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Second Order Slope Rotatable Designs under Intra-Class Correlated Error Structure Using Balanced Incomplete Block Designs

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Abstract

In this paper, following the works of Das [9] and Das et al. [10], construction of second order slope rotatable designs (SOSRD) under intra-class correlated error structure using balanced incomplete block designs (BIBD) is suggested. We may point out here that this SOSRD under intra-class correlated error structure using BIBD with parameters ($v=7$, $b=7$, $r=3$, $k=3$, $\lambda=1$) has only 71 design points for 7 factors, where as the corresponding SOSRD under intra-class correlated error structure using central composite designs (CCD) needs 79 design points. Thus this method sometimes leads to a SOSRD under intra-class correlated error structure using BIBD in less number of design points than the corresponding SOSRD under intra-class correlated error structure using CCD.

Keywords: second order slope rotatable designs (SOSRD), correlated errors.

1. Introduction

Box and Hunter [1] introduced rotatable designs for the exploration of response surface designs. Das and Narasimham [2] constructed second order rotatable designs (SORD) through balanced incomplete block designs (BIBD). Panda and Das [3] introduced first order rotatable designs with correlated errors. Das [4] introduced robust

second order rotatable designs (RSORD) assuming that the errors in the linear model are correlated. Das [5] examined the conditions of second order model when the errors have different variance-covariance structures.

In response surface methodology, good estimation of the derivatives of the response function may be as important or perhaps more important than estimation of mean response. Estimation of differences in responses at two different points in the factor space will often be of great importance. If difference in responses at two points close together is of interest then estimation of local slope (rate of change) of the response is required. Estimation of slopes occurs frequently in practical situations. For instance, there are cases in which we want to estimate rate of reaction in chemical experiment, rate of change in the yield of a crop to various fertilizer doses, rate of disintegration of radioactive material in animal etc, Park [6].

Hader and Park [7] introduced slope rotatable central composite designs (SRCCD). Park [6] studied a class of multifactor designs for estimating the slope of response surfaces. Victorbabu and Narasimham [8] constructed second order slope rotatable design (SOSRD) using BIBD. In agricultural experiments or chemical experiments generally authors are assuming that the errors are uncorrelated and homoscedastic. In some practical situations the assumptions of the linear model are not valid if the errors are correlated. In such situations, different correlated error structures may reduce the error in experimental designs. Das [9] introduced slope rotatability with correlated errors. Das et al. [10] studied on D-optimal robust second order slope rotatable designs (RSOSRD). In this paper, following the works of Das [9] and Das et al. [10], a study on SOSRDs under intra-class correlated error structure using BIBDs is suggested. The variance of the estimated slopes for different values of the intra-class correlation coefficient (ρ) and the distance from the centre (d) for "v factors ($3 \leq v \leq 8$)" are studied empirically. It is shown that this method sometimes leads to designs with less number of design points when we compared with the help of SOSRD under intra-class correlated error structure using CCD.

2. Conditions for SOSRD with uncorrelated errors

A second order response surface design $D = ((x_{ui}))$ for fitting

$$Y_u = b_0 + \sum_{i=1}^v b_i x_{ui} + \sum_{i=1}^v b_{ii} x_{ui}^2 + \sum_{i=1}^{v-1} \sum_{i < j=2}^v b_{ij} x_{ui} x_{uj} + e_u \quad (1)$$

where x_{ui} denotes the level of the i^{th} factor ($i=1,2,\dots,v$) in the u^{th} run ($u=1,2,\dots,N$) of the experiment, e_u 's are uncorrelated random errors with mean zero and variance σ^2 , is said to be a SOSRD if the variance of the estimate of first order partial derivative of $Y_u(x_1, x_2, \dots, x_v)$ with respect to each of independent variable (x_i) is only a function of the distance ($d^2 = \sum_{i=1}^v x_i^2$) of the point ($x_1, x_2, \dots, x_v$) from the origin (centre of the design). Such a spherical variance function for estimation of slopes in the second order response surface is achieved if the design points satisfy the following conditions [7, 8].

$$\sum_{u=1}^N \prod_{i=1}^v x_{ui}^{\alpha_i} = 0 \text{ if any } \alpha_i \text{ is odd, for } \sum \alpha_i \leq 4 \quad (2)$$

$$\sum_{u=1}^N x_{ui}^2 = \text{constant} = N\lambda_2 \quad (3)$$

$$\sum_{u=1}^N x_{ui}^4 = \text{constant} = cN\lambda_4, \text{ for all } i \quad (4)$$

$$\sum_{u=1}^N x_{ui}^2 x_{uj}^2 = \text{constant} = N\lambda_4, \text{ for all } i \neq j \quad (5)$$

$$\sum_{u=1}^N x_{ui}^4 = c \sum_{u=1}^N x_{ui}^2 x_{uj}^2 \quad (6)$$

$$\frac{\lambda_4}{\lambda_2^2} > \frac{v}{c+v-1} \quad (\text{non-singularity condition}) \quad (7)$$

$$\lambda_4[v(5-c) - (c-3)^2] + \lambda_2^2[v(c-5) + 4] = 0 \quad (\text{SOSRD condition}) \quad (8)$$

where c , λ_2 and λ_4 are constants and the summation is over the design points.

The variances and covariances of the estimated parameters are,

$$V(\hat{b}_0) = \frac{\lambda_4(c+v-1)\sigma^2}{N[\lambda_4(c+v-1) - v\lambda_2^2]}$$

$$V(\hat{b}_i) = \frac{\sigma^2}{N\lambda_2}$$

$$V(\hat{b}_{ij}) = \frac{\sigma^2}{N\lambda_4}$$

$$V(\hat{b}_{ii}) = \frac{\sigma^2[\lambda_4(c+v-2) - (v-1)\lambda_2^2]}{(c-1)N\lambda_4[\lambda_4(c+v-1) - v\lambda_2^2]}$$

$$\begin{aligned} \text{Cov}(\hat{b}_0, \hat{b}_{ii}) &= \frac{-\lambda_2 \sigma^2}{N[\lambda_4(c+v-1) - v\lambda_2^2]} \\ \text{Cov}(\hat{b}_{ii}, \hat{b}_{jj}) &= \frac{(\lambda_2^2 - \lambda_4) \sigma^2}{(c-1)N\lambda_4[\lambda_4(c+v-1) - v\lambda_2^2]} \end{aligned} \quad (9)$$

and other covariances are vanish.

$$\text{Further, } V\left(\frac{\partial \hat{y}_u}{\partial x_i}\right) = \frac{1}{N} \left(\frac{1}{\lambda_2} + \frac{d^2}{\lambda_4} \right) \sigma^2. \quad (10)$$

Result [8]: The design points,

$[1 - (v, b, r, k, \lambda)]2^{t(k)} \cup (a, 0, \dots, 0)2^1 \cup (n_0)$ will give a v -dimensional SOSRD in $N = b2^{t(k)} + 2v + n_0$ design points, where a^2 is a positive real root of the fourth degree polynomial equation,

$$\begin{aligned} (8v - 4N)a^8 + 8vr2^{t(k)}a^6 + [2vr^22^{2t(k)} + \{(12 - 2v)\lambda - 4r\}N + (16\lambda - 20v\lambda + \\ 4vr)2^{t(k)}]a^4 + [4vr^2 + (16 - 20v)r\lambda]2^{2t(k)}a^2 + [(5v - 9)\lambda^2 + (6 - v)r\lambda - r^2]N2^{2t(k)} + \\ [(vr + 4\lambda - 5v\lambda)r^22^{3t(k)}] = 0 \end{aligned} \quad (11)$$

Note: Values of SOSRD using BIBD can be obtained by solving the above equation.

3. Conditions for SOSRD under intra-class correlated error structure [9]

A second order response surface design $D = ((x_{ui}))$ for fitting

$$Y_u = b_0 + \sum_{i=1}^v b_i x_{ui} + \sum_{i=1}^v b_{ii} x_{ui}^2 + \sum_{i=1}^{v-1} \sum_{j=i+1}^v b_{ij} x_{ui} x_{uj} + e_u \quad (12)$$

where x_{ui} denotes the level of the i^{th} factor ($i=1,2,\dots,v$) in the u^{th} run ($u=1,2,\dots,N$) of the experiment, e_u 's are correlated random errors, is said to be a SOSRD under intra-class correlated error structure, if the variance of the estimate of first order partial derivative of $Y_u(x_1, x_2, \dots, x_v)$ with respect to each of independent variable (x_i) is only a function of the distance ($d^2 = \sum_{i=1}^v x_i^2$) of the point $(x_1, x_2, \dots, x_v)$ from the origin (centre of the design). Such a spherical variance function for estimation of slopes in the second order response surface is achieved if the design points satisfy the following conditions [9].

$$\sum_{u=1}^N \prod_{i=1}^v x_{ui}^{\alpha_i} = 0 \text{ if any } \alpha_i \text{ is odd, for } \sum \alpha_i \leq 4 \quad (13)$$

$$\sum_{u=1}^N x_{ui}^2 = \text{constant} = N\lambda_2 \quad (14)$$

$$\sum_{u=1}^N x_{ui}^4 = \text{constant} = cN\lambda_4, \text{ for all } i \quad (15)$$

$$\sum_{u=1}^N x_{ui}^2 x_{uj}^2 = \text{constant} = N\lambda_4, \text{ for all } i \neq j \quad (16)$$

$$\sum_{u=1}^N x_{ui}^4 = c \sum_{u=1}^N x_{ui}^2 x_{uj}^2 \quad (17)$$

$$\frac{\lambda_4}{\lambda_2^2} > \frac{v}{(c+v-1)} \quad (\text{non-singularity condition}) \quad (18)$$

$$\begin{aligned} & \left(\frac{AcN\lambda_4 - B}{(1-\rho)} \right) \left[4N - \left(\frac{AcN\lambda_4 - B}{A\lambda_4} \right) + v \left(\frac{N\lambda_2^2(1-\rho)}{A\lambda_4} \right) - (v-2) \left(\frac{AN\lambda_4 - B}{A\lambda_4} \right) \right] + \\ & \frac{N[AN\lambda_4 - B]}{(1-\rho)} \left[4(v-2) + (v-1) \left(\frac{AN\lambda_4 - B}{AN\lambda_4} \right) \right] - N^2\lambda_2^2 \left[4(v-1) + v \left(\frac{AN\lambda_4 - B}{AN\lambda_4} \right) \right] = 0 \quad (19) \end{aligned}$$

where c , λ_2 and λ_4 are constants and the summation is over the design points.

The variances and covariances of the estimated parameters are,

$$\begin{aligned} V(\hat{b}_0) &= \frac{[\lambda_4(c+v-1)A - v\rho N\lambda_2^2]\sigma^2 A}{N\Delta} \\ V(\hat{b}_i) &= \frac{\sigma^2(1-\rho)}{N\lambda_2} \\ V(\hat{b}_{ij}) &= \frac{\sigma^2(1-\rho)}{N\lambda_4} \\ V(\hat{b}_{ii}) &= \frac{\sigma^2(1-\rho)[\lambda_4(c+v-2)A - (v-1)\rho N\lambda_2^2 - (v-1)\lambda_2^2(1-\rho)]}{(c-1)N\lambda_4\Delta} \\ \text{Cov}(\hat{b}_0, \hat{b}_{ii}) &= -\frac{\lambda_2\sigma^2(1-\rho)A}{N\Delta} \\ \text{Cov}(\hat{b}_{ii}, \hat{b}_{jj}) &= \frac{\sigma^2(1-\rho)[\lambda_2^2(1-\rho) - \lambda_4A + N\rho\lambda_2^2]}{(c-1)N\lambda_4\Delta} \quad (20) \end{aligned}$$

where $A = \{1 + (N - 1)\rho\}$, $B = \rho N^2 \lambda_2^2$.

$\Delta = [\lambda_4(c + v - 1)A - v\rho N\lambda_2^2 - v\lambda_2^2(1 - \rho)]$ and other covariances are zero.

$$\text{Further, } V\left(\frac{\partial \hat{y}_u}{\partial x_i}\right) = \frac{(1 - \rho)}{N} \left(\frac{1}{\lambda_2} + \frac{d^2}{\lambda_4} \right) \sigma^2 \quad (21)$$

Note: For SOSRD with uncorrelated errors case, simply set $\rho=0$.

Intra-class correlation structure of errors: Intra-class structure is the variance-covariance structure, when errors of any two observations have the same correlation and each has the same variance. It is known that the SORD preserves the property of rotatability under the intra-class structure of errors [10].

4. Construction of SOSRD under intra-class correlated error structure using BIBD

Following the methods of constructions, of Victorbabu and Narasimham [8], Das [9] and Das et al. [10], construction of SOSRD under intra-class correlated error structure using BIBD is studied. Let ρ ($-\frac{1}{N-1} < \rho < 1$) be the correlation between errors of any two observations, each having the same variance σ^2 . In this paper we follow the notations of Das and Narasimham [2] and Victorbabu and Narasimham [8].

Let (v, b, r, k, λ) denote a BIBD, $2^{t(k)}$ denote a fractional replicate of 2^k in $+1$ and -1 levels, in which no interaction with less than five factors is confounded. $[1-(v, b, r, k, \lambda)]$ denote the design points generated from the transpose of incidence matrix of BIBD. $[1-(v, b, r, k, \lambda)]2^{t(k)}$ are the $b2^{t(k)}$ design points generated from the BIBD by "multiplication" [11], $(a, 0, 0, \dots, 0)2^1$ denote the design points generated from $(a, 0, 0, \dots, 0)$ point set, and $\cup$ denotes combination of the design points generated from different sets of points. n_0 denote the number of central points [2].

Theorem (4.1): The design points, $[1-(v, b, r, k, \lambda)]2^{t(k)} \cup (a, 0, 0, \dots, 0)2^1 \cup (n_0)$ will give a v -dimensional SOSRD under intra-class correlated error structure using BIBD in $N=b2^{t(k)} + 2v + n_0$ design points, where a^2 is positive real root of the fourth degree polynomial equation,

$$\begin{aligned}
& [(8v - 4N)]A^2a^8 + [8vr2^{t(k)}A^2]a^6 + \\
& [2vr^22^{2t(k)} + \{(12 - 2v)\lambda - 4r\}N2^{t(k)} + (16\lambda - 20v\lambda + 4vr)\}2^{t(k)}]A^2a^4 + \\
& [4vr^2 + (16 - 20v)r\lambda]A^22^{2t(k)}a^2 + [(5v - 9)\lambda^2 + (6 - v)r\lambda - r^2]A^2N2^{2t(k)} + \\
& (vr + 4\lambda - 5v\lambda)A^2r^22^{3t(k)} = 0.
\end{aligned} \tag{22}$$

where $A = \{1 + (N - 1)\rho\}$. If at least one positive real root for a^2 exists in (22) then the design exists.

Proof: For the design points generated from the BIBD, simple symmetry conditions are true. Further we have,

$$\sum_{u=1}^N x_{ui}^2 = r2^{t(k)} + 2a^2 = \text{constant} = N\lambda_2, \text{ for all } i \tag{23}$$

$$\sum_{u=1}^N x_{ui}^4 = r2^{t(k)} + 2a^4 = \text{constant} = cN\lambda_4, \text{ for all } i \tag{24}$$

$$\sum_{u=1}^N x_{ui}^2 x_{uj}^2 = \lambda 2^{t(k)} = \text{constant} = N\lambda_4, \text{ for all } i \neq j \tag{25}$$

$$\Rightarrow \lambda_2 = \frac{r2^{t(k)} + 2a^2}{N}, \lambda_4 = \frac{\lambda 2^{t(k)}}{N}, c = \frac{r2^{t(k)} + 2a^4}{\lambda 2^{t(k)}}, A = \{1 + (N - 1)\rho\}, B = \rho N^2 \lambda_2^2.$$

Substituting λ_2 , λ_4 , c , A and B in equation (19) and on simplification, we get equation (22). Solving equation (22) we get 'a' value for the 'v' different factors of SOSRD under intra-class correlated error structure using BIBD. The variance of estimated slope of the second order response surface designs under intra-class correlated error structure using BIBD for $0 \leq \rho \leq 0.9$ and for $3 \leq v \leq 8$ are given in Table 1.

Note: It is observed that for SOSRDs preserves the property of slope rotatability under the intra-class correlation structure of errors.

Example (4.1): We illustrate the above method with construction of SOSRD under intra-class correlated error structure for 3-factors with the help of a BIBD with parameters ($v=3$, $b=3$, $r=2$, $k=2$, $\lambda=1$). The design points, $[1 - (3, 3, 2, 2, 1)]2^2 \cup (a, 0, \dots, 0)2^1 \cup (n_0 = 1)$ will give a SOSRD under intra-class correlated error structure using BIBD in $N=19$ design points for three factors. We have,

No. of runs	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
x_1	1	1	-1	-1	0	0	0	0	1	1	-1	-1	a	-a	0	0	0	0	0
x_2	1	-1	1	-1	1	1	-1	-1	0	0	0	0	0	0	a	-a	0	0	0
x_3	0	0	0	0	1	-1	1	-1	1	-1	1	-1	0	0	0	0	a	-a	0

$$\sum_{u=1}^N x_{ui}^2 = 8 + 2a^2 = N\lambda_2 \quad (26)$$

$$\sum_{u=1}^N x_{ui}^4 = 8 + 2a^4 = c N\lambda_4 \quad (27)$$

$$\sum_{u=1}^N x_{ui}^2 x_{uj}^2 = 4 = N\lambda_4 \quad (28)$$

from (27) and (28), we get $c = \frac{8+2a^4}{4}$. Substituting for λ_2 , λ_4 and c in (19) and on simplification, we get the following biquadratic equation in a^2 (viz.)

$$(1 + 18\rho)^2(52a^8 - 192a^6 - 152a^4 + 640a^2 - 1152) = 0. \quad (29)$$

Equation (29) has only one positive real root $a^2 = 4.0000$ ($\forall \rho(-\frac{1}{N-1} < \rho < 1)$). It can be verified that condition (18) is also satisfied. This can be alternatively written directly from equation (22). Solving (22), we get $a = 2.0000$, $\forall \rho$. Substituting this value of 'a' in (26), (27) and (28), we obtain $\lambda_2 = 0.8421$, $\lambda_4 = 0.2105$ and $c = 10$. From (20), we can obtain the variances and covariances. Further from (21), we have,

$$V\left(\frac{\partial \hat{y}_u}{\partial x_i}\right) = (0.0625 + 0.2500d^2)(1 - \rho)\sigma^2.$$

Example (4.2):

We illustrate the above method with the construction of SOSRD under intra-class correlated error structure for 7- factors using BIBD with parameters ($v=7$, $b=7$, $r=3$, $k=3$, $\lambda=1$). The design points, $[1 - (7, 7, 3, 3, 1)]2^3 \cup (a, 0, \dots, 0)2^1 \cup (n_0 = 1)$ will give a

SOSRD under intraclass correlated error structure using BIBD in N=71 design points for seven factors.

$$\sum_{u=1}^N x_{ui}^2 = 24 + 2a^2 = N\lambda_2 \quad (30)$$

$$\sum_{u=1}^N x_{ui}^4 = 24 + 2a^4 = c N\lambda_4 \quad (31)$$

$$\sum_{u=1}^N x_{ui}^2 x_{uj}^2 = 8 = N\lambda_4 \quad (32)$$

From (31) and (32), we get $c = \frac{24+2a^4}{8}$. Substituting for λ_2 , λ_4 and c in (19) and on simplification, we get the biquadratic equation in a^2 (viz.)

$$(1 + 70\rho)^2(228a^8 - 1344a^6 + 208a^4 + 7680a^2 - 17536) = 0. \quad (33)$$

(33) has only one positive real root $a^2=4.9751$. It can be verified that non-singularity condition (18) is satisfied.

We may point out here that this SOSRD under intra-class correlated error structure using BIBD with parameters ($v=7$, $b=7$, $r=3$, $k=3$, $\lambda=1$) has only 71 design points for 7 factors, where as the corresponding SOSRD under intra-class correlated error structure using CCD needs 79 design points. Thus the method leads to a 7-factor SOSRD under intra-class correlated error structure using BIBD in less number of design points than the corresponding SOSRD under intra-class correlated error structure using CCD.

5. A study of dependence of the variance function of the response at different design points

In this section we study the dependence of variance function of response at different design points of SOSRD under intra-class correlated error structure using BIBD. Given 'v' factors and different values of intra-class correlation coefficient 'p' and distance 'd' between 0 and 1, the variances are tabulated.

From (3.10) the variance of the estimated derivative is obtained by,

$$V\left(\frac{\partial \hat{y}_u}{\partial x_i}\right) = V(\hat{b}_i) + 4x_i^2 V(\hat{b}_{ii}) + \sum_{j=1, j \neq i}^v x_j^2 V(\hat{b}_{ij}) \quad (34)$$

$$= \frac{\sigma^2(1-\rho)}{N\lambda_2} + \frac{\sigma^2(1-\rho)}{N\lambda_4} d^2 \quad (35)$$

The numerical calculations are given in Table 2.

6. Conclusions

In this paper, construction of SOSRD under intra-class correlated error structure using BIBD is suggested. The variance of the estimated slopes for different values of the intra-class correlation coefficient (ρ) and the distance from the centre (d) for “v factors ($3 \leq v \leq 8$)” are studied empirically. Further, it is observed that for SOSRDs preserves the property of slope rotatability under the intra-class correlated errors.

From the Table 1 and 2, we conclude that,

- (i) For given v and ρ , $V\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$ increases as d increases.
- (ii) For given v and d, $V\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$ decreases as ρ increases.
- (iii) We observed that, in some cases SOSRD under intra-class correlated error structure using BIBD method leads to designs with less number of design points compared to designs which constructed with the help of SOSRD under intra-class correlated error structure using CCD.

Table 1. The variance of estimated derivatives (slopes) for the factors $3 \leq v \leq 8$.

ρ	v=3, b=3, r=2, k=2, $\lambda=1$ N=19, a=2.000	v=4, b=6, r=3, k=2, $\lambda=1$ N=33, a=1.9350	v=5, b=10, r=4, k=2, $\lambda=1$ N=51, a=1.8836
	$V\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$	$V\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$	$V\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$
0	$0.0625\sigma^2 + 0.2500\sigma^2 d^2$	$0.0513\sigma^2 + 0.2500\sigma^2 d^2$	$0.0433\sigma^2 + 0.2501\sigma^2 d^2$
0.1	$0.0562\sigma^2 + 0.2250\sigma^2 d^2$	$0.0462\sigma^2 + 0.2250\sigma^2 d^2$	$0.0390\sigma^2 + 0.2251\sigma^2 d^2$
0.2	$0.0500\sigma^2 + 0.2000\sigma^2 d^2$	$0.0411\sigma^2 + 0.2000\sigma^2 d^2$	$0.0346\sigma^2 + 0.2001\sigma^2 d^2$
0.3	$0.0438\sigma^2 + 0.1750\sigma^2 d^2$	$0.0359\sigma^2 + 0.1750\sigma^2 d^2$	$0.0303\sigma^2 + 0.1751\sigma^2 d^2$
0.4	$0.0375\sigma^2 + 0.1500\sigma^2 d^2$	$0.0308\sigma^2 + 0.1500\sigma^2 d^2$	$0.0260\sigma^2 + 0.1501\sigma^2 d^2$
0.5	$0.0313\sigma^2 + 0.1250\sigma^2 d^2$	$0.0257\sigma^2 + 0.1250\sigma^2 d^2$	$0.0216\sigma^2 + 0.1251\sigma^2 d^2$
0.6	$0.0250\sigma^2 + 0.1000\sigma^2 d^2$	$0.0205\sigma^2 + 0.1000\sigma^2 d^2$	$0.0173\sigma^2 + 0.1000\sigma^2 d^2$
0.7	$0.0188\sigma^2 + 0.0750\sigma^2 d^2$	$0.0154\sigma^2 + 0.0750\sigma^2 d^2$	$0.0130\sigma^2 + 0.0750\sigma^2 d^2$
0.8	$0.0125\sigma^2 + 0.0500\sigma^2 d^2$	$0.0103\sigma^2 + 0.0500\sigma^2 d^2$	$0.0087\sigma^2 + 0.0500\sigma^2 d^2$
0.9	$0.0063\sigma^2 + 0.0250\sigma^2 d^2$	$0.0051\sigma^2 + 0.0250\sigma^2 d^2$	$0.0043\sigma^2 + 0.0250\sigma^2 d^2$

Table 1. (Continued).

ρ	$v=6, b=15, r=5, k=2, \lambda=1$ $N=73, a=1.8419$	$v=7, b=7, r=3, k=3, \lambda=1$ $N=71, a=2.2305$	$v=8, b=28, r=7, k=2, \lambda=1$ $N=129, a=1.7782$
	$V\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$	$V\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$	$V\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$
0	$0.0373\sigma^2+0.2500\sigma^2d^2$	$0.0295\sigma^2+0.1250\sigma^2d^2$	$0.0291\sigma^2+0.2501\sigma^2d^2$
0.1	$0.0336\sigma^2+0.2250\sigma^2d^2$	$0.0265\sigma^2+0.1125\sigma^2d^2$	$0.0262\sigma^2+0.2251\sigma^2d^2$
0.2	$0.0299\sigma^2+0.2000\sigma^2d^2$	$0.0236\sigma^2+0.1000\sigma^2d^2$	$0.0233\sigma^2+0.2001\sigma^2d^2$
0.3	$0.0261\sigma^2+0.1750\sigma^2d^2$	$0.0206\sigma^2+0.0875\sigma^2d^2$	$0.0204\sigma^2+0.1750\sigma^2d^2$
0.4	$0.0224\sigma^2+0.1500\sigma^2d^2$	$0.0177\sigma^2+0.0750\sigma^2d^2$	$0.0175\sigma^2+0.1500\sigma^2d^2$
0.5	$0.0187\sigma^2+0.1250\sigma^2d^2$	$0.0147\sigma^2+0.0625\sigma^2d^2$	$0.0146\sigma^2+0.1250\sigma^2d^2$
0.6	$0.0149\sigma^2+0.1000\sigma^2d^2$	$0.0118\sigma^2+0.0500\sigma^2d^2$	$0.0117\sigma^2+0.1000\sigma^2d^2$
0.7	$0.0112\sigma^2+0.0750\sigma^2d^2$	$0.0088\sigma^2+0.0375\sigma^2d^2$	$0.0087\sigma^2+0.0750\sigma^2d^2$
0.8	$0.0075\sigma^2+0.0500\sigma^2d^2$	$0.0059\sigma^2+0.0250\sigma^2d^2$	$0.0058\sigma^2+0.0500\sigma^2d^2$
0.9	$0.0037\sigma^2+0.0250\sigma^2d^2$	$0.0029\sigma^2+0.0125\sigma^2d^2$	$0.0029\sigma^2+0.0250\sigma^2d^2$

NB: $V\left(\frac{\partial \hat{y}_u}{\partial x_i}\right)$ is obtained by using the equation (35).

Table 2. Study of dependence of estimated slope of second order response surface design under intra-class correlation error structure using BIBD at different design points for $3 \leq v \leq 8$ for different values of ' ρ ' and d .

(v=3, b=3, r=2, k=2, $\lambda=1$)										
ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0650	0.0725	0.0850	0.1025	0.1250	0.1525	0.1850	0.2225	0.2650	0.3125
0.1	0.0585	0.0652	0.0765	0.0922	0.1125	0.1372	0.1665	0.2002	0.2385	0.2812
0.2	0.0520	0.0580	0.0680	0.0820	0.1000	0.1220	0.1480	0.1780	0.2120	0.2500
0.3	0.0456	0.0508	0.0596	0.0718	0.0876	0.1068	0.1296	0.1558	0.1856	0.2188
0.4	0.0390	0.0435	0.0510	0.0615	0.0750	0.0915	0.1110	0.1335	0.1590	0.1875
0.5	0.0326	0.0363	0.0426	0.0513	0.0626	0.0763	0.0926	0.1113	0.1326	0.1563
0.6	0.0260	0.0290	0.0340	0.0410	0.0500	0.0610	0.0740	0.0890	0.1060	0.1250
0.7	0.0196	0.0218	0.0256	0.0308	0.0376	0.0458	0.0556	0.0668	0.0796	0.0938
0.8	0.0130	0.0145	0.0170	0.0205	0.0250	0.0305	0.0370	0.0445	0.0530	0.0625
0.9	0.0066	0.0073	0.0086	0.0103	0.0126	0.0153	0.0186	0.0223	0.0266	0.0313
(v=4, b=6, r=3, k=2, $\lambda=1$)										
ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0538	0.0613	0.0738	0.0913	0.1138	0.1413	0.1738	0.2113	0.2538	0.3013
0.1	0.0485	0.0552	0.0665	0.0822	0.1025	0.1272	0.1565	0.1902	0.2285	0.2712
0.2	0.0431	0.0491	0.0591	0.0731	0.0911	0.1131	0.1391	0.1691	0.2031	0.2411
0.3	0.0377	0.0429	0.0517	0.0639	0.0797	0.0989	0.1217	0.1479	0.1777	0.2109
0.4	0.0323	0.0368	0.0443	0.0548	0.0683	0.0848	0.1043	0.1268	0.1523	0.1808
0.5	0.0270	0.0307	0.0370	0.0457	0.0570	0.0707	0.0870	0.1057	0.1270	0.1507
0.6	0.0215	0.0245	0.0295	0.0365	0.0455	0.0565	0.0695	0.0845	0.1015	0.1205
0.7	0.0162	0.0184	0.0222	0.0274	0.0342	0.0424	0.0522	0.0634	0.0762	0.0904
0.8	0.0108	0.0123	0.0148	0.0183	0.0228	0.0283	0.0348	0.0423	0.0508	0.0603
0.9	0.0054	0.0061	0.0074	0.0091	0.0114	0.0141	0.0174	0.0211	0.0254	0.0301

Table 2. (Continued).

(v=5, b=10, r=4, k=2, $\lambda=1$)										
ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0458	0.0533	0.0658	0.0833	0.1058	0.1333	0.1658	0.2034	0.2459	0.2934
0.1	0.0413	0.0480	0.0593	0.0750	0.0953	0.1200	0.1493	0.1831	0.2213	0.2641
0.2	0.0366	0.0426	0.0526	0.0666	0.0846	0.1066	0.1326	0.1627	0.1967	0.2347
0.3	0.0321	0.0373	0.0461	0.0583	0.0741	0.0933	0.1161	0.1464	0.1721	0.2054
0.4	0.0275	0.0320	0.0395	0.0500	0.0635	0.0800	0.0995	0.1221	0.1476	0.1761
0.5	0.0229	0.0266	0.0329	0.0416	0.0529	0.0666	0.0829	0.1017	0.1229	0.1467
0.6	0.0183	0.0213	0.0263	0.0333	0.0423	0.0533	0.0663	0.0813	0.0983	0.1173
0.7	0.0138	0.0160	0.0198	0.0250	0.0318	0.0400	0.0498	0.0610	0.0738	0.0880
0.8	0.0092	0.0107	0.0132	0.0167	0.0212	0.0267	0.0332	0.0407	0.0492	0.0587
0.9	0.0046	0.0053	0.0066	0.0083	0.0106	0.0133	0.0166	0.0203	0.0246	0.0293
(v=6, b=15, r=5, k=2, $\lambda=1$)										
ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0398	0.0473	0.0598	0.0773	0.0998	0.1273	0.1598	0.1973	0.2398	0.2873
0.1	0.0359	0.0426	0.0539	0.0696	0.0899	0.1146	0.1439	0.1776	0.2159	0.2586
0.2	0.0319	0.0379	0.0479	0.0619	0.0799	0.1019	0.1279	0.1579	0.1919	0.2299
0.3	0.0279	0.0331	0.0419	0.0541	0.0699	0.0891	0.1119	0.1381	0.1679	0.2011
0.4	0.0239	0.0284	0.0359	0.0464	0.0599	0.0764	0.0959	0.1184	0.1439	0.1724
0.5	0.0200	0.0237	0.0300	0.0387	0.0500	0.0637	0.0800	0.0987	0.1200	0.1437
0.6	0.0159	0.0189	0.0239	0.0309	0.0399	0.0509	0.0639	0.0789	0.0959	0.1149
0.7	0.0120	0.0142	0.0180	0.0232	0.0300	0.0382	0.0480	0.0592	0.0720	0.0862
0.8	0.0080	0.0095	0.0120	0.0155	0.0200	0.0255	0.0320	0.0395	0.0480	0.0575
0.9	0.0040	0.0047	0.0060	0.0077	0.0100	0.0127	0.0160	0.0197	0.0240	0.0287

Table 2. (Continued).

(v=7, b=7, r=3, k=3, $\lambda=1$)										
ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0308	0.0345	0.0408	0.0495	0.0608	0.0745	0.0908	0.1095	0.1308	0.1545
0.1	0.0276	0.0310	0.0366	0.0445	0.0546	0.0670	0.0816	0.0985	0.1176	0.1390
0.2	0.0246	0.0276	0.0326	0.0396	0.0486	0.0596	0.0726	0.0876	0.1046	0.1236
0.3	0.0215	0.0241	0.0285	0.0346	0.0425	0.0521	0.0635	0.0766	0.0915	0.1081
0.4	0.0185	0.0207	0.0245	0.0297	0.0365	0.0447	0.0545	0.0657	0.0785	0.0927
0.5	0.0153	0.0172	0.0203	0.0247	0.0303	0.0372	0.0453	0.0547	0.0653	0.0772
0.6	0.0123	0.0138	0.0163	0.0198	0.0243	0.0298	0.0363	0.0438	0.0523	0.0618
0.7	0.0092	0.0103	0.0122	0.0148	0.0182	0.0223	0.0272	0.0328	0.0392	0.0463
0.8	0.0062	0.0069	0.0082	0.0099	0.0122	0.0149	0.0182	0.0219	0.0262	0.0309
0.9	0.0030	0.0034	0.0040	0.0049	0.0060	0.0074	0.0090	0.0109	0.0130	0.0154
(v=8, b=28, r=7, k=2, $\lambda=1$)										
ρ	d=0.1	d=0.2	d=0.3	d=0.4	d=0.5	d=0.6	d=0.7	d=0.8	d=0.9	d=1
0	0.0316	0.0391	0.0516	0.0691	0.0916	0.1191	0.1516	0.1892	0.2317	0.2792
0.1	0.0285	0.0352	0.0465	0.0622	0.0825	0.1072	0.1365	0.1703	0.2085	0.2513
0.2	0.0253	0.0313	0.0413	0.0553	0.0733	0.0953	0.1213	0.1514	0.1854	0.2234
0.3	0.0222	0.0274	0.0362	0.0484	0.0642	0.0834	0.1062	0.1324	0.1622	0.1954
0.4	0.0190	0.0235	0.0310	0.0415	0.0550	0.0715	0.0910	0.1135	0.1390	0.1675
0.5	0.0159	0.0196	0.0259	0.0346	0.0459	0.0596	0.0759	0.0946	0.1159	0.1396
0.6	0.0127	0.0157	0.0207	0.0277	0.0367	0.0477	0.0607	0.0757	0.0927	0.1117
0.7	0.0095	0.0117	0.0155	0.0207	0.0275	0.0357	0.0455	0.0567	0.0695	0.0837
0.8	0.0063	0.0078	0.0103	0.0138	0.0183	0.0238	0.0303	0.0378	0.0463	0.0558
0.9	0.0032	0.0039	0.0052	0.0069	0.0092	0.0119	0.0152	0.0189	0.0232	0.0279

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