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The Comparison of Efficiency of Control Chart by Weighted Variance Method, Scaled Weighted Variance Method, Empirical Quantiles Method and Extreme-value Theory for Skewed Populations

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Abstract

The objective of this study is to compare the efficiency of control chart by Weighted Variance Method, Scaled Weighted Variance Method, Empirical Quantiles Method and Extreme-value Theory for skewed populations. The efficiencies of control chart are determined by average run length. The control chart in the study is $\overline{X}$ chart. Various values of the coefficient of skewness are 0.1, 0.5, 1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0 and 9.0. Various values of the level of the mean shift equals to 0σ, 0.5σ, 1.0σ, 1.5σ, 2.0σ, 2.5σ, 3.0σ. The sample sizes are 3, 5 and 7. The data for the experiment are obtained through the Monte Carlo Simulation Technique and the experiment were constructed from 10,000 samples and repeated 1,000 times for each case. The result of the study is that the data have Weibull distribution at coefficient of skewness 0.1, 0.5, 1.0, 2.0 and 3.0. The Scaled Weighted Variance Method has the most efficiency sample size of 3 at coefficient of skewness 4.0, 5.0, 6.0, 7.0, 8.0 and 9.0. Extreme-value Theory

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has the most efficiency sample size of 3, with Lognormal distribution at coefficient of skewness 0.1, 0.5 and 1.0. The Weighted Variance Method has the most efficiency sample size of 3 at coefficient of skewness 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0 and 9.0. The Scaled Weighted Variance Method has the most efficiency sample size of 3, with Burr's distribution at coefficient of skewness 0.1 and 0.5. The Weighted Variance Method has the most efficiency sample size of 3, at coefficient of skewness 1.0, 2.0, 3.0, 4.0 and 5.0. The Scaled Weighted Variance Method has the most efficiency sample size of 3.

Keyword: average run length (ARL), control chart, empirical quantiles, extreme-value Theory (EV), skewness, scaled weighted variance (SWV), weighted variance (WV).

1. Introduction

A control chart originated in early 1920s, it has become a powerful tool in statistical process control (SPC). The control chart has two types : parametric and non-parametric control charts. The parametric one can be made immediately after data gathering, while the non-parametric one needs to have the average values and standard deviations of data simulated before making a chart. Non-parametric control chart is used for both unknown parametric distribution and normal distribution which are suitable for the large data set. Woodall and Montgomery [9] claimed that the distribution function has only one format called Unimodal with either increasing or decreasing density. This type of control chart depends on the model constructed by "Empirical Quantiles" which involves the Bootstrap method [6,7], Kernel estimators and Extreme-value theory [8]. Adisak et.al. [2] had studied the efficiency of control chart by weight method i.e., SWV, and WV, which considering the weight. Thus, it may not be the best way to construct a control chart.

This study present the methods to construct the control charts in case of non-normality distributions. We use WV, SWV, Empirical Quantiles and Extreme-value Theory methods for skewed populations in three different distributions: Weibull distribution, Lognormal distribution and Burr's distribution. We then compare the efficiencies of control charts constructed by four different methods above for each distribution.

2. Materials and methods

1. Distribution

1. Weibull distribution

Density function

$$f(x;\theta,\beta)=\frac{\beta}{\theta^{\beta}}x^{\beta-1}e^{-(x/\theta)^{\beta}} \qquad x>0$$

Mean

$$\mu=E(X)=\frac{\theta}{\beta}\Gamma\left(\frac{1}{\beta}\right)$$

Variance

$$\sigma^{2}=V(X)=\frac{\theta^{2}}{\beta}\left\{2\Gamma\left(\frac{2}{\beta}\right)-\frac{1}{\beta}\left[\Gamma\left(\frac{1}{\beta}\right)\right]^{2}\right\}$$

Where θ : scale parameter

β : shape parameter

In this study θ = 0.1, 0.5, 1, 2, 3, 4, 5, 6, 7, 8, 9 and

β = 3.2219, 2.211, 1.563, 1.0, 0.7686, 0.6478, 0.5737, 0.5237, 0.4873, 0.4596, 0.4376

that relevant with coefficient of skewness at $\alpha_3 \in \{0.1, 0.5, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

2. Lognormal distribution

Density function

$$f(x;\mu,\sigma)=\frac{1}{x\sigma(2\pi)^{1/2}}\exp\left[\frac{-(\ln x-\mu)^{2}}{2\sigma^{2}}\right],\ x>0, \qquad (1)$$

Mean

$$\mu=E(X)=e^{\mu+\frac{\sigma^{2}}{2}}$$

Variance

$$\sigma^{2}=V(X)=e^{2\mu+\sigma^{2}}\left(e^{\sigma^{2}}-1\right)$$

Where $\exp(\mu)$: scale parameter

σ : shape parameter

In this study μ = 0.1, 0.5, 1, 2, 3, 4, 5, 6, 7, 8, 9 and

σ = 0.0334, 0.1641, 0.3142, 0.5513, 0.7156, 0.8326, 0.9202, 0.9889, 1.0446, 1.0911, 1.1307

that relevant with coefficient of skewness at

$\alpha_3 \in \{0.1, 0.5, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

3. Burr's distribution

Density function

$$f(x) = \begin{cases} \dfrac{kcx^{c-1}}{\left(1 + x^c\right)^{k+1}} & , \quad x \ge 0 \\ 0 & , \quad x < 0 \end{cases} \tag{2}$$

Cumulative function

$$F(x) = 1 - \frac{1}{(1 + x^c)^k}$$

Let

$$M_j = \frac{\Gamma\left(\frac{j+1}{c}\right)\Gamma\left(k - \frac{j+1}{c}\right)}{c\Gamma(k)}$$

Mean

$$\mu = E(X) = M_1$$

Variance

$$\sigma^2 = V(X) = M_2$$

2. Control chart

2.1 Weighted Variance : WV Control Charts

The control charts, Choobinech and Ballard [4] proposed the theory of WV method for skewness distribution data. This theory separate distribution into two parts which are the mean of process and another one for constructing symmetrical distribution. These distributions have the same mean but difference standard deviation. Hence,

$\overline{X}$ Control Chart is:

Upper Control Limit: $UCL_{\overline{X}} = \overline{\overline{X}} + W_U \overline{R}$

Central Limit: $CL_{\overline{X}} = \overline{\overline{X}}$

Lower Control Limit: $LCL_{\overline{X}} = \overline{\overline{X}} - W_L \overline{R}$

Where W_U and W_L are the constant based on the sample size and $\hat{P}_X$ estimator

$$\hat{P}_X = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n} \delta\left(\overline{\overline{X}} - X_{ij}\right)}{n \times k} \tag{3}$$

$$\delta(X) = \begin{cases} 1 & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

2.2 Scaled Weighted Variance : SWV $\overline{X}$ Control Chart

Castagliola [3] said that the scale weighted variance method separate function into two parts $f_L(x)$ and $f_u(x)$ which are $\psi(x, \mu, \sigma_L', 2P_x)$ and which are $\psi(x; \mu, \sigma_L', 2P_X)$ and $\psi(x; \mu, \sigma_U', 2(1 - P_X))$ respectively with Bell-shape function. The center of bell shape function is μ and the second moments are $\sigma_L'^2$ and $\sigma_U'^2$ and the area under curves are equal to $2P_x$ and $2(1 - P_x)$.

Bell – shape probability function is

$$\psi(x,\mu,t,\kappa) = \frac{\kappa^{3/2}}{t}\varphi\left(\frac{(x-\mu)\sqrt{\kappa}}{t}\right)$$

So, the control chart is

Upper Control Limit: $$UCL_{\overline{X}} = \overline{\overline{X}} + \frac{W_U}{3}\sqrt{\frac{1}{2\left(1-\hat{P}_X\right)}}\Phi^{-1}\left(1-\frac{\alpha}{4\left(1-\hat{P}_X\right)}\right)\overline{R}$$

Central Limit: $$CL_{\overline{X}} = \overline{\overline{X}}$$

Lower Control Limit: $$LCL_{\overline{X}} = \overline{\overline{X}} - \frac{W_L}{3}\sqrt{\frac{1}{2\hat{P}_X}}\ \Phi^{-1}\left(1-\frac{\alpha}{4\hat{P}_X}\right)\overline{R} \qquad (4)$$

Where W_U and W_L are the constant of WV method ; α is type I error.

2.3 Empirical Quantile $\overline{X}$ Control Chart

The bootstrap method, introduced by Efron [6], is a powerful tool for estimating the sampling distribution of statistic. Let X_1^*, X_2^*, …, X_N^* be an independent and identically distributed sample with mean and variance. The standard bootstrap procedure is to draw with replacement a random sample of size N from X_1 , X_2 ,… , X_N .Let x_1^*, x_2^* , … , x_n^* are Bootstrap Sample

$\overline{X}_n^*$ is mean of subgroup

$\overline{\overline{X}}_N^*$ is mean of Bootstrap Sample

S_N^* is standard deviation of Bootstrap Sample

F_N is distribution Empirical Quantiles of x_1^*, x_2^* , … , x_n^*

Let N = kn, with is n the subgroup size and k is the number of subgroups. Because the $\overline{X}$ chart plots the subgroup sample means, the control limits should be obtained from $\frac{\alpha}{2}$ and $1-\frac{\alpha}{2}$ quantiles of the sampling distribution of $\sqrt{n}(\overline{X}_N^* - \overline{\overline{X}}_N)$. Hence, this sampling distribution can be approximated by bootstrap from any observation.

$$P(\sqrt{n}(\overline{X}_n^* - \overline{\overline{X}}_N) \le x \mid F_N) \approx P(\sqrt{n}(\overline{X}_n - \mu) \le x \mid F) \quad (5)$$

From equation (5) leads to an alternative approach to constructing and $\overline{X}$ chart for iid observation by repeating the bootstrap procedure k times and form a histogram of the resulting k terms of $\sqrt{n}(\overline{X}_N^* - \overline{\overline{X}}_N)$, and then locate the $\frac{\alpha}{2}$ and $1-\frac{\alpha}{2}$ quantiles.

These are then used as the estimated $\frac{\alpha}{2}$ and $1-\frac{\alpha}{2}$ quantiles to obtain $\tau_{\alpha/2}$ So,

$$\alpha/2 = P(\sqrt{n}(\overline{X}_n^* - \overline{\overline{X}}_N) \le \tau_{\alpha/2} \mid F_N)$$

$$\alpha/2 \approx P(\overline{X} \le \mu + \tau_{\alpha/2}/\sqrt{n} \mid F)$$

In summary, we conclude that the Empirical Quantiles control chart obtained by $\tau_{\alpha/2}$ which are constructed from repeating $\sqrt{n}(\overline{X}_N^* - \overline{\overline{X}}_N)$ of random sample of the distribution (As shown in appendix equation(16),(18),(20))

The control limits of Weibull distribution is

Upper Control Limit: $UCL = \overline{\theta} + \tau_{(1-\alpha/2)}/\sqrt{n}$

Central Limit: $CL = \overline{\theta}$

Lower Control Limit: $LCL = \overline{\theta} + \tau_{\alpha/2}/\sqrt{n}$ (6)

where

θ is sample mean of each subgroup

$\overline{\theta}$ is sample mean

$$\overline{\theta} = \frac{\sum_{i=1}^{k}\theta_i}{k}$$

k is the number of sample class.

The control limits of Lognormal distribution is

Upper Control Limit: $UCL = \overline{\mu} + \tau_{(1-\alpha/2)} / \sqrt{n}$

Central Limit: $CL = \overline{\mu}$

Lower Control Limit: $LCL = \overline{\mu} + \tau_{\alpha/2} / \sqrt{n}$ (7)

where

μ is sample mean of each subgroup

$\overline{\mu}$ is sample mean

$$\overline{\mu} = \frac{\sum_{i=1}^{k} \mu_i}{k} \quad (8)$$

k is the number of sample class.

The control limits of Burr's distribution is

Upper Control Limit: $UCL = \overline{k} + \tau_{(1-\alpha/2)} / \sqrt{n}$

Central Limit: $CL = \overline{k}$

Lower Control Limit: $LCL = \overline{k} + \tau_{\alpha/2} / \sqrt{n}$ (9)

where

k is sample mean of each subgroup

$\overline{k}$ is sample mean

$$\overline{k} = \frac{\sum_{i=1}^{k} k_i}{m} \quad (10)$$

m is the number of sample class

2.4 Extreme-value Theory $\overline{X}$ Control Chart

To obtain Extreme value of $\overline{X}$ Chart, it must be estimated $M_k^{(r)}$ by moment method (As shown in appendix (24), (26), (28)). $\tau\left(\frac{k+\beta-i}{\beta}\right)$ from 2-8 is the gamma function which

$$\tau(z) = \int_0^\infty t^{z-1} e^{-t} dt \ , \ z > 0$$

and z is the real number in gamma function table. Simulating data to estimate $\hat{\gamma}_k$ from equation (14), we obtain Extreme value theory control chart. Hence, The control limits of Weibull distribution is

Upper Control Limit:

$$UCL = X_{(k-m)} + \frac{(m/(kq))^{\hat{\gamma}_k} - 1}{\hat{\gamma}_k}(1 - (\hat{\gamma}_k \wedge 0))\, X_{(k-m)} M_k^{(1)}$$

Lower Control Limit:

$$LCL = X_{(m+1)} + \frac{(m/(k\alpha/2))^{\overline{\gamma}_k} - 1}{\overline{\gamma}_k}(1 - (\overline{\gamma}_k \wedge 0))\, X_{(m+1)} \overline{M}_k^{(1)} \qquad (11)$$

where

$$\overline{M}_k^{(r)} = \frac{\sum_{k=1}^{n} M_k^{(r)}}{n}$$

n is the number of class.

The control limits of Lognormal distribution is

Upper Control Limit :

$$UCL = X_{(k-m)} + \frac{(m/(kq))^{\hat{\gamma}_k} - 1}{\hat{\gamma}_k}(1 - (\hat{\gamma}_k \wedge 0))\, X_{(k-m)} M_k^{(1)}$$

Lower Control Limit:

$$LCL = X_{(m+1)} + \frac{(m/(k\alpha/2))^{\overline{\gamma}_k} - 1}{\overline{\gamma}_k}(1-(\overline{\gamma}_k \wedge 0))X_{(m+1)}\overline{M}_k^{(1)} \quad (12)$$

where

$$\overline{M}_k^{(r)} = \frac{\sum_{k=1}^{n} M_k^{(r)}}{n}$$

n is the number of class.

The control limits of Burr's distribution is

Upper Control Limit:

$$UCL = X_{(k-m)} + \frac{(m/(kq))^{\hat{\gamma}_k} - 1}{\hat{\gamma}_k}(1-(\hat{\gamma}_k \wedge 0))X_{(k-m)}M_k^{(1)}$$

Lower Control Limit:

$$LCL = X_{(m+1)} + \frac{(m/(k\alpha/2))^{\overline{\gamma}_k} - 1}{\overline{\gamma}_k}(1-(\overline{\gamma}_k \wedge 0))X_{(m+1)}\overline{M}_k^{(1)} \quad (13)$$

where

$$\overline{M}_k^{(r)} = \frac{\sum_{k=1}^{n} M_k^{(r)}}{n}$$

$$\hat{\gamma}_k = M_k^1 + 1 - \frac{1}{2}\left\{1 - \frac{(M_k^{(1)})^2}{M_k^{(2)}}\right\}^{-1} \quad (14)$$

$$\overline{\gamma}_k = \overline{M}_k^1 + 1 - \frac{1}{2}\left\{1 - \frac{(\overline{M}_k^{(1)})^2}{\overline{M}_k^{(2)}}\right\}^{-1}$$

n is the number of class.

The sample sizes of this study are 3, 5 and 7. The value of the coefficient of skewness. $\alpha_3 \in \{0.1, 0.5, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The values of the level of the mean shift equal to 0σ to 3σ. The results of this study are simulated under:

1. For WV method, Let α = 0.0027 for comparison the efficiency of control chart with Weibull distribution, lognormal distribution and burr's distribution.
2. For SWV method, Let α = 0.0027 for comparison the efficiency of control chart with Weibull distribution, lognormal distribution and Burr's distribution.
3. For Empirical using Weibull distribution, lognormal distribution and Burr's distribution for method comparison the efficiency of control chart.
4. For Extreme method, let α = 0.0027 for comparison the efficiency of control chart with Weibull distribution, lognormal distribution and Burr's distribution.

3. Results and discussion

3.1 The propose of this study is to compare the efficiency of control chart by WV, SWV for skewed populations i.e., Weibull distribution, lognormal distribution and burr's distribution. From the study, the results are as a the following;

Weibull distribution data of skewness more than one has the shape of curve like normal distribution and right skew, Shewhart chart gives the same result**.** Extreme-value is good detect data which agree with M.B.Vermaat et.al. [8], is an extreme-value appropriate non-normal distribution. At coefficient slightly less than or equal to 1 the shape of curve like exponential distribution, studied weight variance method which agree with Adisak et.al.. [2]. Examine control chart is appeared at coefficient 0.1, 0.5, 1.0, 2.0 and 3.0 by SWV method is efficient with most width of control limits and ARL but at coefficients 4.0, 5.0, 6.0, 7.0, 8.0 and 9.0 by EV method is efficiency with most width of control limits and ARL is maximum, see Figure 1.

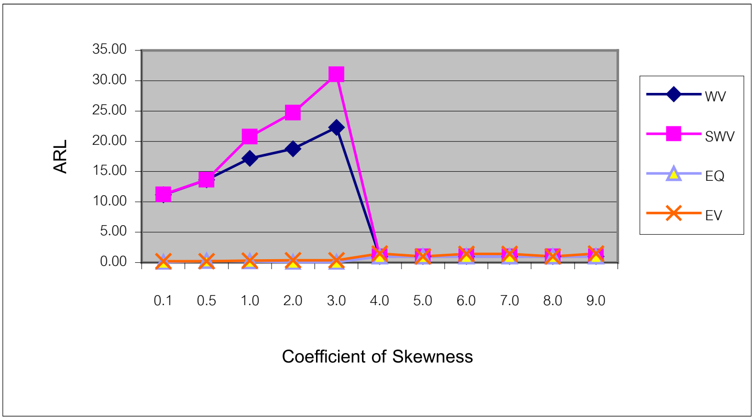

Figure1. Comparison ARL of Average Control Chart (n=3:0 σ) between WV, SWV,EQ and EV for data from Weibull distribution.

Lognormal distribution data of skewness more than one has the shape of the curve like normal distribution and right skew, Shewhart chart give the same result, but detect data not good as non-normal distribution theory which agree with Adisak et.al.. [1] .Examine control chart is appeared at coefficients 0.1, 0.5 and 1.0 by WV method is efficient with most width of control limits and ARL is maximum, but at coefficients 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0 and 9.0 by WV method is efficient with most width of control limits and ARL is maximum, see Figure 2.

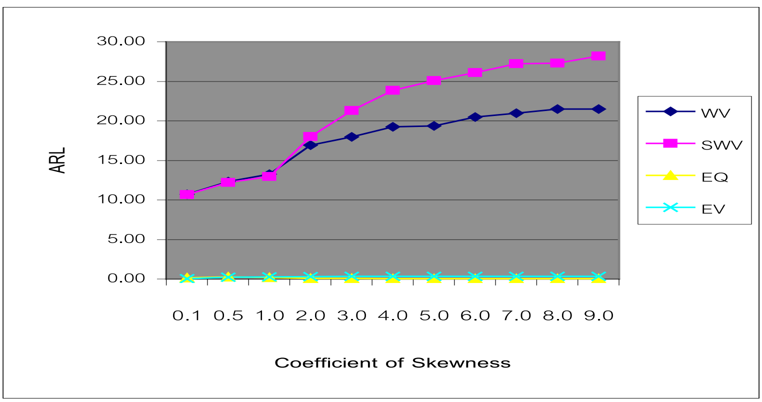

Figure 2. Comparison ARL of Average Control Chart (n=3:0 σ) between WV, SWV,EQ and EV for data from Lognormal distribution.

Burr's distribution data of skewness more than one has the very skew shape of curve, when k increase shape of curve like Weibull distribution lead to weight variance method which agree with Adisak et al. [2]. Examine control chart is appeared at coefficients 0.1 and 0.5 by WV method is efficient with most width of control limits and ARL is maximum, but at coefficients 1.0, 2.0, 3.0, 4.0 and 5.0 by SWV method is efficient with most width of control limits and with maximum ARL, see Figure 3.

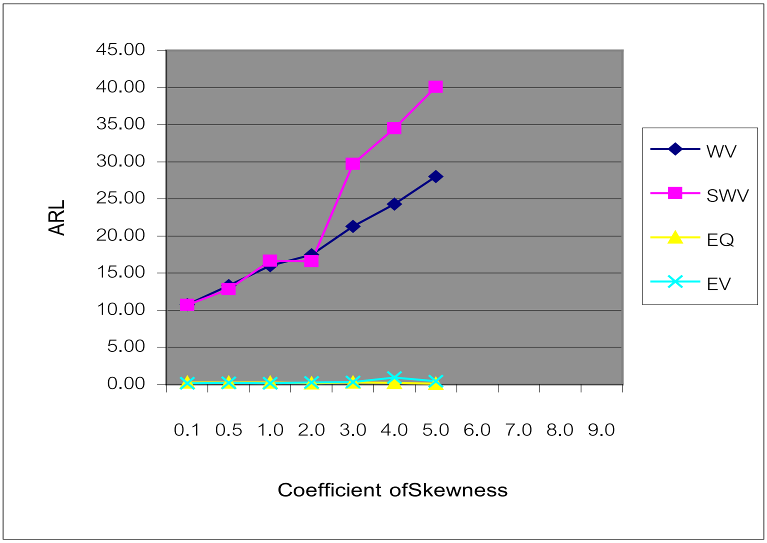

Figure 3. Comparison ARL of Average Control Chart (n=3:0 σ) between WV, SWV,EQ and EV for data from Burr's distribution.

3.2 Data are shifted right skewed increasing and average run length decrease. Then control chart increase in efficiency .In this study Weibull distribution with coefficients of skewness 0.1, 0.5, 1.0, 2.0 and 3.0, by Scaled Weighted Variance Method is the most efficient. At coefficients of skewness 0.4, 5.0, 6.0, 7.0, 8.0 and 9.0, Extreme-value Theory is the most efficient. Data with Lognormal distribution at coefficients of skewness 0.1, 0.5, 3.0 and 1.0, Weighted Variance Method has the most efficient. At coefficients of skewness 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0 and 9.0, Scaled Weighted Variance Method is the most efficient. Data with Burr's distribution at coefficient of skewness 0.1 and 0.5, Weighted Variance Method is the most efficient at sample size of 3. At coefficients of skewness 1.0, 2.0, 3.0, 4.0 and 5.0, Scaled Weighted Variance Method is the most efficient.

4. Conclusion

Studied ($\overline{X}$) control charts are constructed by Weighted Variance Method, Scaled Weighted Variance Method, Empirical Quantiles Method and Extreme-Value Theory for skewed populations. We found that all distributions give the best results when each sample size is three and the results are as follows:

- Under Weibull distribution

Scaled weighted variance method is the most efficient one when coefficients of skewness are 0.1, 0.5, 1.0, 2.0 and 3.0. At the rest of coefficients of skewness: 4.0, 5.0, 6.0, 7.0, 8.0 and 9.0, extreme-value theory is the most efficient.

- Under Log-normal distribution

Weighted variance method is the most efficient one when coefficients of skewness are 0.1, 0.5, and 1.0. At the rest coefficients of skewness: 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0 and 9.0, scaled weighted variance method is the most efficient.

- Under Burr's distribution

Weighted variance method is the most efficient one when coefficients of skewness are 0.1 and 0.5. At the rest coefficients of skewness: 1.0, 2.0, 3.0, 4.0 and 5.0, scaled weighted variance method is the most efficient.

SUGGESTIONS

1. For Weibull distribution data, we ought to use scaled weighted variance method if the coefficient of skewness is 0.1, 0.5, 1.0, 2.0 or 3.0, but use extreme-value method if the coefficient of skewness is 4.0, 5.0, 6.0, 7.0, 8.0 or 9.0.
2. For Log-normal distribution data, we ought to use weighted variance method if the coefficient of skewness is 0.1, 0.5 or 1.0, but use scaled weighted variance method if the coefficient of skewness is 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0 or 9.0.
3. For Burr's distribution data, we ought to use weighted variance method if the coefficient of skewness is 0.1 or 0.5, but use scaled weighted variance method if the coefficient of skewness is 1.0, 2.0, 3.0, 4.0, or 5.0.
4. We may study data under other distributions such as student's *t* distribution.
5. Control charts may be constructed by other methods such as Example by Cowden [5], and Kernel.

APPENDIX

Estimator of $\sqrt{n}(\overline{X}_N^* - \overline{\overline{X}}_N)$ of Weibull distribution

$$f(x;\theta,\beta) = \frac{\beta}{\theta^{\beta}} x^{\beta-1} e^{-(x/\theta)^{\beta}} \qquad ; x > 0 \tag{15}$$

$$L(x;\theta,\beta) = \frac{\beta^{n} \sum_{i=1}^{n} x_i^{\beta-1} e^{-\sum_{i=1}^{n}(\frac{x_i}{\theta})^{\beta}}}{\theta^{n\beta}} \tag{16}$$

$$\ln L(x;\theta,\beta) = n\ln\beta + (\beta-1)\ln\sum_{i=1}^{n} x_i - \sum_{i=1}^{n}(\frac{x_i}{\theta})^{\beta} - n\beta\ln\theta$$

$$\frac{\partial}{\partial\theta}\ln L(x;\theta,\beta) = -\beta\sum_{i=1}^{n}(\frac{x_i}{\theta})^{\beta-1}(\sum_{i=1}^{n}\frac{x_i}{-\theta^{2}}) - \frac{n\beta}{\theta}$$

$$= \frac{\beta}{\theta}\sum_{i=1}^{n}(\frac{x_i}{\theta})^{\beta} - \frac{n\beta}{\theta} \qquad = 0$$

$$\frac{\beta}{\theta}\sum_{i=1}^{n}(\frac{x_i}{\theta})^{\beta} = \frac{n\beta}{\theta}$$

$$\sum_{i=1}^{n}(\frac{x_i}{\theta})^{\beta} = \frac{n\beta}{\theta}\times\frac{\theta}{\beta} = n$$

$$\sum_{i=1}^{n}(\frac{x_i}{\theta}) = n^{1/\beta}$$

$$\theta = \frac{\sum_{i=1}^{n} x_i}{n^{1/\beta}} = \left(\frac{\sum_{i=1}^{n} x_i^{\beta}}{n}\right)^{1/\beta}$$

$$\frac{\partial}{\partial\beta}\ln L(x;\theta,\beta) = \frac{n}{\beta} + \sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right) - n\ln\theta = 0$$

$$= \frac{1}{\beta} + \frac{\sum_{i=1}^{n} \ln x_i}{n} - \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta} \ln\left(\frac{x_i}{\theta}\right) - \ln\theta = 0 \qquad (17)$$

Estimator of $\sqrt{n}\,(\overline{X}_N^{*} - \overline{\overline{X}}_N)$ of Lognormal distribution

$$f(x;\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\left(\frac{\ln x-\mu}{2\sigma^2}\right)^2} \quad ; \quad x > 0 \qquad (18)$$

$$L(x;\mu,\sigma) = \frac{1}{\sum_{i=1}^{n} x_i \sigma^n (2\pi)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(\ln x_i-\mu)^2} \qquad (19)$$

$$\ln L(x;\mu,\sigma) = -\ln\sum_{i=1}^{n} x_i - n\ln\sigma - \frac{n}{2}\ln(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(\ln x_i - \mu)^2$$

$$\frac{\partial}{\partial\mu}\ln L(x;\mu,\sigma) = \frac{\partial}{\partial\mu}\frac{1}{\sigma^2}\sum_{i=1}^{n}(\ln x_i - \mu) = 0$$

$$\sum_{i=1}^{n}(\ln x_i - \mu) = 0$$

$$n\mu = \sum_{i=1}^{n}\ln x_i$$

$$\mu = \frac{\sum_{i=1}^{n}\ln x_i}{n}$$

$$\frac{\partial}{\partial\sigma}\ln L(x;\mu,\sigma) = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(\ln x_i - \mu)^2 = 0$$

$$\frac{1}{\sigma^3}\sum_{i=1}^{n}(\ln x_i - \mu)^2 = \frac{n}{\sigma}$$

$$\sum_{i=1}^{n}(\ln x_i - \mu)^2 = n\sigma^2$$

$$\sigma^2 = \frac{\sum_{i=1}^{n} (\ln x_i - \mu)^2}{n}$$

$$\sigma = \left[\frac{\sum_{i=1}^{n} (\ln x_i - \mu)^2}{n} \right]^{1/2}$$

Estimator of $\sqrt{n}\,(\overline{X}_N^* - \overline{\overline{X}}_N)$ of Burr's distribution

$$f(x; c, k) = \frac{kcx^{c-1}}{(1 + x^c)^{k+1}} \quad ; \quad x \geq 0 \qquad (20)$$

$$L(x; c, k) = \frac{k^n c^n \sum_{i=1}^{n} x_i^{c-1}}{\sum_{i=1}^{n} (1 + x^c)^{k+1}} \qquad (21)$$

$$\ln L(x; c, k) = n \ln k + n \ln c + (c - 1) \ln \sum_{i=1}^{n} x_i - (k + 1) \ln \sum_{i=1}^{n} (1 + x^c)$$

$$\frac{\partial}{\partial k} \ln L(x; c, k) = \frac{n}{k} - \ln \sum_{i=1}^{n} (1 + x_i^c) = 0$$

$$\frac{n}{k} = \ln \sum_{i=1}^{n} (1 + x_i^c)$$

$$k = \frac{n}{\ln \sum_{i=1}^{n} (1 + x_i^c)}$$

$$\frac{\partial}{\partial c} \ln L(x; c, k) = \frac{n}{c} + \sum_{i=1}^{n} \ln x_i - (k + 1) \ln \sum_{i=1}^{n} (1 + x_i^c) x_i^c \ln x_i = 0$$

$$= \frac{1}{c} + \frac{\sum_{i=1}^{n} \ln x_i}{n} - \frac{(k+1)\ln \sum_{i=1}^{n} \left(1 + x_i^c\right) x_i^c \ln x_i}{n} = 0 \tag{22}$$

Estimator of $M_k^{(r)}$ of Weibull distribution

$$M_k^{(r)} = \left[E(x-\mu)^k\right]^r$$

By Binomial Theorem so

$$M_k^{(r)} = \left[\sum_{i=0}^{k} \binom{k}{i} (-\mu)^i E\left(x^{k-i}\right)\right]^r \tag{23}$$

$$E\left(x^{k-i}\right) = \int_0^\infty x^{k-i} \frac{\beta}{\theta^\beta} x^{\beta-1} e^{-(x/\theta)^\beta} dx$$

$$= \frac{\beta}{\theta^\beta} \int_0^\infty x^{k+\beta-i-1} e^{-(x/\theta)^\beta} dx$$

give $$y = \left(\frac{x}{\theta}\right)^\beta = \frac{x^\beta}{\theta^\beta}$$

$$x^\beta = y\theta^\beta$$

$$x = y^{1/\beta}\theta$$

$$dx = \frac{1}{\beta} y^{\frac{1}{\beta}-1} \theta dy$$

$$E\left(x^{k-i}\right) = \frac{\beta}{\theta^\beta} \int_0^\infty \left(y^{\frac{1}{\beta}}\theta\right)^{k+\beta-i-1} e^{-y} \frac{1}{\beta} y^{\frac{1}{\beta}-1} \theta dy$$

$$= \frac{1}{\theta^{\beta-1}} \int_0^\infty \left(y^{\frac{1}{\beta}}\theta\right)^{k+\beta-i-1} e^{-y} y^{\frac{1}{\beta}-1} dy$$

$$= \frac{\theta^{k+\beta-i-1}}{\theta^{\beta-1}} \int_0^\infty \left(y^{\frac{1}{\beta}} \right)^{k+\beta-i-1} e^{-y} y^{\frac{1}{\beta}-1} dy$$

$$= \theta^{k-i} \int_0^\infty \left(y^{\frac{k+\beta-i-1}{\beta}} \times y^{\frac{1}{\beta}-1} \right) e^{-y} dy$$

$$= \theta^{k-i} \int_0^\infty \left(y^{\frac{k+\beta-i-1}{\beta}+\frac{1}{\beta}-1} \right) e^{-y} dy$$

$$= \theta^{k-i} \int_0^\infty \left(y^{\frac{k+\beta-i}{\beta}-1} \right) e^{-y} dy$$

$$= \theta^{k-i} \tau\left(\frac{k+\beta-i}{\beta} \right)$$

from eqution (2-7) so

$$M_k^{(r)} = \left[\sum_{i=0}^{k} \binom{k}{i} (-\mu)^i \theta^{k-i} \tau\left(\frac{k+\beta-i}{\beta} \right) \right]^r \qquad (24)$$

Estimator of $M_k^{(r)}$ of Weibull distribution

$$M_k^{(r)} = \left[E(x-\mu)^k \right]^r$$

By Binomial Theorem so

$$M_k^{(r)} = \left[\sum_{i=0}^{k} \binom{k}{i} (-\mu)^i E\left(x^{k-i} \right) \right]^r \qquad (25)$$

$$E\left(x^{k-i} \right) = \int_0^\infty x^{k-i} \frac{1}{x \hat{\sigma} \sqrt{2\pi}} e^{-\left(\frac{\left(\ln x - \hat{\mu} \right)^2}{2 \hat{\sigma}^2} \right)} dx$$

give
$$y = \ln x$$
$$x = e^{y}$$
$$dx = e^{y} dy$$

so

$$E\left(x^{k-i}\right) = \int_{0}^{\infty} \left(e^{y}\right)^{k-i} \frac{1}{e^{y}\,\hat{\sigma}\sqrt{2\pi}} e^{-\left(\frac{\left(y-\hat{\mu}\right)^2}{2\hat{\sigma}^2}\right)} e^{y} dx$$

$$= M_y\left(k-i\right)$$

$$= e^{\hat{\mu}(k-i)+\frac{\hat{\sigma}^2 (k-i)^2}{2}}$$

from equation (25) so

$$M_k^{(r)} = \left[\sum_{i=0}^{k} \binom{k}{i} (-\mu)^i \theta^{k-i} e^{\hat{\mu}(k-i)+\frac{\hat{\sigma}^2 (k-i)^2}{2}}\right]^r \qquad (26)$$

Estimator of $M_k^{(r)}$ of Weibull distribution

$$M_k^{(r)} = \left[E\left(x-\mu\right)^k\right]^r$$

By Binomial Theorem so

$$M_k^{(r)} = \left[\sum_{i=0}^{k} \binom{k}{i} (-\mu)^i E\left(x^{k-i}\right)\right]^r \qquad (27)$$

$$E\left(x^{k-i}\right)=\int_0^\infty x^{k-i}\,\frac{mcx^{c-1}}{\left(1+x^c\right)^{m+1}}\,dx$$

give $k-i=v$

$$y=1+x^c \qquad 0<y<1$$

$$x=\left(\frac{1-y}{y}\right)^{\frac{1}{c}}$$

$$|J|=|dx|=\left|\frac{1}{c}\left(\frac{(1-y}{y}\right)^{\frac{1}{c}-1}\frac{1}{y^2}\,dy\right|$$

$$=\int_0^1 mcy^{m+1}\left(\frac{1-y}{y}\right)^{\frac{1}{c}(v+c-1)}\frac{1}{c}\left(\frac{1-y}{y}\right)^{\frac{1}{c}-1}\frac{1}{y^2}\,dy$$

$$=\int_0^1 my^{m+1}\left(\frac{1-y}{y}\right)^{\frac{v}{c}}\frac{1}{y^2}\,dy$$

$$=\int_0^1 my^{m+1-\frac{v}{c}-2}(1-y)^{\frac{v}{c}}\,dy$$

$$=\int_0^1 my^{m-\frac{v}{c}-1}(1-y)^{\frac{v}{c}+1-1}\,dy$$

$$=m\beta\left(m-\frac{v}{c},1+\frac{v}{c}\right)$$

from $v=k-i$

$$=m\beta\left(m-\frac{k-i}{c},1+\frac{k-i}{c}\right)$$

$$= \frac{m\tau\left(m - \frac{k-i}{c}\right)\tau\left(1 + \frac{k-i}{c}\right)}{\tau(m+1)}$$

from eqution (2-7) so

$$M_k^{(r)} = \left[\sum_{i=0}^{k}\binom{k}{i}(-\mu)^i \frac{m\tau\left(m - \frac{k-i}{c}\right)\tau\left(1 + \frac{k-i}{c}\right)}{\tau(m+1)}\right]^r \qquad (28)$$

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