



Thailand Statistician  
January 2008; 6(1) : 75-90  
<http://statassoc.or.th>  
Contributed paper

## **A Case of Constant Returns to Scale in Fuzzy Stochastic Data Envelopment Analysis: Chance-Constrained Programming and Possibility Approach**

**Varathorn Punyangarm\*[a], Patcharaporn Yanpirat [b], Peerayuth Charnsethikul[b], and Saowanee Lertworasirikul [c]**

[a] Department of Industrial Engineering, Srinakharinwirot University, Ongkarak, Thailand.

[b] Department of Industrial Engineering, Kasetsart University, Thailand.

[c] Department of Product Development, Faculty of Agro-Industry, Kasetsart University, Thailand.

**\*Author for correspondence;** e-mail: [v\\_pou@yahoo.com](mailto:v_pou@yahoo.com)

Received: 1 September 2007

Accepted: 5 November 2007.

### **Abstract**

The fuzzy stochastic data envelopment analysis (FSDEA) model is a mathematical programming for estimating production frontiers and measuring the relative efficiencies of a set of homogenous decision making units (DMUs) that use multiple fuzzy random inputs to produce multiple fuzzy random outputs. Since the conventional data envelopment analysis (DEA) model is built based on linear programming (LP) concept, then exact (refer to as crisp in fuzzy terminology) and deterministic inputs and outputs are required. In this paper, the concept of chance-constrained programming (CC) and the possibility approach are proposed to confront randomness and vagueness in inputs and outputs. Since many DEA models are introduced by researchers, then the most basic DEA model in a case of constant return to scales (CRS), which is called an input-oriented CCR envelopment (DCCR-I) model, is focused in this paper. The result of two steps for transforming, the first one is to convert a fuzzy stochastic DCCR-I (FSDCCR-I) model to a fuzzy deterministic DCCR-I (FDDCCR-I) model by using both of the CC and the linearization approach, then both of the CC and the possibility approach

are used convert the FDDCCR-I model into basic assumptions of traditional DEA concepts, referred as the crisp deterministic DCCR-I (CDDCCR-I) model.

---

**Keyword:** chance-constrained programming, data envelopment analysis, possibility approach, possibility theory.

## 1. Introduction

The traditional DEA model is a widely applied non-parametric mathematical programming technique for estimating production frontiers and measuring the relative efficiencies of a set of homogenous DMUs that use multiple inputs to produce multiple outputs. The first DEA model is a CCR model, which was proposed by Charnes *et al.* [4], is used to evaluating the performance of a set of comparable DMUs based on a case of the CRS of efficiency production frontiers for input or output oriented models, and extended to a case of the variable returns to scale (VRS) by Banker *et al.* [1], which is called BCC model, the non-increasing and non-decreasing returns to scale (NIRS and NDRS) which were imposed by Zhu [14], the additive (ADD) model, which is used to combine input and output orientations in a single model [3], the slacks-based measure of efficiency (SBM), which is in form of a single scalar in ADD model based on an units invariant and monotone properties [12], and many others. One of the advantages of the DEA model is that it does not require either priori weights or explicit specification of functional relations between the multiple outputs and inputs. Numerous research papers on efficiency measurement using DEA have been conducted in a number of contexts including education systems, healthcare units, productions, and military logistics (see [11], for a bibliography of more than 800 articles on DEA applications). However, DEA modeling in the real world is based decision on fuzzily imprecise and probabilistically uncertain information, which cannot be handled by the traditional DEA model, so the FSDEA model is proposed in this paper. Since many DEA models are introduced by researchers, then the most basic DEA model, which is DCCR-I model, is focused in this paper. Based on FSDCCR-I model, there are two uncertainties in both of input and output data. Confronting uncertainty, the concept of CC, which was proposed by Charnes and Cooper [2], is used to deal with randomness in data. CC is a kind of stochastic optimization approaches. It is suitable for solving optimization problems with random variables included in constraints and sometimes in the objective function as well. The constraints are guaranteed to be satisfied with a specified probability or confidence

level at the optimal solution found. Subsequently, some researchers established some theoretical results in the field of stochastic DEA [8, 9]. To deal with vagueness in data, the fuzzy set theory and possibility theory which were introduced by Zadeh [13] are used (good references on possibility theory can be found in [7] and [15]). Subsequently, Lertworasirikul *et al.* [9] adopted this theory to solve a fuzzy DEA (FDEA) model.

This paper is organized as follows. The traditional DEA and possibility theory are summarized in Section 2. In Section 3, the Fuzzy Stochastic DEA and the Chance-Constrained DEA model are introduced. In Section 4 and 5, the chance-constrained fuzzy stochastic DEA (CC-FSDEA) is transformed into the equivalent fuzzy deterministic DEA model and crisp deterministic DEA model, respectively. A numerical example and conclusion of this paper are proposed in Section 6 and 7, respectively.

## 2. Background

### 2.1 Data Envelopment Analysis (DEA)

In the usual setting, suppose that there are  $n$  evaluated DMUs, each of which consumes the same type of  $m$  inputs and produces the same type of  $s$  outputs. All inputs and outputs are assumed to be nonnegative, but at least one input and one output are positive. If the input and output data for DMU <sub>$j$</sub> ,  $j = 1, \dots, n$  are  $(x_{1j}, \dots, x_{mj})$  and  $(y_{1j}, \dots, y_{sj})$  respectively, then the efficiency of an evaluated DMU (DMU <sub>$o$</sub> ) where  $o$  ranges over  $1, \dots, n$  is measured by solving the following fractional programming (FP) problem to obtain the values of input weights,  $v_i$ , and output weights  $u_r$  for  $i = 1, \dots, m$  and  $r = 1, \dots, s$  as decision variables.

$$(FP) \max \theta = \frac{\sum_{r=1}^s u_r y_{ro}}{\sum_{i=1}^m v_i x_{io}} \tag{1}$$

$$\text{Subject to } \frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{i=1}^m v_i x_{ij}} \leq 1 \text{ for } j = 1, \dots, n \tag{2}$$

$$u_r, v_i \geq 0 \tag{3}$$

If the denominator of (1) is set as 1 and moved to be a constraint, after multiplying both sides of constraints in (2) by the corresponding denominator then the fractional programming problem is equivalent to the following linear programming (LP) problem.

$$(CCR) \max \theta = \sum_{r=1}^s u_r y_{ro} \tag{4}$$

$$\text{Subject to } \sum_{i=1}^m v_i x_{io} = 1 \quad (5)$$

$$\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0; j = 1, \dots, n \quad (6)$$

$$u_r, v_i \geq 0 \quad (7)$$

For every LP, there exists the related dual problem, in which the roles of variables and constraints are reversed. Suppose that real variables  $\theta$  and  $\lambda_j$  for  $j = 1, \dots, n$  are dual variables. From Pareto-Koopmans efficiency, a DMU is fully efficient if and only if it is not possible to improve any input or output without worsening some other input or output [6] Therefore, the dual problem of CCR model (DCCR) or envelopment model is the following multi-objective LP problem.

$$\text{(Phase I-DCCR) min } \theta \quad (8)$$

$$\text{(Phase II-DCCR) max } \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \quad (9)$$

$$\text{Subject to } \theta x_{io} - \sum_{j=1}^n \lambda_j x_{ij} - s_i^- = 0; i = 1, \dots, m \quad (10)$$

$$\sum_{j=1}^n \lambda_j y_{rj} - y_{ro} - s_r^+ = 0; r = 1, \dots, s \quad (11)$$

$$\theta \text{ Unrestricted, } \lambda_j \geq 0, s_i^- \geq 0, s_r^+ \geq 0 \quad (12)$$

In the first phase, LP problem is solved to find the optimal  $\theta^*$ . Then the second phase is to find a solution that maximizes the sum of all slacks with assurance that  $\theta^*$  will not be possible to improve any input or output with worsening some other inputs or outputs. The DMU<sub>o</sub> is determined to be radial efficient if and only if an optimal solution satisfies  $\theta^* = 1$ , and will be called CCR-efficiency if and only if an optimal solution satisfies  $\theta^* = 1$  and all slacks are zero. An optimal solution  $(\lambda_j^*, s_i^{-*}, s_r^{+*})$  from (8)-(12) is called the max-slack solution. If the max-slack solution satisfies both  $s_i^{-*} = 0$  and  $s_r^{+*} = 0$ , then it is called zero slack. For an inefficient DMU<sub>o</sub>, its efficiency of  $(x_{io}, y_{ro})$  can be improved by projecting DMU<sub>o</sub> into its reference set which is defined by  $E_o = \{j / \lambda_j^* > 0\}$  for  $j \in \{1, \dots, n\}$ . Projections of the input oriented DCCR model are given in the following formula.

$$x_{io,improve} = \theta^* x_{io} - s_i^{-*} \text{ and } y_{ro,improve} = y_{ro} + s_r^{+*} \tag{13}$$

Note that, there are three important reasons for solving the envelopment model instead of solving its primal model. First, the number of DMUs ( $n$ ) is larger than the number of inputs and outputs ( $m + s$ ) and hence it takes more time and larger memory to solve primal problem or multiplier model with  $n$  constraints than to solve the envelopment model with  $m + s$  constraints. Second, the activities of inefficient DMU cannot be improved because the reference set and max-slack solution cannot be found in multiplier models. Finally, the interpretations of envelopment models are more straightforward than these of multiplier models. [6]

**2.2 Possibility Theory**

Possibility theory in the context of the fuzzy set theory was introduced by Zadeh [13] which was dealing with non-stochastic imprecision and vagueness. Suppose that  $(\Theta_i, P(\Theta_i), \pi_i)$  for  $i = 1, \dots, n$  is a possibility space with  $\Theta_i$  being the nonempty set of interest,  $P(\Theta_i)$  is the collection of all subsets of  $\Theta_i$ , and  $\pi_i$  is the possibility measure from  $P(\Theta_i)$  to  $[0, 1]$ , then

- (i)  $\pi(\emptyset) = 0$  and  $\pi(\Theta_i) = 1$ , and
- (ii)  $\pi(\cup_i A_i) = \sup\{\pi(A_i)\}$  with each  $A_i \in P(\Theta_i)$ .

Let  $\tilde{\xi}$  be fuzzy variable as a real-valued function defined over  $\Theta_i$ , therefore the membership function of  $\tilde{\xi}$  is given by

$$\mu_{\tilde{\xi}}(s) = \pi(\{\theta_i \in \Theta_i / \tilde{\xi}(\theta_i) = s\}) = \sup_{\theta_i \in \Theta_i} \{\pi(\{\theta_i\} / \tilde{\xi}(\theta_i) = s)\}, \forall s \in \mathcal{R}. \tag{14}$$

Let  $(\Theta, P(\Theta), \pi)$  be a product possibility space such that  $\Theta = \Theta_1 \times \dots \times \Theta_n$  then

$$\pi(A) = \min\{\pi_i(A_i) / A = A_1 \times \dots \times A_n, A_i \in P(\Theta_i)\}. \tag{15}$$

To compare fuzzy variables, let  $\tilde{a}_1, \dots, \tilde{a}_n$  be fuzzy variables and  $f_j: \mathcal{R}^n \rightarrow \mathcal{R}$  be a real-valued function for  $j = 1, \dots, m$ . The possibility measure of fuzzy event is given by

$$\pi(f_j(\tilde{a}_1, \dots, \tilde{a}_n) \leq 0) = \sup_{s_1, \dots, s_n \in \mathcal{R}} \{\min\{\mu_{\tilde{a}_i}(s_i)\} / f_j(s_1, \dots, s_n) \leq 0\}. \tag{16}$$

**3. FSDCCR-I Model**

In this paper, a fuzzy random variable (which is a measurable function from a probability space to the set of fuzzy variables) first proposed by Kwakernaak [8] is used

in the DCCR model under random and vague environment. The fuzzy stochastic DCCR (FSDCCR) and chance-constrained FSDCCR (CC-FSDCCR) models are as follows.

$$(FSDCCR) \min \theta \tag{17}$$

$$\text{Subject to } \theta \tilde{x}_{io} - \sum_{j=1}^n \lambda_j \tilde{x}_{ij} \geq 0; i = 1, \dots, m \tag{18}$$

$$\sum_{j=1}^n \lambda_j \tilde{y}_{rj} - \tilde{y}_{ro} \geq 0; r = 1, \dots, s \tag{19}$$

$$\theta \text{ Unrestricted, } \lambda_j \geq 0 \tag{20}$$

where  $\tilde{\mathbf{x}}_i = (\tilde{x}_{1j}, \dots, \tilde{x}_{mj})^T$  and  $\tilde{\mathbf{y}}_j = (\tilde{y}_{1j}, \dots, \tilde{y}_{sj})^T$  respectively represent  $(m \times 1)$  and  $(s \times 1)$  fuzzy random input and output vectors.

$$(CC-FSDCCR) \min \theta \tag{21}$$

$$\text{Subject to } \pi \left\{ Pr \left\{ \sum_{j=1}^n \lambda_j \tilde{x}_{ij} - \theta \tilde{x}_{io} \leq 0 \right\} \geq 1 - \alpha_i \right\} \geq \varpi_i \tag{22}$$

$$\pi \left\{ Pr \left\{ \tilde{y}_{ro} - \sum_{j=1}^n \lambda_j \tilde{y}_{rj} \leq 0 \right\} \geq 1 - \beta_r \right\} \geq \omega_r \tag{23}$$

$$\theta \text{ Unrestricted, } \lambda_j \geq 0 \tag{24}$$

where “Pr” means probability,  $1 - \alpha_i$  and  $1 - \beta_r$  is a pre-specified minimum probability. “ $\pi$ ” means possibility,  $\varpi_i$  and  $\omega_r$  are pre-specified acceptable levels of possibility.

**4. The Equivalent Fuzzy Deterministic DEA Model**

The CC-FSDCCR model can be transformed into the equivalent fuzzy deterministic DCCR (FDDCCR) model. Let  $\rho_i$  and  $\tau_r$  for  $i = 1, \dots, m$  and  $r = 1, \dots, s$  be slacks which can be inserted in inequality outside braces to achieve equality, therefore

$$\pi \left\{ Pr \left\{ \sum_{j=1}^n \lambda_j \tilde{x}_{ij} - \theta \tilde{x}_{io} \leq 0 \right\} = (1 - \alpha_i) + \rho_i \right\} \geq \varpi_i \tag{25}$$

$$\pi \left\{ Pr \left\{ \tilde{y}_{ro} - \sum_{j=1}^n \lambda_j \tilde{y}_{rj} \leq 0 \right\} = (1 - \beta_r) + \tau_r \right\} \geq \omega_r. \tag{26}$$

Let  $s_i^-$  and  $s_r^+$  be positive variables such that

$$\pi \left\{ Pr \left\{ \sum_{j=1}^n \lambda_j \tilde{x}_{ij} - \theta \tilde{x}_{i0} \leq -s_i^- \right\} = 1 - \alpha_i \right\} \geq \varpi_i \quad (27)$$

$$\pi \left\{ Pr \left\{ \tilde{y}_{ro} - \sum_{j=1}^n \lambda_j \tilde{y}_{rj} \leq -s_r^+ \right\} = 1 - \beta_r \right\} \geq \omega_r \quad (28)$$

In this paper, fuzzy random input and output variables are assumed to be normal distributed, therefore (27) and (28) are respectively normalized by

$$\pi \left\{ Pr \left\{ \frac{\sum_{j=1}^n \lambda_j \tilde{x}_{ij} - \theta \tilde{x}_{i0} - \sum_{j=1}^n \lambda_j E(\tilde{x}_{ij}) + \theta E(\tilde{x}_{i0}) - s_i^- - \sum_{j=1}^n \lambda_j E(\tilde{x}_{ij}) + \theta E(\tilde{x}_{i0})}{\sqrt{[-\theta, \lambda]^T \mathbf{Cov}_i[-\theta, \lambda]}} \leq \frac{-s_i^- - \sum_{j=1}^n \lambda_j E(\tilde{x}_{ij}) + \theta E(\tilde{x}_{i0})}{\sqrt{[-\theta, \lambda]^T \mathbf{Cov}_i[-\theta, \lambda]}} \right\} = 1 - \alpha_i \right\} \geq \varpi_i \quad (29)$$

$$\pi \left\{ Pr \left\{ \frac{\tilde{y}_{ro} - \sum_{j=1}^n \lambda_j \tilde{y}_{rj} - E(\tilde{y}_{ro}) + \sum_{j=1}^n \lambda_j E(\tilde{y}_{rj}) - s_r^+ - E(\tilde{y}_{ro}) + \sum_{j=1}^n \lambda_j E(\tilde{y}_{rj})}{\sqrt{[\mathbf{1}, -\lambda]^T \mathbf{Cov}_r[\mathbf{1}, -\lambda]}} \leq \frac{-s_r^+ - E(\tilde{y}_{ro}) + \sum_{j=1}^n \lambda_j E(\tilde{y}_{rj})}{\sqrt{[\mathbf{1}, -\lambda]^T \mathbf{Cov}_r[\mathbf{1}, -\lambda]}} \right\} = 1 - \beta_r \right\} \geq \omega_r \quad (30)$$

where  $[-\theta, \lambda] = (-\theta, \lambda_1, \dots, \lambda_n)^T$ ,  $[\mathbf{1}, -\lambda] = (1, -\lambda_1, \dots, -\lambda_n)^T$ ,  $\mathbf{Cov}_i$  and  $\mathbf{Cov}_r$  are  $(n + 1) \times (n + 1)$  matrices respectively indicating variance and covariance of fuzzy output and input random variables for the  $j$ th DMU in which the symbol “V” and “Cov” refer to variance and covariance operators, respectively.

$$\mathbf{Cov}_i = \begin{pmatrix} V(\tilde{x}_{i0}) & \text{Cov}(\tilde{x}_{i0}, \tilde{x}_{i1}) & \cdots & \text{Cov}(\tilde{x}_{i0}, \tilde{x}_{in}) \\ \text{Cov}(\tilde{x}_{i0}, \tilde{x}_{i1}) & V(\tilde{x}_{i1}) & \cdots & \text{Cov}(\tilde{x}_{i1}, \tilde{x}_{in}) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(\tilde{x}_{i0}, \tilde{x}_{in}) & \text{Cov}(\tilde{x}_{i1}, \tilde{x}_{in}) & \cdots & V(\tilde{x}_{in}) \end{pmatrix} \quad (31)$$

$$\mathbf{Cov}_r = \begin{pmatrix} V(\tilde{y}_{ro}) & \text{Cov}(\tilde{y}_{ro}, \tilde{y}_{r1}) & \cdots & \text{Cov}(\tilde{y}_{ro}, \tilde{y}_{rm}) \\ \text{Cov}(\tilde{y}_{ro}, \tilde{y}_{r1}) & V(\tilde{y}_{r1}) & \cdots & \text{Cov}(\tilde{y}_{r1}, \tilde{y}_{rm}) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(\tilde{y}_{ro}, \tilde{y}_{rm}) & \text{Cov}(\tilde{y}_{r1}, \tilde{y}_{rm}) & \cdots & V(\tilde{y}_{rm}) \end{pmatrix} \quad (32)$$

Let the left hand side term inside “Pr” braces of (29) and (30) are respectively denoted by  $z_i$  for  $i = 1, \dots, m$  and  $z_r$  for  $r = 1, \dots, s$  which are assumed to be standard normal distributed with mean zero and variance one.

$$\pi \left\{ Pr \left\{ z_i \leq \frac{-s_i^- - \sum_{j=1}^n \lambda_j E(\tilde{x}_{ij}) + \theta E(\tilde{x}_{i0})}{\sqrt{[-\theta, \lambda]^T \mathbf{Cov}_i[-\theta, \lambda]}} \right\} = 1 - \alpha_i \right\} \geq \varpi_i \quad (33)$$

$$\pi \left\{ Pr \left\{ z_r \leq \frac{-s_r^+ - E(\tilde{y}_{ro}) + \sum_{j=1}^n \lambda_j E(\tilde{y}_{rj})}{\sqrt{[\mathbf{1}, -\boldsymbol{\lambda}]^T \mathbf{Cov}_r [\mathbf{1}, -\boldsymbol{\lambda}]}} \right\} = 1 - \beta_r \right\} \geq \omega_r \tag{34}$$

Therefore, (33) and (34) can be respectively reformulated to be

$$\pi \left\{ \theta E(\tilde{x}_{io}) - \sum_{j=1}^n \lambda_j E(\tilde{x}_{ij}) - s_i^- = (\Phi^{-1}(1 - \alpha_i)) \sqrt{[-\boldsymbol{\theta}, \boldsymbol{\lambda}]^T \mathbf{Cov}_i [-\boldsymbol{\theta}, \boldsymbol{\lambda}]} \right\} \geq \varpi_i \tag{35}$$

$$\pi \left\{ \sum_{j=1}^n \lambda_j E(\tilde{y}_{rj}) - E(\tilde{y}_{ro}) - s_r^+ = (\Phi^{-1}(1 - \beta_r)) \sqrt{[\mathbf{1}, -\boldsymbol{\lambda}]^T \mathbf{Cov}_r [\mathbf{1}, -\boldsymbol{\lambda}]} \right\} \geq \omega_r \tag{36}$$

where  $\Phi$  represents the normal cumulative distribution function and  $\Phi^{-1}$  is its inverse. Since (35) and (36) are expressed by the expected value and the quadratic terms of variance-covariance matrices,  $\mathbf{Cov}_i$  and  $\mathbf{Cov}_r$ , solving the FDDCCR model is a non-trivial task. In this paper, the linearization approach to obtain a linear deterministic equivalent model, which was introduced by Cooper *et al.* [5] and Li [10], is used. Let  $\tilde{x}_{io}$ ,  $\tilde{x}_{ij}$ ,  $\tilde{y}_{ro}$  and  $\tilde{y}_{rj}$  respectively represent fuzzy means of fuzzy input and output variables.  $\tilde{a}_{io}$ ,  $\tilde{a}_{ij}$ ,  $\tilde{b}_{ro}$  and  $\tilde{b}_{rj}$  respectively represent fuzzy standard deviations of fuzzy input and output variables.  $\zeta_{io}$ ,  $\zeta_{ij}$ ,  $\zeta_{ro}$  and  $\zeta_{rj}$  represent error structures which are assumed to be standard normal distributed.  $\tilde{a}_{io}\zeta_{io}$ ,  $\tilde{a}_{ij}\zeta_{ij}$ ,  $\tilde{b}_{ro}\zeta_{ro}$  and  $\tilde{b}_{rj}\zeta_{rj}$  represent symmetric disturbance or error terms of fuzzy input and output. Then input and output data structures can be written as follows.

$$\tilde{x}_{io} = \tilde{x}_{io} + \tilde{a}_{io}\zeta_{io}, \tilde{x}_{ij} = \tilde{x}_{ij} + \tilde{a}_{ij}\zeta_{ij}, \tilde{y}_{ro} = \tilde{y}_{ro} + \tilde{b}_{ro}\zeta_{ro} \text{ and } \tilde{y}_{rj} = \tilde{y}_{rj} + \tilde{b}_{rj}\zeta_{rj} \tag{37}$$

Therefore, fuzzy expected value and variance are given by

$$E(\tilde{x}_{io}) = \tilde{x}_{io}, E(\tilde{x}_{ij}) = \tilde{x}_{ij}, E(\tilde{y}_{ro}) = \tilde{y}_{ro} \text{ and } E(\tilde{y}_{rj}) = \tilde{y}_{rj} \tag{38}$$

$$\text{Var}(\tilde{x}_{io}) = \tilde{a}_{io}^2, \text{Var}(\tilde{x}_{ij}) = \tilde{a}_{ij}^2, \text{Var}(\tilde{y}_{ro}) = \tilde{b}_{ro}^2 \text{ and } \text{Var}(\tilde{y}_{rj}) = \tilde{b}_{rj}^2. \tag{39}$$

If (31) and (32) are substituted by fuzzy variances in (39) and correlations of all input and output data are assumed to be 1, then variance and covariance matrices of fuzzy inputs and outputs are given in (40) and (41), respectively.

$$\mathbf{Cov}_i = \begin{pmatrix} \tilde{a}_{io}^2 & \tilde{a}_{io}\tilde{a}_{i1} & \cdots & \tilde{a}_{io}\tilde{a}_{in} \\ \tilde{a}_{io}\tilde{a}_{i1} & \tilde{a}_{i1}^2 & \cdots & \tilde{a}_{i1}\tilde{a}_{in} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{io}\tilde{a}_{in} & \tilde{a}_{i1}\tilde{a}_{in} & \cdots & \tilde{a}_{in}^2 \end{pmatrix} \quad (40)$$

$$\mathbf{Cov}_r = \begin{pmatrix} \tilde{b}_{ro}^2 & \tilde{b}_{ro}\tilde{b}_{r1} & \cdots & \tilde{b}_{ro}\tilde{b}_{rm} \\ \tilde{b}_{ro}\tilde{b}_{r1} & \tilde{b}_{r1}^2 & \cdots & \tilde{b}_{r1}\tilde{b}_{rm} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{b}_{ro}\tilde{b}_{rm} & \tilde{b}_{r1}\tilde{b}_{rm} & \cdots & \tilde{b}_{rm}^2 \end{pmatrix} \quad (41)$$

Fuzzy standard deviation terms in (35) and (36) are respectively reformulated to be

$$\sqrt{[-\mathbf{0}, \boldsymbol{\lambda}]^T \mathbf{Cov}_i [-\mathbf{0}, \boldsymbol{\lambda}]} = \sum_{j=1}^n \lambda_j \tilde{a}_{ij} - \theta \tilde{a}_{io} \quad (42)$$

$$\sqrt{[\mathbf{1}, -\boldsymbol{\lambda}]^T \mathbf{Cov}_r [\mathbf{1}, -\boldsymbol{\lambda}]} = \sum_{j=1}^n \lambda_j \tilde{b}_{rj} - \tilde{b}_{ro}. \quad (43)$$

By substituting of fuzzy standard deviation in (42) and (43) into (35) and (36), respectively, the equivalent fuzzy deterministic DCCR model (FDDCCR) in terms of fuzzy LP problem can be formulated as follows.

$$\text{(FDDCCR) min } \theta \quad (44)$$

$$\text{Subject to } \pi \left\{ \sum_{j=1}^n \lambda_j (\tilde{x}_{ij} + (\Phi^{-1}(1 - \alpha_i))\tilde{a}_{ij}) - (\tilde{x}_{io} + (\Phi^{-1}(1 - \alpha_i))\tilde{a}_{io})\theta \leq 0 \right\} \geq \varpi_i \quad (45)$$

$$\pi \left\{ (\tilde{y}_{ro} - (\Phi^{-1}(1 - \beta_r))\tilde{b}_{ro}) - \sum_{j=1}^n \lambda_j (\tilde{y}_{rj} - (\Phi^{-1}(1 - \beta_r))\tilde{b}_{rj}) \leq 0 \right\} \geq \omega_r \quad (46)$$

$$\theta \text{ Unrestricted, } \lambda_j \geq 0. \quad (47)$$

### 5. The Equivalent Crisp Deterministic DEA Model

Based on Possibility theory, Lertworasirikul *et al.* [9] proved and proposed Lemma 1 for solving the fuzzy multiplier form of input-oriented CCR model.

**Lemma 1.** Let  $\tilde{a}_1, \dots, \tilde{a}_n$  be fuzzy variables with normal and convex membership functions and  $b$  be a crisp variable. Let  $(\bullet)_{\alpha_i}^L$  and  $(\bullet)_{\alpha_i}^U$  denote the lower and upper bounds of the  $\alpha$ -level set of  $\tilde{a}_i$  for  $i = 1, \dots, n$ . Then, for any given possibility levels  $\alpha_1, \alpha_2$  and  $\alpha_3$  with  $0 \leq \alpha_1, \alpha_2, \alpha_3 \leq 1$

- (i)  $\pi(\tilde{a}_1 + \dots + \tilde{a}_n \leq b) \geq \alpha_1$  if and only if  $(\tilde{a}_1)_{\alpha_1}^L + \dots + (\tilde{a}_n)_{\alpha_1}^L \leq b$ ,
- (ii)  $\pi(\tilde{a}_1 + \dots + \tilde{a}_n \geq b) \geq \alpha_2$  if and only if  $(\tilde{a}_1)_{\alpha_2}^U + \dots + (\tilde{a}_n)_{\alpha_2}^U \geq b$ ,
- (iii)  $\pi(\tilde{a}_1 + \dots + \tilde{a}_n = b) \geq \alpha_3$  if and only if  $(\tilde{a}_1)_{\alpha_3}^L + \dots + (\tilde{a}_n)_{\alpha_3}^L \leq b$  and  $(\tilde{a}_1)_{\alpha_3}^U + \dots + (\tilde{a}_n)_{\alpha_3}^U \geq b$ .

To built the equivalent CDDCCR model, Lemma 1 is used to transform the fuzzy deterministic constraints in (45) and (46) into equivalent crisp deterministic constraints as given by

$$\left( \sum_{j=1}^n \lambda_j (\tilde{x}_{ij} + (\Phi^{-1}(1 - \alpha_i))\tilde{a}_{ij}) \right)_{\bar{\omega}_i}^L - \left( (\tilde{x}_{io} + (\Phi^{-1}(1 - \alpha_i))\tilde{a}_{io})\theta \right)_{\bar{\omega}_i}^U \leq 0 \tag{47}$$

$$\left( (\tilde{y}_{ro} - (\Phi^{-1}(1 - \beta_r))\tilde{b}_{ro}) \right)_{\omega_r}^L - \left( \sum_{j=1}^n \lambda_j (\tilde{y}_{rj} - (\Phi^{-1}(1 - \beta_r))\tilde{b}_{rj}) \right)_{\omega_r}^U \leq 0. \tag{48}$$

Let  $s_i^-$  for  $i = 1, \dots, m$  and  $s_r^+$  for  $r = 1, \dots, s$  are slack variables which can be inserted into inequalities (47) and (48) to achieve equalities and assume that fuzzy standard deviations are approximated by crisp standard deviations. Therefore the equivalent CDDCCR model becomes the following multi-objective linear programming problem.

(Phase I-CDDCCR)  $\min \theta$  (49)

(Phase II-CDDCCR)  $\max \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+$  (50)

Subject to

$$\sum_{j=1}^n \lambda_j ((\tilde{x}_{ij})_{\bar{\omega}_i}^L + (\Phi^{-1}(1 - \alpha_i))\tilde{a}_{ij}) - ((\tilde{x}_{io})_{\bar{\omega}_i}^U + (\Phi^{-1}(1 - \alpha_i))\tilde{a}_{io})\theta + s_i^- = 0; i = 1, \dots, m \tag{51}$$

$$(\tilde{y}_{ro})_{\omega_r}^L - (\Phi^{-1}(1 - \beta_r))\tilde{b}_{ro} - \sum_{j=1}^n \lambda_j ((\tilde{y}_{rj})_{\omega_r}^U - \Phi^{-1}(1 - \beta_r)\tilde{b}_{rj}) + s_r^+ = 0; r = 1, \dots, s \tag{52}$$

$\theta$  Unrestricted,  $\lambda_j \geq 0, s_i^- \geq 0, s_r^+ \geq 0$  (53)

**6. Numerical Example**

Suppose that there are 4 evaluated DMUs, each of which uses the same type of 4 inputs and produces the same type of 1 fuzzy random output (see Table 1).

**Table 1.** Inputs and the Output Data

DMUs	Input 1	Input 2	Input 3	Input 4		Output			
				Points at membership function = 1	Spread		Points at membership function = 1	Spread	
					L	U		L	U
DMU 1	12	(4, 5, 7)	{18, 18, 21, 23, 24}	{8, 8, 9, 9, 9}	2	2	{32, 32, 34, 35, 37}	5	5
DMU 2	15	(8, 11, 15)	{18, 18, 21, 23, 24}	{8, 8, 9, 9, 9}	2	2	{32, 32, 34, 35, 37}	5	5
DMU 3	15	(8, 11, 15)	{25, 25, 26, 28, 29}	{10, 13, 14, 14, 16}	1	1	{26, 26, 28, 28, 29}	3	5
DMU 4	18	(10, 12, 15)	{27, 28, 29, 29, 29}	{9, 10, 10, 11, 11}	1	2	{30, 30, 30, 31, 31}	3	5

From table 1, input 1 is a crisp deterministic data, input 2 is a triangular fuzzy deterministic data, input 3 is a crisp stochastic data with 5 observations, input 4 is a triangular fuzzy stochastic data with 5 observations, and output is a triangular fuzzy stochastic data with 5 observations. To evaluate DMU 1, 2, 3 and 4, first, minimum probability and acceptable levels of possibility are specified by a decision maker. Let  $\alpha_3 = \alpha_4 = \beta = 0.05$  or  $\Phi^{-1}(1 - \alpha_3) = \Phi^{-1}(1 - \alpha_4) = \Phi^{-1}(1 - \beta) = 1.645$  and  $\varpi_2 = \varpi_4 = \omega = 0.5$ . Then the CDCCR model becomes the following multi-objective linear programming problem.

$$(Phase\ I-CDDCCR)\ \min\ \theta \tag{54}$$

$$(Phase\ II-CDDCCR)\ \max\ s_1^- + s_2^- + s_3^- + s_4^- + s^+ \tag{55}$$

Subject to

$$\text{Input 1 constraint; } (12\lambda_1 + 15\lambda_2 + 15\lambda_3 + 18\lambda_4) + s_1^- = x_{1o}\theta \tag{56}$$

$$\text{Input 2 constraint; } (4.5\lambda_1 + 9.5\lambda_2 + 9.5\lambda_3 + 11\lambda_4) + s_2^- = 0.5((\tilde{x}_{2o})_0^U + (\tilde{x}_{2o})_1^{LU})\theta \tag{57}$$

Input 3 constraint;

$$(25.3647\lambda_1 + 25.3647\lambda_2 + 29.5883\lambda_3 + 29.8713\lambda_4) + s_3^- = (\bar{x}_{3o} + 1.645a_{3o})\theta \tag{58}$$

Input 4 constraint;

$$(8.5010\lambda_1 + 8.5010\lambda_2 + 16.5040\lambda_3 + 11.0763\lambda_4) + s_4^- = (0.5((\tilde{x}_{4o})_0^U + (\tilde{x}_{4o})_1^{LU}) + 1.645\tilde{a}_{4o})\theta \tag{59}$$

Output constraint;

$$(33.0104\lambda_1 + 33.0104\lambda_2 + 28.4980\lambda_3 + 27.9990\lambda_4) - s^+ = 0.5((\tilde{y}_o)_0^L + (\tilde{y}_o)_1^{LU}) - 1.645\tilde{b}_o \tag{60}$$

$$\theta \text{ Unrestricted, } \lambda_j \geq 0, s_1^- \geq 0, s_2^- \geq 0, s_3^- \geq 0, s_4^- \geq 0, s^+ \geq 0 \tag{61}$$

To evaluate DMU 1, substitute  $x_{1o} = 12$ ,  $0.5((\tilde{x}_{2o})_0^U + (\tilde{x}_{2o})_1^{LU}) = 6$ ,  $\bar{x}_{3o} + 1.645a_{3o} = 25.3647$ ,  $(0.5((\tilde{x}_{4o})_0^U + (\tilde{x}_{4o})_1^{LU}) + 1.645\tilde{a}_{4o})\theta = 10.5010$  and  $0.5((\tilde{y}_o)_0^L + (\tilde{y}_o)_1^{LU}) - 1.645\tilde{b}_o = 28.0104$  in (56)-(61). The optimal solution of the LP problem is  $\theta^* = 0.8485$ ,  $\lambda_1^* = 0.8485$ ,  $\lambda_2^* = 0$ ,  $\lambda_3^* = 0$ ,  $\lambda_4^* = 0$ ,  $s_1^{-*} = 0$ ,  $s_2^{-*} = 1.2728$ ,  $s_3^{-*} = 0$ ,  $s_4^{-*} = 1.6971$  and  $s^{+*} = 0$ . In a similar manner of DMU 2, 3 and 4, optimal solution  $(\theta^*, \lambda_j^*, s_i^{-*}, s_r^{+*})$  for  $j = 1, \dots, 4$ , and  $i = 1, \dots, 4$ , and  $r = 1$  are  $(0.8485, 0.8485, 0, 0, 0, 2.5456, 7.2125, 0, 1.6971, 0)$ ,  $(0.6362, 0.7421, 0, 0, 0, 0.6374, 4.9309, 0, 4.8271, 0)$  and  $(0.7202, 0.8482, 0, 0, 0, 2.7858, 5.9062, 0, 1.8473, 0)$ , respectively. Since  $\theta^* \leq 1$  for all DMUs, then all DMUs are determined to be the CCR-inefficient DMUs at possibility level  $\varpi_2 = \varpi_4 = \omega = 0.5$ . Results of efficiency evaluation at possibility level 0, 0.25, 0.5, 0.75 and 1 are shown in Table 2.

From table 2, Consider DMU 1, since  $\theta^* = 1$  at  $\varpi_2 = \varpi_4 = \omega = 1$  and max slack solution is the zero slack solution then DMU 1 is found to be CCR-efficient. However, DMU 2 is the technical efficiency, which is not CCR-efficient, because the max slack solution is not the zero slack. For DMU 3 and 4, there are  $\theta^* < 1$  therefore all of them are evaluated to be CCR-inefficient. Consider optimal solution  $\theta^*$  of all DMUs at  $\varpi_2 = \varpi_4 = \omega < 1$ , there are not optimal solutions which are equal to 1. Therefore, optimal solution must be unitized by maximum division, for example, if  $\theta^*$  of DMU 1, 2, 3 and 4 at  $\varpi_2 = \varpi_4 = \omega = 0$  are 0.7184, 0.7184, 0.5552 and 0.6336, respectively, are multiplied by  $1/\max\{0.7184, 0.7184, 0.5552 \text{ and } 0.6336\}$ , then we have technical efficiency of DMU 1, 2, 3 and 4 are 1, 1, 0.7728 and 0.88196.

**Table 2.** Results of efficiency evaluation

DMU <sub>o</sub>	Optimal solution	Significant level 0.05 and Possibility level				
		0	0.25	0.5	0.75	1
DMU 1	$\theta^*$	0.7184	0.7811	0.8485	0.9213	1
	$\lambda_1^*$	0.7184	0.7811	0.8485	0.9213	0
	$\lambda_2^*$	0	0	0	0	0
	$\lambda_3^*$	0	0	0	0	0
	$\lambda_4^*$	0	0	0	0	0
	$s_1^{-*}$	0	0	0	0	0
	$s_2^{-*}$	2.1552	1.7574	1.2728	0.6910	0
	$s_3^{-*}$	0	0	0	0	0
	$s_4^{-*}$	2.8736	2.3433	1.6971	0.9213	0
	$s^{+*}$	0	0	0	0	0
DMU 2	$\theta^*$	0.7184	0.7811	0.8485	0.9213	1
	$\lambda_1^*$	0.7184	0.7811	0.8485	0.9213	0
	$\lambda_2^*$	0	0	0	0	0
	$\lambda_3^*$	0	0	0	0	0
	$\lambda_4^*$	0	0	0	0	0
	$s_1^{-*}$	2.1552	2.3433	2.5456	2.7639	3
	$s_2^{-*}$	7.9023	7.6156	7.2125	6.6793	6
	$s_3^{-*}$	0	0	0	0	0
	$s_4^{-*}$	2.8736	2.3433	1.6971	0.9213	0
	$s^{+*}$	0	0	0	0	0

DMU <sub>o</sub>	Optimal solution	Significant level 0.05 and Possibility level				
		0	0.25	0.5	0.75	1
DMU 3	$\theta^*$	0.5552	0.5942	0.6362	0.6815	0.7305
	$\lambda_1^*$	0.6476	0.6932	0.7421	0.7950	0.8521
	$\lambda_2^*$	0	0	0	0	0
	$\lambda_3^*$	0	0	0	0	0
	$\lambda_4^*$	0	0	0	0	0
	$s_1^{-*}$	0.5562	0.5953	0.6374	0.6827	0.7318
	$s_2^{-*}$	5.7373	5.3731	4.9309	4.4017	3.7746
	$s_3^{-*}$	0	0	0	0	0
	$s_4^{-*}$	5.1377	5.0037	4.8271	4.6028	4.3251
	$s^{+*}$	0	0	0	0	0
DMU 4	$\theta^*$	0.6336	0.6754	0.7202	0.7686	0.8210
	$\lambda_1^*$	0.7462	0.7953	0.8482	0.9052	0.9668
	$\lambda_2^*$	0	0	0	0	0
	$\lambda_3^*$	0	0	0	0	0
	$\lambda_4^*$	0	0	0	0	0
	$s_1^{-*}$	2.4509	2.6122	2.7858	2.9730	3.1755
	$s_2^{-*}$	6.5198	6.2436	5.9062	5.5003	5.0175
	$s_3^{-*}$	0	0	0	0	0
	$s_4^{-*}$	3.0051	2.4676	1.8473	1.1345	0.3179
	$s^{+*}$	0	0	0	0	0

## 7. Conclusion

This paper has presented the FSDCCR-I model which allow for the fuzzily imprecise and probabilistically uncertain. Confronting both fuzzy and probabilistic uncertainties, the concept of chance-constrained programming and the possibility approach are used to transform FSDCCR-I model to be the FDDCCR-I and the CDDCCR-I model, respectively. However, since the DCCR-I model is an only simplest DEA model, then this procedure will be adopted to solve other DEA model in future study.

## References

- [1] Banker, R. D., Charnes, A., and Cooper, W.W., Some models for estimating technical and scale inefficiency in data envelopment analysis, *Management Science*, 1984; 30: 1078-1092.
- [2] Charnes, A., and Cooper, W.W., Chance constrained programming, *Management Science*, 1959; 6(1): 73-79.
- [3] Charnes, A., Cooper, W.W., Golany, B., Seiford, L., and Stutz, J., Foundations of data envelopment analysis for Pareto-Koopmans efficient empirical production functions", *Journal of Econometrics*, 1985; 30: 91-107.
- [4] Charnes, A., Cooper, W.W. and Rhodes, E., Measuring the efficiency of decision making units, *European Journal of Operational Research*, 1978; 2: 429-444.
- [5] Cooper, W.W., Huang, Z.M., Lelas, V., Li, S.X., and Olesen, O.B., Chance constrained programming formulations for stochastic characterizations of efficiency and dominance in DEA, *Journal of Productivity Analysis*, 1998; 9: 53-79.
- [6] Cooper, W.W., Seiford L.M., and Tone. K., *Data envelopment analysis: A comprehensive text with models, applications, reference and DEA-solver software*, Kluwer Academic Publishers, London, 2000.
- [7] Dubois, D., and Prade, H., *Fuzzy sets and systems: Theory and applications*, Academic Press, New York, 1980.
- [8] Kwakernaak, H., Fuzzy random variables-I: Definitions and theorems, *Information Science*, 1978; 15: 1-29.
- [9] Lertworasirikul, S., Fang, S.C., Joines, J.A., and Nuttle, H.L.W., Fuzzy data envelopment analysis (DEA): A possibility approach, *Fuzzy Sets and Systems*, 2003; 139: 379-394.
- [10] Li, S.X. (1998). "Stochastic models and variable returns to scales in data envelopment analysis," *European Journal of Operational Research* 104, 532-548.

- [11] Seiford, L., Data envelopment analysis: The evolution of the state of the art, *Journal of Productivity Analysis*, 1996; 7: 99-137.
- [12] Tone, K., A slacks-based measure of efficiency in data envelopment analysis, *European Journal of Operational Research*, 2001; 130(3): 498-509.
- [13] Zadeh, L.A., Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems*, 1978; 1: 3-28.
- [14] Zhu, J., *Quantitative models for performance evaluation and benchmarking: data envelopment analysis with spreadsheets and DEA excel solver*, Kluwer Academic Publishers, Boston, 2003.
- [15] Zimmermann, H.J., *Fuzzy Set Theory and Its Application*, Kluwer Academic Publishers, London, 1996.