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Parametric Estimation for the Birnbaum-saunders Lifetime Distribution Based on a New Parametrization

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Abstract

In this article, we consider a new parametrization of the two-parameter Birnbaum-Saunders lifetime distribution. Importantly, this re-parametrization fits the physics of studying phenomena since the proposed parameters characterize or specify the thickness of the sample and the nominal treatment loading on the sample, respectively. The usual shape and scale parameters of the distribution do not offer this physical interpretation. Instead of substitution method of the parameter estimators of the original Birnbaum-Saunders model into the new model, the statistical properties of the direct application of the standard methods of point estimation to the new parameters are investigated. In an effort to appraise the performance of proposed estimators in a practical setting, Monte-Carlo simulations are conducted for small, moderate and large sample sizes. Two real life examples based on published data are used to illustrate the suggested estimation methods. Some concluding remarks and areas for further research are also presented.

Keywords:

1. Introduction

The two-parameter Birnbaum-Saunders distribution was introduced by Birnbaum and Saunders [2] as a failure time distribution for fatigue failure caused under cyclic loading. This distribution is widely used as a lifetime distribution in the various models of reliability theory in the case when a failure of the object under consideration appears to be due to the development of fatigue cracks. Desmond [10,11] provided a more general derivation based on a biological model and strengthened the physical justification for the use of this distribution. This derivation follows from considerations of renewal theory for the number of cycles needed to force a fatigue crack extension to exceed a critical value. Birnbaum and Saunders [3] presented a comprehensive review, both theoretical and practical, of the fitting of this family of distributions to the solution of the problem of crack development. Desmond [11] considered estimation of the parameters for censored data. Ahmad [1] considered the estimation of the scale parameter (which overestimates the median life) by the jackknife method to eliminate first-order bias. This estimate has the same limiting behavior as that of Birnbaum and Saunders [3]. Rieck [18] derived asymptotically optimal linear estimator for symmetrically type II censored samples. We refer to the monograph by Bogdanoff and Kozin [4] for motivating examples of probabilistic models of cumulative damage. A more recent view on the problem of fatigue crack damages based on stochastic differential equations is suggested by Singpurwalla [20]. Some recent work on Birnbaum-Saunders distribution can be found in Chang and Tang [5,6], Dupis and Mills [12], Rieck [19], and a review of these developments can be found in Johnson *et al.*[14].

A continuous random variable X has a Birnbaum-Saunders distribution if X has the following cumulative distribution function

$$F_X(x;\alpha,\beta)=1-\Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{\beta}{x}}-\sqrt{\frac{x}{\beta}}\right)\right],\quad x>0,\alpha>0,\beta>0, \tag{1}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. The parameters α and β are the shape and scale parameters, respectively. The probability density function (pdf) is a mixture (with equal weights) of the inverse Gaussian pdf and reciprocal inverse Gaussian pdf. It is well know that the pdf of Birnbaum-Saunders is unimodal and although the hazard is not an increasing function of x but the average hazard rate is nearly a non-decreasing function of x.

The maximum likelihood estimators (MLE) were first discussed by Birnbaum and Saunders [3] and suggested some iterative schemes to solve the required non-linear equation. Engelehardt *et al.* [13] established the asymptotic distribution of the MLE. Further, the conventional moment estimators also have a difficulty in that they may not always exist and even if this the case, they may not be unique. Ng *et al.* [16] considered the modified moment estimators for the parameters to overcome this problem. However, Wu and Wong [21] reported that those expressions for the intervals of estimators for β suggested by Ng et al.[16] are presented incorrectly. Furthermore, there is no guarantee that the upper bounds of those intervals are always positive.

Our contribution in this article suggests a new parametrization of Birnbaum-Saunders distribution and develops the estimation scheme for the new parameters which are meaningful in a practical setting. Based on the reviewed literature, this kind of study is not available for the model under consideration.

1.1. A New Parametrization

Birnbaum and Saunders [2] considered a probability model of a fatigue crack development under cyclic loading in the framework of renewal theory. However, a more general model of such phenomena can be described by recurrence equations that have a similar form to ones that produce the lognormal distribution. We refer to Cramer [8], Parzen [17] and Desmond [11] for a description of the recurrence equations methods in connection with the lognormal distribution. Indeed, the later approach gives a richer picture of the physical phenomena of fatigue cracks. Moreover, the distribution of the size of a crack by a fixed moment of time can be arbitrary, while in the Birnbaum-Saunders method it has to be normal. The use of recurrence relations is given in Birnbaum and Saunders [2]. Interestingly, these recurrence relations lead to a new parametrization of the model. For example, on a metallic sample, which has the form of a rectangular plate of thickness h and is fixed on two sides, suppose there is a cyclic loading, which results in the development of a crack. Let X_k be the size of the crack at time $k = 1, 2, \ldots,$ that is, after the k th loading cycle. The following recurrence equations are derived [11]:

$$X_{k+1} = X_k + Y_k g(X_k), k = 0, 1, \ldots, \quad Y_0 = X_0 = 0, g(0) \neq 0.$$

These equations connect the crack size in previous and next moments of time by a positive continuous function g and a sequence of random variables $Y_1, Y_2, \ldots,$ that take care of variations in values of loading and some other physical factors that influence the development of the crack. Assume that the random variables Y_k are nonnegative, independent and identically distributed. Further, we assume the existence of the second moment of Y_k. We are interested in the time $U(=1,2,...),$ at which the crack achieves the critical value h. By the recurrence equations we obtain

$$\sum_{k=0}^{U-1} Y_k = \sum_{k=0}^{U-1} \frac{X_{k+1} - X_k}{g(X_k)} \approx \int_0^{X_U} \frac{dx}{g(x)}.$$

Here we assume, of course, that the increments $X_{k+1} - X_k$ are sufficiently small.

For each sufficiently large value t of the variable $U,$ the random variable on the left hand side of the equality can be approximated by a normal distribution with mean tm and variance $t\sigma^2,$ where $m = E(Y_1), \sigma^2 = Var(Y_1).$ Hence, for large $t,$

$$P(U > t) = P\left(\int_0^{X_t} \frac{dx}{g(x)} < \int_0^{h} \frac{dx}{g(x)}\right) \approx P\left(\sum_{k=0}^{U-1} Y_k < a(h)\right)$$

$$\approx \Phi\left(\frac{a(h) - mt}{\sigma\sqrt{t}}\right),$$

where

$$a(h) = \int_0^h \frac{dx}{g(x)}$$

is a strictly increasing function of the upper limit $h(>0),$ since $g(x) \geq 0.$

We obtain a natural re-parametrization by letting

$$\lambda = a(h)/\sigma, \; \mu = m/\sigma.$$

Importantly, this re-parameterization fits the physics of studying phenomena since the proposed parameters λ and μ correspond to the thickness of the sample and nominal treatment loading on the sample, respectively.

Thus, in terms of mu and λ the cumulative distribution function of U is given by

$$F_{\mu,\lambda}(x) = 1 - \Phi\left(\frac{\lambda}{\sqrt{x}} - \mu\sqrt{x}\right), \quad x > 0, \lambda > 0, \mu > 0. \tag{2}$$

Finally, we find the interrelations between the usual parameters α, β and new parameters μ and λ are as follows:

$$\begin{aligned} \mu &= \frac{1}{\alpha\sqrt{\beta}}, \quad & \alpha &= \frac{1}{\sqrt{\mu\lambda}} \\ \lambda &= \frac{\sqrt{\beta}}{\alpha}, \quad & \beta &= \frac{\lambda}{\mu}. \end{aligned} \tag{3}$$

One may attempt to estimate μ and λ by considering the above relations with the use of existing estimators of α and β. However, the relationship is highly nonlinear which may complicate the inference process. For this reason we direct develop estimation methods for the parameters of interest, that is, μ and λ.

1.2. Outline of the Paper

Our contribution in this article is to establish various estimation strategies for the new well-defined and meaningful parameters μ and λ for Birnbaum-Saunders distribution. The article is organized as follows. In Section 2 we derive maximum likelihood estimators for the new parametrization. Further, expression for asymptotic mean squared error (AMSE) of the proposed estimators are derived analytically and some computational aspects are discussed. In Section 3, we propose the method of moment estimation for the parameters of interest and derive the expression for the AMSE of the estimators. The regression-quantile estimation is discussed in Section 4. The strong consistency of the estimators are established. In Section 5, numerical values of the biases and mean squared errors (MSE) of the estimators are calculated by numerical methods and compared with each other. A detailed Bias and MSE analysis is also provided. Concluding remarks are offered in Section 6, and areas for further research are also discussed.

2. Maximum Likelihood Estimation

The probability density function of the Birnbaum-Saunders distribution after the re-parameterization is as follows:

$$f(x;\mu,\lambda)=\frac{1}{2\sqrt{2\pi}}\left(\frac{\lambda}{x\sqrt{x}}+\frac{\mu}{\sqrt{x}}\right)\exp\left\{-\frac{1}{2}\left(\frac{\lambda}{\sqrt{x}}-\mu\sqrt{x}\right)^2\right\},\quad x>0.$$

The observed likelihood function is

$$\begin{aligned}L(\mu,\lambda)&=\sum_{k=1}^{n}\ln f(X_k,\mu,\lambda)\\&\approx\sum_{k=1}^{n}\ln\left(\frac{\lambda}{X_k\sqrt{X_k}}+\frac{\mu}{\sqrt{X_k}}\right)-\frac{1}{2}\sum_{k=1}^{n}\left(\frac{\lambda}{\sqrt{X_k}}-\mu\sqrt{X_k}\right)^2.\end{aligned}$$

We obtain the system of maximum likelihood equations by evaluating derivatives with respect to μ and λ.

$$\frac{\partial L(\mu,\lambda)}{\partial\mu}=\sum_{k=1}^{n}\frac{X_k}{\lambda+\mu X_k}+\sum_{k=1}^{n}(\lambda-\mu X_k),$$

$$\frac{\partial L(\mu,\lambda)}{\partial\lambda}=\sum_{k=1}^{n}\frac{1}{\lambda+\mu X_k}+\sum_{k=1}^{n}\left(\frac{\lambda}{X_k}-\mu\right).$$

Hence the MLE $\hat{\mu}^{(MLE)}$ and $\hat{\lambda}^{(MLE)}$ of μ and λ can be obtained by simultaneously solving $\frac{\partial L(\mu,\lambda)}{\partial\mu}=0$ and $\frac{\partial L(\mu,\lambda)}{\partial\lambda}=0$, i.e.,

$$\mu=\frac{\lambda}{n}\sum_{k=1}^{n}\frac{1}{X_k}-\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\lambda+\mu X_k}\ (=f_1(\mu,\lambda)),$$

$$\lambda=\frac{\mu}{n}\sum_{k=1}^{n}X_k-\frac{1}{n}\sum_{k=1}^{n}\frac{X_k}{\lambda+\mu X_k}\ (=f_2(\mu,\lambda)).\qquad(4)$$

It seems to be natural to use certain iteration methods to obtain a solution of the above system of equations. However, for a parametric space $\mu > 0$ and $\lambda > 0$ there are no initial points that can ensure the convergence of the iteration process.

A necessary (but not sufficient) condition for convergence of the iteration process is the inequality $\|A(\mu,\lambda)\| < 1$, where the operator $A(\mu,\lambda) = (f_1, f_2)$. In the L_2-metric the norm of operator A is equal to the largest eigenvalue of the Jacobian matrix

$$G = G(\mu,\lambda) = \begin{pmatrix} \dfrac{\partial f_1}{\partial \mu} & \dfrac{\partial f_1}{\partial \lambda} \\ \dfrac{\partial f_2}{\partial \mu} & \dfrac{\partial f_2}{\partial \lambda} \end{pmatrix}.$$

The equation for eigenvalues is

$$\left(\frac{1}{n}\sum_{k=1}^{n}\frac{X_k}{(\lambda+\mu X_k)^2} - a^2\right)^2 - \frac{1}{n}\left[\left(\sum_{k=1}^{n}\frac{1}{X_k} + \sum_{k=1}^{n}\frac{1}{(\lambda+\mu X_k)^2}\right)\left(\sum_{k=1}^{n}X_k + \sum_{k=1}^{n}\frac{X_k^2}{(\lambda+\mu X_k)^2}\right)\right] = 0,$$

and its solutions are

$$a_1(\mu,\lambda) = \frac{1}{n}\sum_{k=1}^{n}\frac{X_k}{(\lambda+\mu X_k)^2} + \frac{1}{\sqrt{n}}\left(\sum_{k=1}^{n}\frac{1}{X_k} + \sum_{k=1}^{n}\frac{1}{(\lambda+\mu X_k)^2}\right)\left(\sum_{k=1}^{n}X_k + \sum_{k=1}^{n}\frac{X_k^2}{(\lambda+\mu X_k)^2}\right),$$

$$a_2(\mu,\lambda) = \frac{1}{n}\sum_{k=1}^{n}\frac{X_k}{(\lambda+\mu X_k)^2} - \frac{1}{\sqrt{n}}\left(\sum_{k=1}^{n}\frac{1}{X_k} + \sum_{k=1}^{n}\frac{1}{(\lambda+\mu X_k)^2}\right)\left(\sum_{k=1}^{n}X_k + \sum_{k=1}^{n}\frac{X_k^2}{(\lambda+\mu X_k)^2}\right).$$

Note that $a_1 > a_2$. So $A \equiv a_1(\mu,\lambda)$. It is safe to conclude that $a_1(\mu,\lambda) > 1$ for all $\mu > 0$ and $\lambda > 0$. Hence, most likely there do not exist the initial values of parameters that will ensure the convergence of the iteration process.

An alternative method will be discussed in Section 5 to obtain a solution to the maximum likelihood equations. In passing, we would like to remark here that the MLE of original parameters α and β suffer similar problems. Birnbaum-Saunders [3] proposed two iterative procedures to compute MLE of $\hat{\beta}$. However both procedures fail to work in the entire range of sample space.

3. Moment Estimation

To obtain the point estimators canonical parameters μ and λ by the method of moment, let

$$T_1 = \frac{1}{n}\sum_{k=1}^{n} X_k, \quad T_2 = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{X_k}, \quad T = T_1 T_2. \tag{5}$$

Then, by using the results of Birnbaum and Saunders [3], we get

$$E_1 = E(T_1) = \frac{\lambda\mu + 1/2}{\mu^2}, \quad E_2 = E(T_2) = \frac{\lambda\mu + 1/2}{\lambda^2},$$

$$Var(T_1) = \frac{\lambda\mu + 5/4}{n\mu^4}, Var(T_2) = \frac{\lambda\mu + 5/4}{n\lambda^4},$$

$$E(T) = \frac{1}{n} + \frac{n-1}{n}\left(\frac{\lambda\mu + 1/2}{\lambda\mu}\right)^2. \tag{6}$$

Now, by equating these expectations with their sample values, we find the moment method estimators (MME)

$$\hat{\mu}_n^{(MME)} = \sqrt{\frac{\sqrt{T}}{2T_1(\sqrt{T}-1)}}, \quad \hat{\lambda}_n^{(MME)} = \sqrt{\frac{T_1}{2\sqrt{T}(\sqrt{T}-1)}}.$$

The expressions for the asymptotic MSE of MME are given in the following theorem.

THEOREM. *As* $n \to \infty$*, the MSE of* $\hat{\mu}_n^{(MME)}$ *and* $\hat{\lambda}_n^{(MME)}$ *are given by*

$$MSE(\hat{\mu}_n^{(MME)}) = \frac{(\lambda\mu+5/4)\left(\mu^2(\lambda\mu+1)^2+\lambda^2\mu^4\right)}{n(2\lambda\mu+1)^2} - \frac{2\lambda\mu^3(\lambda\mu+1/4)(\lambda\mu+1)}{n(2\lambda\mu+1)^2} + O(n^{-2}), (7a)$$

$$MSE(\hat{\lambda}_n^{(MME)}) = \frac{(\lambda\mu+5/4)\left(\lambda^2(\lambda\mu+1)^2+\lambda^4\mu^2\right)}{n(2\lambda\mu+1)^2} - \frac{2\lambda^3\mu(\lambda\mu+1/4)(\lambda\mu+1)}{n(2\lambda\mu+1)^2} + O(n^{-2})(7b)$$

PROOF. Following the method outlined in [8] (p.352-358), we introduce functions $M(T_1,T_2) = \hat{\mu}_n$ and $L(T_1,T_2) = \hat{\lambda}_n$. Rewriting the functions M and L in the following canonical forms

$$M(x,y) = 2^{-1/2}x^{-1/4}y^{1/4}\left(x^{1/2}y^{1/2}-1\right)^{1/2},$$

$$L(x,y) = 2^{-1/2}x^{1/4}y^{-1/4}\left(x^{1/2}y^{1/2}-1\right)^{1/2}.$$

Since the statistics T_1 and T_2 will have all moments, the following asymptotic representation can be written

$$Var(\hat{\mu}_n^{(MME)}) = Var(T_1)\left(\frac{\partial M(E_1,E_2)}{\partial x}\right)^2 + 2cov(T_1,T_2)\frac{\partial M(E_1,E_2)}{\partial x}\frac{\partial M(E_1,E_2)}{\partial y} + Var(T_2)\left(\frac{\partial M(E_1,E_2)}{\partial y}\right)^2 + O(n^{-2}),$$

$$Var(\hat{\lambda}_n^{(MME)}) = Var(T_1)\left(\frac{\partial L(E_1,E_2)}{\partial x}\right)^2 + 2cov(T_1,T_2)\frac{\partial L(E_1,E_2)}{\partial x}\frac{\partial L(E_1,E_2)}{\partial y} + Var(T_2)\left(\frac{\partial L(E_1,E_2)}{\partial y}\right)^2 + O(n^{-2}).$$

Direct evaluation of derivatives gives

$$\frac{\partial M(E_1,E_2)}{\partial x} = \frac{\mu^3(\lambda\mu+1)}{2\lambda\mu+1}, \quad \frac{\partial M(E_1,E_2)}{\partial y} = \frac{\lambda^3\mu^2}{2\lambda\mu+1},$$

$$\frac{\partial L(E_1,E_2)}{\partial x} = \frac{\lambda^2\mu^3}{2\lambda\mu+1}, \quad \frac{\partial L(E_1,E_2)}{\partial y} = \frac{\lambda^3(\lambda\mu+1)}{2\lambda\mu+1}.$$

Finally, using relations in (6), we obtain the required asymptotic expansions which completes the proof.

On the other hand, one can construct the moment estimators of μ and λ directly by using moment estimator of α and β. Having said that, the difficulty is that the moment estimator for α and β may not always exist, and even if they do, the moment estimator may not be unique. It can be seen (by equating first and second populations with the sample moments) that if the sample coefficient of variation (CV) is greater than $\sqrt{5}$, then the moment estimators do not exist. If the sample CV is less than $\sqrt{5}$, then the moment estimator is tractable. In any event, the moment estimator of β may not be unique. Thus, the estimators of α and β may be constructed by using moment estimates of μ and λ accordingly. Ng et al. [16] suggested a modified version to deal with this problem. However, Wu and Wong [21] reported that those expressions for the intervals of estimators for β are presented incorrectly in [16]. Furthermore, there is no guarantee that the upper bounds of those intervals are always positive.

Again, the focus of this paper is on the estimation of new parameters based on a re-parametrization which fits the physics of studying phenomena as opposed to usual shape and scale parameters which do not provide the physical interpretation.

4. Regression-quantile Estimation

In this section, we propose the regression-quantile (least square) method which is based on the minimization of the quadratic measure of the difference between the empirical distribution function $F_n(x)$ and the theoretical cumulative distribution function

$$F(x) = 1 - \Phi(\lambda/\sqrt{x} - \mu\sqrt{x}).$$

If $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ are order statistics of $X_1, X_2, \ldots, X_n$, then by definition the empirical distribution function is given by $F_n(X_{(k)}) = k/n, k = 1, \ldots, n.$ Consider the following asymptotic equality

$$\Phi^{-1}\left(1-\frac{k}{n}\right) \approx \frac{\lambda}{\sqrt{X_{(k)}}} - \mu\sqrt{X_{(k)}}, \quad k = 1, \ldots, n-1,$$

which can be used for the parameter estimation. Hence, estimations of parameters are obtained by finding the minimum of the function

$$G(\lambda,\mu) = \sum_{k=1}^{n}\left(\frac{\lambda}{\sqrt{X_{(k)}}} - \mu\sqrt{X_{(k)}} - t_k\right)^2,$$

where $t_k = \Phi^{-1}(1-k/n)$ for $k = 1, \ldots, n-1.$ Since $\Phi^{-1}(0) = -\infty, t_n$ is chosen by the condition of further minimization of the function $G.$ It is interesting to note that the simulation study (Section 5) indicates that the optimal choice of t_n is close to $t_{n-1} - 1$ for nearly all $\mu,$ λ and $n.$

Rewriting the statistics T_1 and T_2 in (5) in the following form:

$$T_1 = \frac{1}{n}\sum_{k=1}^{n} X_{(k)}, \quad T_2 = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{X_{(k)}}.$$

Further,

$$T_3 = \frac{1}{n}\sum_{k=1}^{n} t_k\sqrt{X_{(k)}}, \quad T_4 = \frac{1}{n}\sum_{k=1}^{n}\frac{t_k}{\sqrt{X_{(k)}}}.$$

Hence, the regression-quantile estimators (RQE) of μ and λ, can be written respectively as

$$\tilde{\mu}_n = \frac{T_2T_3 - T_4}{1 - T_1T_2}, \qquad \tilde{\lambda}_n = \frac{T_3 - T_1T_4}{1 - T_1T_2}.$$

The consistency of the RQE is readily obtained from the application of the Glivenko-Cantelli theorem, which states that an empirical distribution function is a strongly (even uniformly) consistent estimator of a true distribution function. However, the evaluation of asymptotic MSE seems to be mathematically intractable.

5. Computation and simulation

In order to compare the performance of all above estimators, we performed a numerical study for different sample sizes and for different parameter values. Thus, in this section we will study the statistical properties of the proposed estimators by numerical methods. For the MLE and the MME we will numerically compute the asymptotic MSE of theses estimators using direct computations. Further, the simulated biases and MSE are also presented. On the other hand, the behavior of the RQE is investigated only via a simulation study by calculating bias and MSE.

5.1. Computational Study

As pointed out in Section 2, the iteration process may diverge for a solution of the system of maximum likelihood equations (cf. Section 2, (4)) in the region $\mu > 0, \lambda > 0$. For our numerical work, we chose a rectangle, which is divided into 100 congruent rectangles. Then we find a point $(\mu_., \lambda_.)$, for which the sum of squares of differences of the left hand and right hand sides of the equations (4) of the maximum likelihood system is obtained. The point $(\mu_., \lambda_.)$ is surrounded by a rectangle of smaller size, which is also divided into 100 parts and the process is repeated until the required accuracy 10^{-3} is achieved. Hence, numerical values of asymptotic MSE of MLE are computed.

On the other hand, the numerical values of asymptotic MSE of MME are computed using the relation (7) given in Theorem 1.

The results are reported in Tables 1-3 and Tables 4-6 for bias and MSE of the estimators, respectively.

5.2. Simulation Study

If Y has the standard normal distribution, then the root

$$X = \left(\frac{-Y + \sqrt{Y^2 + 4\mu\lambda}}{2\mu} \right)^2$$

of the equation $\lambda/\sqrt{X} = Y + \mu\sqrt{X}$ will have the Birnbaum-Saunders distribution with parameters μ and λ. Noting the second root $X = \left(\frac{-Y - \sqrt{Y^2 + 4\mu\lambda}}{2\mu} \right)^2$ of the equation will not produce the Birnbaum-Saunders distribution. First, we must generate a sample $Y_1, \ldots, Y_n$, of given size n from the standard normal distribution, and then a sample $X_1, \ldots, X_n$, is obtained from $Y_1, \ldots, Y_n$.

From $5{,}000$ simulated values of $(\hat{\mu}, \hat{\lambda})$, $(\hat{\mu}^{(MME)}, \hat{\lambda}^{(MME)})$, and $(\tilde{\mu}, \tilde{\lambda})$, we calculated the biases and the MSE of suggested estimators at selected values of μ and λ and for given sample size n. These simulated results are reported in Tables 1-9.

5.3. Bias Analysis

Note that all six estimators are consistent. Therefore, all proposed estimators are asymptotically unbiased. The simulated bias analysis is in agreement with the theoretical result since bias is a decreasing function of n. In other words, as sample size increases, the magnitude of the bias decreases and approaches to 0 as $n \to \infty$. Furthermore, we make the following observations from Tables 1-9.

• It is evident from Tables $1-3$ that the MLE has a systematic positive bias (under-estimation) when the true value of at least one parameter is sufficiently large and the bias is an increasing function of the value of the parameters. For example, for $n = 50$ and $\lambda = 5, \mu = 10$, the bias of λ is 0.003, while for $\lambda = 50, \mu = 50$ the bias of λ is 0.084. Having said that, the amount of the bias may be considered negligible overall.

• Tables $4-6$ reveal that the MME has a systematic negative bias (over-estimation). For fixed sample size n, the value of the absolute bias increases as the values the parameters increases. For example, for $n=50$ and $\lambda=5, \mu=10$, the bias of λ is -0.129, while for $\lambda=50, \mu=50$ the bias of λ is $-1.277.$ The magnitude of bias seems to be significant, and may have an adverse effect on the MSE behavior.

• Finally, it is seen from Tables 7-9 that the RQE has a positive bias for the parameter λ and a negative bias for μ when the values of the parameters are small. For example, for $n=50$ and $\lambda=0.5, \mu=0.5$, the bias of λ is 0.009 and the bias for μ is -0.047. For larger values of parameters, a systematic over-estimation is observed. Further, as the value of the parameters increase, the bias also increases. For $n=50$ and $\lambda=5, \mu=10$, the bias of λ is -0.117 and the bias for μ is $-0.318.$

Based on the results of the Monte Carlo simulation study, we observed that the MLEs and MMEs performance are very similar in terms of bias, however, MLE is less biased than that of MME in almost all instances. On the other hand, the RQE procedure gives more bias than the other two. It is possible to inspect the pattern of the bias functions of all the estimators and to suggest bias-reduced estimators. Alternatively, one can consider Jackknife estimation for the parameter of interest.

5.4. MSE Analysis

The the numerical values of the MSE of all the three estimators are reported in Tables 1-9. Clearly, MLE outperforms the other two estimators in the simulated parameter space. The simulation study also reveals that the MME performs better than the RQE for small samples. However, for large samples the performance of the RQE and the MME is similar.

Based on the results of our Monte Carlo simulation, we observed following interesting points:

• The MSE functions for all estimators are a decreasing function of n. In other words, the larger the sample size, the smaller is the MSE.

• On the other hand, it is seen from the Tables that in most of the cases the MSE increases as the values of the parameters increase.

• The asymptotic MSE and the simulated MSE of MLE and MME are comparable, except in a few instances.

• The numerical values of MSE are larger when the parameters are disproportionate, that is, one parameter is much bigger than the other. For example, for $n = 50$ and $\lambda = 0.5, \mu = 0.5$, the MSE of $\hat{\lambda}$ for MLE is 0.005, while for $\lambda = 50, \mu = 0.5$, the MSE of $\hat{\lambda}$ is $1.942.$

• The difference between the asymptotic MSE and the simulated MSE is small for proportional values of μ and λ.

• It is noted that the asymptotic MSE of the MME is much higher than the MSE of the MLE, particularly for large values of the parameters. For example, for $n = 50, \lambda = 50,$ and $\mu = 50$ the asymptotic MSE for the MLE of λ is $0.020,$ while for MME it is $25.005.$

• The simulated MSE of all the estimators MSE tend to zero as $n \to \infty.$

The practical application of the proposed estimators is illustrated in the following section.

6. Illustrative Examples

We consider two examples, one involving a small sample $(n = 10)$ and the other with a relatively large sample $(n = 101)$.

Example 1. The data is given by Birnbaum and Saunders [3] on the fatigue life of 6061-T6 aluminum coupons cut parallel to the direction of the roll and oscillated at 18 cycles/s (cps). The data set has 101 observations with maximum stress per cycle $31{,}000$ psi. For this example, the point estimates of μ and λ obtained by three methods are summarized in Table 10. Further, the point estimates for the parameters α and β are also calculated. Interestingly, the point estimates of α and β are comparable with those of Ng *et al.* [16] and Wu and Wong [21]. More importantly, estimators $\hat{\mu}$ and $\hat{\lambda}$ correspond to the thickness of the sample and nominal treatment loading on the sample, respectively. The estimators $\hat{\alpha}$ and $\hat{\beta}$ lack these characteristics.

Example 2. This example is taken from McCool [15] on the fatigue life in hours of 10 bearings of a certain type. The data are

152.7	172.0	172.5	173.5	193.0
204.7	216.5	234.9	262.6	422.6

The above data was used by Cohen *et al.* [7] to illustrate an example for the three parameter Weibull distribution.

For this example, the point estimates of μ and λ are obtained by three methods and summarized in Table 11. Further, using these values the point estimates for the parameters α and β are also calculated. Again, they are comparable with the point estimates of α and β reported in Ng *et al.* [16] and Wu and Wong [21].

7. Summary

A new parametrization of the two-parameter Birnbaum-Saunders lifetime distribution to fit the physics of studying phenomena is tackled. Importantly, the proposed parameters correspond to the thickness of the sample and the nominal treatment loading on the sample, respectively. These usual scale and shape parameters lack these characteristics. Three classical estimation schemes for suggested new parameters are presented and their statistical properties investigated and compared. An extensive sampling experiment is used to investigate the finite-sample performance of the suggested estimation strategy. The numerical study reveals that the performance of maximum likelihood estimators is relatively better than the other two proposed estimators. However, moment estimators may have a more desirable property such as ease of calculation.

In this article we discussed point estimation of new parameters and the question of interval estimation, and the test of hypothesis remains to be considered for future research. However, it is expected that in the asymptotic sequence, tests and confidence estimation procedures based on suggested methods will begin behaving correctly in terms of coverage and size. If a bias correction is applied and an appropriate distribution is used for establishing critical values, then target size and coverage probabilities, and reasonably good power of test, can be achieved for moderate sample sizes. We suggest a more computationally intensive nested bootstrap, which calculates critical values of the test statistic from its bootstrapped distribution rather than using tests on the critical value of the *student-t* distribution for a fruitful testing procedure. To calculate the asymptotically valid variances, covariances, and bias measures, one can use the balanced bootstrapping re-sampling methods. There are several techniques for generating confidence intervals available, for example the percentile methods and bias corrected method with acceleration. However, it is beyond the scope of this paper and will be dealt with in a separate communication.

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Appendix

Bias and MSE for Maximum Likelihood Estimators, $n = 10$

		Simul. Bias		MSE			
λ	μ	$\hat{\lambda}$	$\hat{\mu}$	Asympt.	Simul.	Asympt.	Simul.
				$\hat{\lambda}$	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\mu}$
0.5	0.5	-0.008	-0.010	0.023	0.002	0.023	0.002
0.5	1	-0.007	+0.007	0.019	0.002	0.076	0.010
0.5	5	-0.005	-0.002	0.008	0.002	0.765	0.232
0.5	10	+0.000	+0.012	0.004	0.000	1.736	0.011
0.5	50	+0.001	+0.022	0.000	0.000	9.708	0.248
1	0.5	+0.008	-0.008	0.076	0.001	0.019	0.002
1	1	-0.003	-0.004	0.056	0.009	0.056	0.009
1	5	-0.013	-0.025	0.017	0.007	0.434	0.197
1	10	+0.001	+0.013	0.009	0.000	0.930	0.011
1	50	+0.001	-0.006	0.002	0.000	4.926	0.239
5	0.5	-0.001	-0.006	0.766	0.229	0.008	0.002
5	1	-0.032	-0.011	0.434	0.197	0.017	0.007
5	5	-0.094	-0.090	0.097	0.192	0.097	0.188
5	10	+0.003	+0.009	0.049	0.003	0.197	0.011
5	50	+0.006	+0.043	0.010	0.002	0.997	0.228
10	0.5	+0.012	+0.000	1.736	0.011	0.004	0.000
10	1	+0.015	+0.000	0.930	0.011	0.009	0.000
10	5	+0.007	+0.004	0.197	0.011	0.049	0.003
10	10	+0.005	+0.006	0.099	0.010	0.099	0.010
10	50	+0.009	+0.046	0.020	0.007	0.499	0.206
50	0.5	-0.217	+0.001	9.708	0.243	0.001	0.000
50	1	+0.002	+0.001	4.926	0.242	0.002	0.000
50	5	+0.036	+0.004	0.997	0.230	0.010	0.002
50	10	+0.049	+0.012	0.499	0.204	0.020	0.007
50	50	+0.051	+0.054	0.100	0.194	0.100	0.189

Bias and MSE for Maximum Likelihood Estimators, $n = 50$

		Simul. bias		MSE			
λ	μ	$\hat{\lambda}$	$\hat{\mu}$	Asympt.	Simul.	Asympt.	Simul.
				$\hat{\lambda}$	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\mu}$
0.5	0.5	-0.003	-0.003	0.005	0.002	0.005	0.002
0.5	1	-0.003	+0.001	0.004	0.002	0.015	0.007
0.5	5	-0.003	-0.002	0.002	0.001	0.153	0.125
0.5	10	+0.000	+0.014	0.001	0.000	0.347	0.010
0.5	50	+0.000	+0.023	0.000	0.000	1.942	0.225
1	0.5	-0.000	-0.003	0.015	0.007	0.004	0.002
1	1	-0.004	-0.003	0.011	0.006	0.011	0.006
1	5	-0.006	-0.017	0.004	0.005	0.087	0.109
1	10	+0.001	+0.011	0.002	0.000	0.186	0.010
1	50	+0.001	+0.036	0.000	0.000	0.985	0.214
5	0.5	-0.012	-0.003	0.153	0.126	0.002	0.001
5	1	-0.025	-0.006	0.087	0.114	0.003	0.005
5	5	-0.044	-0.042	0.019	0.141	0.019	0.137
5	10	+0.003	+0.008	0.011	0.002	0.039	0.009
5	50	+0.005	+0.059	0.002	0.002	0.199	0.184
10	0.5	+0.018	+0.000	0.010	0.347	0.001	0.000
10	1	+0.015	+0.001	0.186	0.010	0.002	0.000
10	5	+0.006	+0.003	0.039	0.009	0.010	0.002
10	10	+0.005	+0.006	0.020	0.008	0.020	0.008
10	50	+0.010	+0.055	0.004	0.007	0.100	0.187
50	0.5	+0.018	+0.000	1.942	0.226	0.000	0.000
50	1	+0.015	+0.001	0.985	0.221	0.000	0.000
50	5	+0.034	+0.004	0.199	0.175	0.002	0.002
50	10	+0.054	+0.011	0.100	0.180	0.004	0.007
50	50	+0.084	+0.085	0.020	0.232	0.020	0.227

Bias and MSE for Maximum Likelihood Estimators, $n = 100$

		Simul. bias		MSE			
λ	μ	$\hat{\lambda}$	$\hat{\mu}$	Asympt.	Simul.	Asympt.	Simul.
				$\hat{\lambda}$	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\mu}$
0.5	0.5	-0.001	-0.001	0.002	0.001	0.002	0.001
0.5	1	-0.002	-0.000	0.002	0.001	0.008	0.005
0.5	5	-0.002	-0.006	0.001	0.001	0.077	0.092
0.5	10	+0.000	+0.017	0.000	0.000	0.174	0.010
0.5	50	+0.000	+0.055	0.000	0.000	0.971	0.022
1	0.5	-0.002	-0.001	0.008	0.005	0.002	0.001
1	1	-0.003	-0.002	0.006	0.005	0.006	0.005
1	5	-0.003	-0.006	0.002	0.003	0.043	0.083
1	10	+0.001	+0.014	0.001	0.000	0.093	0.009
1	50	+0.001	+0.034	0.000	0.000	0.493	0.191
5	0.5	-0.015	-0.002	0.076	0.091	0.001	0.001
5	1	-0.011	-0.002	0.043	0.083	0.002	0.003
5	5	-0.034	-0.032	0.010	0.108	0.010	0.110
5	10	+0.004	+0.010	0.005	0.002	0.020	0.008
5	50	+0.005	+0.054	0.001	0.002	0.100	0.184
10	0.5	+0.011	+0.001	0.174	0.975	0.000	0.000
10	1	+0.010	+0.001	0.093	0.009	0.001	0.000
10	5	+0.007	+0.004	0.020	0.008	0.005	0.002
10	10	+0.009	+0.007	0.010	0.007	0.010	0.008
10	50	+0.014	+0.075	0.002	0.007	0.050	0.189
50	0.5	+0.025	+0.001	0.971	0.210	0.000	0.000
50	1	+0.021	+0.001	0.493	0.191	0.000	0.000
50	5	+0.056	+0.050	0.010	0.175	0.001	0.002
50	10	+0.066	+0.014	0.050	0.187	0.002	0.007
50	50	+0.076	+0.078	0.010	0.235	0.010	0.240

Bias and MSE for Moment Method Estimators, $n = 10$

		Simul. bias		MSE			
λ	μ	$\hat{\lambda}$	$\hat{\mu}$	Asympt.	Simul.	Asympt.	Simul.
				$\hat{\lambda}$	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\mu}$
0.5	0.5	-0.107	-0.105	0.024	0.073	0.024	0.067
0.5	1	-0.945	-0.186	0.020	0.053	0.081	0.196
0.5	5	-0.079	-0.777	0.015	0.033	1.476	3.203
0.5	10	-0.076	-1.503	0.014	0.031	11.901	5.475
0.5	50	-0.073	-7.299	0.013	0.028	127.475	279.978
1	0.5	-0.189	-0.093	0.081	0.211	0.020	0.049
1	1	-0.172	-0.169	0.069	0.164	0.069	0.155
1	5	-0.152	-0.751	0.055	0.123	1.369	2.975
1	10	-0.149	-1.476	0.052	0.117	5.244	11.454
1	50	-0.146	-7.274	0.059	0.112	126.244	278.042
5	0.5	-0.788	-0.078	1.476	3.340	0.015	0.032
5	1	-0.760	-0.150	1.369	3.069	0.055	0.119
5	5	-0.734	-0.730	1.275	2.840	1.275	2.780
5	10	-0.730	-1.454	1.262	2.809	5.050	11.122
5	50	-0.727	-7.257	1.252	2.781	125.250	276.804
10	0.5	-1.519	-0.075	5.475	12.277	0.014	0.030
10	1	-1.488	-0.148	5.244	11.715	0.052	0.114
10	5	-1.460	-0.727	5.050	11.236	1.262	2.779
10	10	-1.457	-1.452	5.025	11.168	5.025	11.088
10	50	-1.453	-7.255	5.006	11.106	125.125	276.737
50	0.5	-7.339	-0.073	127.480	284.042	0.013	0.028
50	1	-7.303	-0.145	126.243	280.903	0.051	0.111
50	5	-7.269	-0.726	125.250	278.079	1.252	2.768
50	10	-7.265	-1.451	125.125	277.638	5.005	11.070
50	50	-7.259	-7.255	125.025	277.186	125.025	276.783

Bias and MSE for Moment Method Estimators, $n = 50$

		Simul. bias		MSE			
λ	μ	$\hat{\lambda}$	$\hat{\mu}$	Asympt.	Simul.	Asympt.	Simul.
				$\hat{\lambda}$	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\mu}$
0.5	0.5	-0.018	-0.016	0.005	0.006	0.005	0.006
0.5	1	-0.016	-0.030	0.004	0.005	0.016	0.019
0.5	5	-0.014	-0.134	0.003	0.004	0.295	0.348
0.5	10	-0.014	-0.261	0.003	0.003	1.095	1.299
0.5	50	-0.013	-1.278	0.003	0.003	25.495	30.685
1	0.5	-0.033	-0.015	0.016	0.021	0.004	0.005
1	1	-0.030	-0.028	0.014	0.017	0.014	0.016
1	5	-0.027	-0.131	0.011	0.014	0.274	0.325
1	10	-0.026	-0.258	0.011	0.013	1.049	1.254
1	50	-0.026	-1.276	0.010	0.012	25.250	30.491
5	0.5	-0.140	-0.013	0.295	0.367	0.003	0.003
5	1	-0.135	-0.261	0.274	0.339	0.011	0.013
5	5	-0.130	-0.128	0.255	0.313	0.255	0.307
5	10	-0.129	-0.255	0.252	0.309	1.010	1.220
5	50	-0.128	-1.275	0.250	0.306	25.050	30.386
10	0.5	-0.269	-0.013	1.095	1.356	0.003	0.003
10	1	-0.263	-0.026	1.049	1.294	0.011	0.013
10	5	-0.258	-0.128	1.010	1.238	0.253	0.305
10	10	-0.257	-0.255	1.005	1.229	1.005	1.217
10	50	-0.256	-1.275	1.001	1.221	25.025	30.387
50	0.5	-1.296	-0.013	25.495	31.324	0.003	0.003
50	1	-1.288	-0.026	25.249	30.945	0.010	0.012
50	5	-1.279	-0.127	25.049	30.589	0.249	0.304
50	10	-1.279	-0.255	25.025	30.531	1.001	1.215
50	50	-1.277	-1.276	25.005	30.468	25.005	30.404

Bias and MSE for Moment Method Estimators, $n = 100$

		Simul. bias		MSE			
λ	μ	$\hat{\lambda}$	$\hat{\mu}$	Asympt.	Simul.	Asympt.	Simul.
				$\hat{\lambda}$	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\mu}$
0.5	0.5	-0.007	-0.006	0.002	0.003	0.002	0.003
0.5	1	-0.007	-0.012	0.002	0.002	0.008	0.009
0.5	5	-0.005	-0.049	0.001	0.002	0.148	0.163
0.5	10	-0.005	-0.095	0.001	0.002	0.548	0.604
0.5	50	-0.005	-0.465	0.001	0.001	12.748	14.133
1	0.5	-0.013	-0.006	0.008	0.009	0.002	0.002
1	1	-0.012	-0.011	0.007	0.008	0.007	0.008
1	5	-0.010	-0.047	0.005	0.006	0.137	0.151
1	10	-0.010	-0.093	0.005	0.006	0.524	0.579
1	50	-0.010	-0.465	0.005	0.006	12.624	14.018
5	0.5	-0.055	-0.005	0.148	0.168	0.001	0.002
5	1	-0.052	-0.009	0.137	0.155	0.005	0.006
5	5	-0.049	-0.047	0.127	0.143	0.127	0.141
5	10	-0.048	-0.093	0.126	0.141	0.505	0.561
5	50	-0.047	-0.467	0.125	0.140	12.525	13.939
10	0.5	-0.103	-0.005	0.548	0.620	0.001	0.002
10	1	-0.100	-0.009	0.524	0.591	0.005	0.006
10	5	-0.096	-0.047	0.505	0.566	0.126	0.139
10	10	-0.095	-0.093	0.502	0.562	0.502	0.559
10	50	-0.094	-0.467	0.499	0.559	12.512	13.933
50	0.5	-0.485	-0.005	12.748	14.312	0.001	0.001
50	1	-0.479	-0.009	12.624	14.145	0.005	0.006
50	5	-0.473	-0.047	12.525	13.996	0.139	0.125
50	10	-0.472	-0.093	12.512	13.974	0.499	0.557
50	50	-0.470	-0.468	12.502	13.951	12.502	13.933

Bias and MSE for Regression-Quantile Estimators, $n = 10$

		Bias		MSE	
λ	μ	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\lambda}$	$\hat{\mu}$
0.5	0.5	+0.004	-0.200	0.048	0.115
0.5	1	-0.007	-0.319	0.038	0.296
0.5	5	-0.029	-1.009	0.029	3.885
0.5	10	-0.037	-1.750	0.028	13.599
0.5	50	-0.048	-7.102	0.027	298.306
1	0.5	-0.013	-0.159	0.152	0.074
1	1	-0.034	-0.257	0.130	0.210
1	5	-0.074	-0.875	0.111	3.400
1	10	-0.086	-1.572	0.109	12.619
1	50	-0.103	-6.741	0.110	291.790
5	0.5	-0.295	-0.101	2.880	0.039
5	1	-0.369	-0.175	2.768	0.136
5	5	-0.483	-0.710	2.726	2.983
5	10	-0.514	-1.348	2.738	11.672
5	50	-0.555	-6.264	2.764	284.279
10	0.5	-0.738	-0.088	11.074	0.034
10	1	-0.855	-0.157	10.925	0.126
10	5	-1.027	-0.674	10.954	2.918
10	10	-1.069	-1.296	10.997	11.497
10	50	-1.129	-6.151	11.095	282.945
50	0.5	-4.833	-0.071	272.615	0.030
50	1	-5.136	-0.135	273.840	0.117
50	5	-5.547	-0.626	276.407	2.843
50	10	-5.644	-1.230	277.385	11.318
50	50	-5.782	-6.009	278.701	281.186

Bias and MSE for Regression-Quantile Estimators, $n = 50$

		Bias		MSE	
λ	μ	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\lambda}$	$\hat{\mu}$
0.5	0.5	+0.009	-0.047	0.005	0.009
0.5	1	+0.004	-0.078	0.005	0.027
0.5	5	-0.006	-0.249	0.003	0.401
0.5	10	-0.007	-0.420	0.003	1.443
0.5	50	-0.011	-1.735	0.003	31.649
1	0.5	+0.008	-0.039	0.018	0.007
1	1	-0.002	-0.063	0.015	0.021
1	5	-0.016	-0.215	0.012	0.357
1	10	-0.018	-0.375	0.013	1.361
1	50	-0.024	-1.641	0.012	31.006
5	0.5	-0.055	-0.025	0.329	0.004
5	1	-0.080	-0.043	0.310	0.014
5	5	-0.114	-0.173	0.279	0.316
5	10	-0.117	-0.318	0.315	1.289
5	50	-0.133	-1.519	0.297	30.296
10	0.5	-0.160	-0.022	1.240	0.004
10	1	-0.195	-0.039	1.203	0.013
10	5	-0.233	-0.159	1.259	0.322
10	10	-0.256	-0.315	1.186	1.225
10	50	-0.271	-1.490	1.190	30.171
50	0.5	-1.136	-0.017	29.668	0.003
50	1	-1.219	-0.033	29.617	0.012
50	5	-1.278	-0.147	31.426	0.317
50	10	-1.357	-0.298	29.739	1.207
50	50	-1.393	-1.453	29.830	30.023

Bias and MSE for Regression-Quantile Estimators, $n = 100$

		Bias		MSE	
λ	μ	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\lambda}$	$\hat{\mu}$
0.5	0.5	+0.008	-0.024	0.003	0.003
0.5	1	+0.004	-0.038	0.002	0.011
0.5	5	-0.001	-0.115	0.002	0.167
0.5	10	-0.002	-0.195	0.001	0.603
0.5	50	-0.004	-0.749	0.001	13.642
1	0.5	+0.009	-0.019	0.009	0.003
1	1	+0.003	-0.030	0.008	0.009
1	5	-0.005	-0.097	0.006	0.151
1	10	-0.007	-0.170	0.005	0.568
1	50	-0.010	-0.700	0.005	13.441
5	0.5	-0.010	-0.011	0.166	0.002
5	1	-0.024	-0.019	0.154	0.006
5	5	-0.043	-0.075	0.133	0.136
5	10	-0.048	-0.140	0.132	0.538
5	50	-0.054	-0.636	0.131	13.254
10	0.5	-0.049	-0.010	0.614	0.002
10	1	-0.069	-0.017	0.545	0.006
10	5	-0.095	-0.070	0.528	0.134
10	10	-0.102	-0.133	0.526	0.533
10	50	-0.110	-0.621	0.526	13.224
50	0.5	-0.425	-0.007	14.343	0.001
50	1	-0.477	-0.014	13.200	0.005
50	5	-0.537	-0.064	13.148	0.133
50	10	-0.551	-0.124	13.149	0.529
50	50	-0.562	-0.593	14.132	14.152

Point estimates for Example 1.

Estimator	μ	λ	α	β
MLE	0.511188	67.3842	0.170385	131.81882
MME	0.511187	67.3843	0.170385	131.81927
RQE	0.243114	66.7805	0.171449	131.08943

Point estimates of for Example 2.

Estimator	μ	λ	α	β
MLE	0.243097	51.5486	0.282489	212.04951
MME	0.243114	51.5451	0.282489	212.02028
RQE	0.242734	48.0368	0.292852	197.89893

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