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Confidence Interval for Cp of Non-normal Distributions Using an Adjusted Confidence Interval for Standard Deviation

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Abstract

The capability index Cp is widely used to give a quick indication of process capability in a format that is easy to use and understand (Kotz and Lovelace, [8], pp. 40-41). In this paper a new confidence interval is derived for Cp. The method is based on an adjusted confidence interval for the standard deviation for non-normal data. A Monte Carlo simulation has been used to investigate the coverage probabilities of the new confidence interval for small samples of sizes 10, 25, 50 and 100 for data from a range of non-normal distributions. The simulations show that, for all non-normal distributions studied, the new confidence interval for Cp provides higher values for the coverage probabilities than the confidence interval from Kotz and Lovelace and that for most of the distributions the new coverage probabilities are greater than a nominal value of 0.95.

Keyword: confidence interval, coverage probability, process capability index.

1. Introduction

Statistical process control has been widely applied in many industries. One of the quality measurement tools used for improvement of quality and productivity is a process capability index (PCI) such as Cp. This index is defined by

$$Cp = \frac{USL - LSL}{6\sigma} , \qquad (1)$$

where USL and LSL are upper and lower specified limits of the process, and σ is the process deviation. The estimated capability index Cp is usually computed under the assumption of normal data. However, data from many industrial processes are not normally distributed (see e.g. Chang et al. [4], Chen and Pearn [5], Wu et al. [10], Ding [6], Bittanti [2]). A widely-used (1-α)100% confidence interval for the capability index Cp is constructed from the fact that

$$\frac{(n-1)\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-1)} \qquad (2)$$

where $\hat{\sigma}^2 = (n-1)^{-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. In other words, the distribution of the pivotal quantity, $\frac{(n-1)\hat{\sigma}^2}{\sigma^2}$, is the central chi-square distributed with $n-1$ degrees of freedom.

A (1-α)100% confidence interval for Cp from (1) and (2) is then

$$\left(\frac{USL-LSL}{6\hat{\sigma}}.\sqrt{\frac{\chi^2_{\alpha/2,n-1}}{(n-1)}}, \quad \frac{USL-LSL}{6\hat{\sigma}}.\sqrt{\frac{\chi^2_{1-\alpha/2,n-1}}{(n-1)}}\right). \qquad (3)$$

This index, Cp, is useful for giving a quick indication of process capability in a format that is easy to use and understand (see, e.g., Kotz and Lovelace [8], pp. 40-41).

For data that is normally distributed, the coverage probability of the confidence interval in (3) is close to a nominal value of 0.95. However, for non-normal data, the coverage probability of this confidence interval can be appreciably below 0.95. Balamurali and Kalyanasundaram [1] found that for non-normal data the bootstrap method can help to improve the coverage probability of the confidence interval.

Our aim in this paper is to construct a confidence interval for Cp for non-normal distributions that is not based on the bootstrap method. Instead, we obtain a modified confidence interval for Cp by using a modified confidence interval for σ proposed by Niwitpong and Kirdwichai [9]. They found that for non-normal distributions the coverage probability of their proposed confidence interval for σ was closer to the nominal value of 0.95 than the coverage probability of the confidence interval for σ proposed by Bonett [3].

Section 2 reviews a method for constructing a confidence interval for Cp from the confidence interval for standard deviation. The proposed confidence interval for Cp is then obtained in section 3. Numerical results obtained from Monte Carlo simulations for a range of non-normal distributions are shown in section 4. A discussion of the results and conclusions are presented in section 5.

2. A Confidence Interval for Cp

Suppose $X_i \sim N\left(\mu, \sigma^2\right), i = 1,2,\ldots,n.$ A well-known $(1-\alpha)100\%$ confidence interval for $\sigma,$ using a pivotal quantity (2), is (Bonett, [3])

$$(n-1)s^2 / \chi^2_{1-\alpha/2} < \sigma^2 < (n-1)s^2 / \chi^2_{\alpha/2}\,, \tag{4}$$

where $s^2 = (n-1)^{-1}\sum_{i=1}^{n}(X_i - \bar{X})^2,$ and $\chi^2_{\alpha/2}$ and $\chi^2_{1-\alpha/2}$ are the (α/2)100th and (1-α/2)100th percentiles of the central chi-square distribution with n-1 degrees of freedom. From (4), we have

$$\Pr\left(\frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}}\right) = 1-\alpha$$

$$1-\alpha = \Pr\left(\frac{\chi^2_{\alpha/2,n-1}}{(n-1)s^2} < \frac{1}{\sigma^2} < \frac{\chi^2_{1-\alpha/2,n-1}}{(n-1)s^2}\right)$$

$$= \Pr\left(\sqrt{\frac{\chi^2_{\alpha/2,n-1}}{(n-1)s^2}} < \frac{1}{\sigma} < \sqrt{\frac{\chi^2_{1-\alpha/2,n-1}}{(n-1)s^2}}\right)$$

.

$$= \Pr\left(\frac{(USL-LSL)}{6}\sqrt{\frac{\chi^2_{\alpha/2,n-1}}{(n-1)s^2}} < \frac{(USL-LSL)}{6}\frac{1}{\sigma} < \frac{(USL-LSL)}{6}\sqrt{\frac{\chi^2_{1-\alpha/2,n-1}}{(n-1)s^2}}\right)$$

$$= \Pr\left(\frac{(USL-LSL)}{6s}\sqrt{\frac{\chi^2_{\alpha/2,n-1}}{(n-1)}} < C_p < \frac{(USL-LSL)}{6s}\sqrt{\frac{\chi^2_{1-\alpha/2,n-1}}{(n-1)}}\right) \tag{5}$$

From (5), we obtain a $100(1-\alpha)\%$ confidence interval for *Cp* which is

$$\left(\frac{USL-LSL}{6s} \cdot \sqrt{\frac{\chi^2_{\alpha/2,n-1}}{(n-1)}}, \quad \frac{USL-LSL}{6s} \cdot \sqrt{\frac{\chi^2_{1-\alpha/2,n-1}}{(n-1)}} \right). \tag{6}$$

3. Proposed Confidence Interval for Cp

For the special case of small sample sizes and for non-normally distributed X_i, Bonett [3] has proposed using prior information to obtain an estimate for the kurtosis of the non-normal distribution. In this method, Bonett suggests first selecting one large sample of size $n_0 = 200$ or 500 to estimate the kurtosis. He then suggests using this prior estimate of the kurtosis to obtain a pooled estimator for the small sample sizes of size n = 10, 25, 50, 100 selected in production runs. He then proposes an approximate confidence interval for σ^2 as

$$\text{Exp}\{\ln \text{cS}^2 - Z_{\alpha/2} Se\} < \sigma^2 < \text{Exp}\{\ln \text{cS}^2 + Z_{\alpha/2} Se\}, \tag{7}$$

where Exp is the exponential function and $Se = c[\{\hat{\gamma}_4^*(n-3)/n\}/(n-1)]^{1/2}$ is a small sample adjustment. In (7), *n* is the random sample size, n_0 is the large prior sample size (200 or 500), $\text{c} = \text{n}/(\text{n-} \text{Z}_{\alpha/2})$ and $\hat{\gamma}_4^* = (n_0\tilde{\gamma}_4 + n\bar{\gamma}_4)/(n_0 + n)$, where $\bar{\gamma}_4$ is an estimate of the 4th moment γ_4 defined by

$$\bar{\gamma}_4 = n\sum (X_i - m)^4 / (\sum (X_i - \bar{X})^2)^2 . \tag{8}$$

Here, $\bar{X}$ is the mean and m is a trimmed mean. On combining $\bar{\gamma}_4$ with $\tilde{\gamma}_4$ a pooled estimate of γ_4 is obtained. If $\tilde{\gamma}_4$ is not known then Bonett uses $\hat{\gamma}_4^* = \tilde{\gamma}_4$. Bonett's formula (7) works well for moderately non-normal distributions such as Uniform, Laplace, t(5), Gamma(1,6), Exponential and $\chi^2(1)$.

Niwitpong and Kirdwichai [9] proposed replacing $Z_{\alpha/2}$ in (7) with $t_{\alpha/2}$, the $\alpha/2$ quantile of the T distribution and they obtained a confidence interval for σ^2 with a coverage probability close to a nominal value of 0.95. This confidence interval is given by

$$\text{Exp}\{\ln \text{cS}^2 - t_{\alpha/2} Se^*\} < \sigma^2 < \text{Exp}\{\ln \text{cS}^2 + t_{\alpha/2} Se^*\} . \tag{9}$$

They also proposed using the median, *med,* instead of the trimmed mean *m* as an estimate for the sample mean $\hat{\mu}$ and obtained the estimate for $\overline{\gamma_4}$ and Se in (7) as:

$$\overline{\gamma^*}_4 = n\sum_{i=1}^{n}(X_i - med)^4/(\sum_{i=1}^{n}(X_i - \bar{X})^2)^2 , \qquad (10)$$

$$Se^* = c[\{\hat{\gamma}_4^*(n-3)/n\}/(n-1)]^{1/2} ,$$

and $\hat{\gamma}_4^* = (n_0\tilde{\gamma}_4 + n\overline{\gamma^*}_4)/(n_0 + n)$.

From Equation (9), we can then obtain a confidence interval for Cp as follows:

$$\Pr\left(\text{Exp}\{\ln \text{cS}^2 - t_{\alpha/2}Se^*\} \le \sigma^2 \le \text{Exp}\{\ln \text{cS}^2 + t_{\alpha/2}Se^*\}\right) = 1-\alpha$$

$$1-\alpha = \Pr\left(\frac{1}{\text{Exp}\{\ln \text{cS}^2 + t_{\alpha/2}Se^*\}} < \frac{1}{\sigma^2} < \frac{1}{\text{Exp}\{\ln \text{cS}^2 - t_{\alpha/2}Se^*\}}\right)$$

$$= \Pr\left(\sqrt{\frac{1}{\text{Exp}\{\ln \text{cS}^2 + t_{\alpha/2}Se^*\}}} < \frac{1}{\sigma} < \sqrt{\frac{1}{\text{Exp}\{\ln \text{cS}^2 - t_{\alpha/2}Se^*\}}}\right)$$

$$= \Pr\left(\frac{(USL-LSL)}{6}\sqrt{\frac{1}{\text{Exp}\{\ln \text{cS}^2 + t_{\alpha/2}Se^*\}}} < \frac{USL-LSL}{6\sigma} < \frac{(USL-LSL)}{6}\sqrt{\frac{1}{\text{Exp}\{\ln \text{cS}^2 - t_{\alpha/2}Se^*\}}}\right)$$

$$= \Pr\left(\frac{(USL-LSL)}{6}\sqrt{\frac{1}{\text{Exp}\{\ln \text{cS}^2 + t_{\alpha/2}Se^*\}}} < C_p < \frac{(USL-LSL)}{6}\sqrt{\frac{1}{\text{Exp}\{\ln \text{cS}^2 - t_{\alpha/2}Se^*\}}}\right). \qquad (11)$$

From (11), we obtain a new $(1-\alpha)100\%$ confidence interval for Cp which is

$$\left(\frac{USL-LSL}{6}\frac{1}{\sqrt{\text{Exp}\{\ln \text{cS}^2 + \text{t}_{\alpha/2}Se^*\}}}, \frac{USL-LSL}{6}\frac{1}{\sqrt{\text{Exp}\{\ln \text{cS}^2 - \text{t}_{\alpha/2}Se^*\}}}\right). \qquad (12)$$

4. Numerical Results

We have tested our proposed formulas for the confidence interval for Cp with extensive Monte Carlo simulation runs for a range of non-normal distributions. We used 10,000 Monte Carlo simulation runs for sample sizes n = 10, 25, 50 and 100 for each distribution with a significance level $\alpha = 0.05$ and Cp = 1.

The results are shown in Table 1. We found that the new confidence interval performs well for all of the distributions, t(5), $\chi^2(1)$, Exponential(2), Gamma(1,6), Lognormal(0,1), Beta(3,3) and Beta(1,10). From Table 1, we conclude that, for all distributions studied, the new confidence interval in (12) gives higher coverage probabilities than the basic confidence interval given in (6). Further, for most of the distributions and sample sizes studied the coverage probabilities from the new formula is greater than the nominal value of 0.95.

5. Discussion and Conclusions

We have proposed a new method for obtaining the confidence interval for capability index Cp for data from a process that is not normally distributed. This interval is based on the confidence interval for standard deviation obtained in Niwitpong and Kirdwichai [9] by adjusting the confidence interval of Bonett [3]. This new confidence interval for Cp provides a higher coverage probability than the existing confidence interval given by Kotz and Lovelace [8] when data are t(5), $\chi^2(1)$, Exp(2), Gamma(1,6) and Beta(1,10). In further research we plan to apply the method in this paper to construct a confidence interval for the well known capability index Cpk.

A referee has suggested that we compare the results of coverage probabilities of confidence interval (12) with the confidence intervals of Cp proposed by Heavlin [7] (see Kotz and Lovelace [8, pp. 41-42] for details of the Heavlin method) and with those obtained from the bootstrap method [1]. We have not yet carried out this comparison. We argue that our confidence interval for Cp gives high coverage probabilities and is simple and easy to use in practice. In contrast, the confidence interval for Cp, based on the computational approach, requires more time to compute the confidence interval and its expected length. However, in future research we intend to carry out a detailed comparison of the confidence interval for Cp (12) given in this paper with the confidence interval for Cp based on the method proposed by Heavlin [7].

Table 1. Coverage Probabilities of confidence intervals (6) and (12).

Distribution	N	Kotz and Lovelace (Equ. (5))	New formula (Equ. (11))
N(0,1)	10	0.9519	0.9987
	25	0.9518	0.9865
	50	0.9522	0.9838
	100	0.9525	0.9846
t(5)	10	0.8765	0.9780
	25	0.8329	0.9548
	50	0.8097	0.9500
	100	0.7856	0.9502
$\chi^2(1)$	10	0.6392	0.9006
	25	0.5921	0.9152
	50	0.5673	0.9341
	100	0.5613	0.9467
Exp(2)	10	0.769	0.9429
	25	0.7226	0.9369
	50	0.6998	0.9501
	100	0.6785	0.9589
Gamma(1,6)	10	0.7679	0.9405
	25	0.7224	0.9378
	50	0.707	0.9527
	100	0.6868	0.9608
Lognormal(0,1)	10	0.9312	0.9871
	25	0.9317	0.9835
	50	0.9216	0.9876
	100	0.9264	0.9814
Beta(3,3)	10	0.9779	0.9936
	25	0.9823	0.9925
	50	0.9806	0.9906
	100	0.9845	0.9936
Beta(1,10)	10	0.8255	0.9587
	25	0.8108	0.9585
	50	0.8077	0.9682
	100	0.7925	0.9752

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