rho-1)x_T^{(k)}\right)^2\left(\frac{1}{M}\right)+\sum_{k\in M_a}\left((1-\hat{\rho}^{(k)})v^{(k)}+(\rho-\hat{\rho}^{(k)})x_T^{(k)}\right)^2\left(\frac{1}{M}\right)$$

where M_0 is the set of simulation runs for which H_0 fails to reject and M_a is the set of simulation runs for which H_a fails to rejects accepted. In a similar way to the Monte Carlo simulation estimator used in Niwitpong [2], this Monte Carlo simulation estimator includes some variance reduction.

The relative efficiency of the predictor based on the estimators $(\hat{\mu}_0,\hat{\rho}_W)$ using the scaled $PMSE_p$ compared to the predictor based on the estimators $(\hat{\mu}_0,\hat{\rho}_W)$ using the scaled PMSE is defined to be

$$\frac{scaled\ PMSE_p}{scaled\ PMSE}.$$

We chose $\rho=0.1,...,0.9,0.95,0.97,0.99$ and T= 25, 50, 100 and 250. All simulations were performed using programs written in S-PLUS, with $\mathrm{M}=1000$ at level significance $\alpha=0.05$. The relative efficiencies of the estimators $(\hat{\mu}_0,\hat{\rho}_W)$ using the scaled $PMSE_p$ compared to the estimators $(\hat{\mu}_0,\hat{\rho}_W)$ using the scaled PMSE are reported in Table 1. As can be seen from this table, we have gained the efficiency of the estimators $(\hat{\mu}_0,\hat{\rho}_W)$ using the scaled $PMSE_p$ when ρ approaches 1 and T= 25, 50 and 100 for all preliminary unit root tests considered here. Table 1 also shows that all preliminary unit root tests considered here perform similarly to improve the forecast from an AR(1) process when ρ approaches one. However, the preliminary t-type unit roots tests of Dickey-Fuller and the weighted symmetric estimator are slightly superior to other unit root tests.

Table 1: **The relative efficiency of the estimators** $(\hat{\mu}_0, \hat{\rho}_W)$ **using the scaled** $PMSE_p$ **compared to the estimators** $(\hat{\mu}_0, \hat{\rho}_W)$ **using the scaled PMSE in an AR(1) process, for** $\rho = 0.1, \ldots, 0.9, 0.95, 0.97, 0.99$, T= 25, 50, 100 and 250, $\mathrm{M} = 1000$ and $\alpha = 0.05$.

ρ	$T=25$				$T=50$			
	$\hat{\tau}_0$	$\hat{\kappa}_0$	$\hat{\tau}_W$	$\hat{\kappa}_W$	$\hat{\tau}_0$	$\hat{\kappa}_0$	$\hat{\tau}_W$	$\hat{\kappa}_W$
0.99	**0.8936**	**0.9247**	**0.9142**	**0.9917**	**0.9523**	**0.9573**	**0.9545**	**0.9618**
0.97	**0.9232**	**0.9294**	**0.9271**	**0.9270**	**0.9641**	**0.9670**	**0.9636**	**0.9704**
0.95	**0.9329**	**0.9422**	**0.9334**	**0.9517**	**0.9681**	**0.9780**	**0.9762**	**0.9837**
0.90	**0.9496**	**0.9656**	**0.9646**	**0.9654**	1.0036	1.0032	1.0194	1.0142
0.80	1.0295	1.0385	1.0330	1.0426	1.0682	1.0508	1.0511	1.0523
0.70	1.0937	1.0873	1.0878	1.0948	1.0694	1.0392	1.0386	1.0397
0.60	0.1310	1.0985	1.0967	1.1157	1.0317	1.0141	1.0131	1.0089
0.50	1.1517	1.1141	1.0933	1.0912	1.0045	1.0000	1.0050	1.0000
0.40	1.1457	1.0770	1.0711	1.0501	1.0000	1.0000	1.0000	1.0000
0.30	1.0797	1.0307	1.0284	1.0247	1.0000	1.0000	1.0000	1.0000
0.20	1.0491	1.0231	1.0375	1.0153	1.0000	1.0000	1.0000	1.0000
0.10	1.0447	1.0060	1.0167	1.0095	1.0000	1.0000	1.0000	1.0000
ρ	$T=100$				$T=250$			
	$\hat{\tau}_0$	$\hat{\kappa}_0$	$\hat{\tau}_W$	$\hat{\kappa}_W$	$\hat{\tau}_0$	$\hat{\kappa}_0$	$\hat{\tau}_W$	$\hat{\kappa}_W$
0.99	**0.9747**	**0.9794**	**0.9796**	**0.9917**	**0.9523**	**0.9573**	**0.9545**	**0.9618**
0.97	**0.9869**	**0.9945**	**0.9931**	**0.9933**	1.0077	1.0067	1.0000	1.0077
0.95	1.0057	1.0055	1.0060	1.0091	1.0149	1.0119	1.0000	1.0103
0.90	1.0313	1.0240	1.0257	1.0244	1.0045	1.0002	1.0000	1.0000
0.80	1.0206	1.0059	1.0085	1.0051	1.0000	1.0000	1.0000	1.0000
0.70	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.60	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.40	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.30	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

5. Discussion and Conclusion

The referee comments that all statistics and scaled PMSE considered in this paper are all function of $(X_1, X_2, ..., X_T)$, which is independent of the parameters, is sufficient for the results of Section 3. Therefore the simulation results of Section 4 are valid when $(\mu, \sigma^2) = (0, 1)$. We have shown that preliminary unit root tests improve the forecast from an AR(1) process when ρ approaches one. However, the preliminary t-type unit roots tests of Dickey-Fuller and the weighted symmetric estimator are slightly superior to other unit root tests. The numerical results, for small sample sizes, have shown that the scaled $PMSE_p$ should be chosen when there is strong prior information that ρ approaches one.

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