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Bayesian Estimation of Scale Parameter of Pareto Type-I Distribution by Two Different Methods

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Abstract

Bayesian estimators of the scale parameter of Pareto type I model have been obtained by direct method and Lindley's approach. Further, the expressions for posterior expected loss under squared error loss function (SELF) and asymmetric precautionary loss function (APLF) are obtained. The calculations have been illustrated with the help of numerical example.

Keywords: Bayes estimator, maximum likelihood estimator, prior, scale parameter, squared error loss function and asymmetric precautionary loss function.

1. Introduction

Bayesian statistics, named after the Revd. Thomas Bayes (1702-1761), represents a different approach to statistical inference. Bayesians consider unknown parameters to be random variables and assign prior distributions for the parameters. After this the prior distribution is multiplied by the likelihood to give the posterior distribution of the parameter.

The Pareto distribution was first known as a model for the distribution of income. Later on, it was applied in various other fields such as biological, demographic, economic, linguistic, and sociological etc.

The probability density function of Pareto type I distribution is given by,

$$f\left(x|\theta,p\right)=\frac{p\theta^{p}}{x^{p+1}}\quad p,\theta>0\quad \theta<x<\infty,$$
$$=0\qquad \text{otherwise}\,, \tag{1}$$

or

$$f\left(x|\theta,p\right)=\left(\frac{p\theta^{p}}{x^{p+1}}\right)I_{(\theta,\infty)}\left(x\right)\quad p,\theta>0,$$

where p is the shape parameter and θ is the scale parameter of the distribution.

Ashour et al. (1994) used the quasi-likelihood function to derive Bayesian and non-Bayesian estimates for the unknown parameters of the Pareto distribution. Bodhisuwan and Nanuwong (2014) developed a new four-parameter beta length-biased Pareto distribution, studied various properties of this distribution and obtained estimates of the parameters of the distribution. Finally, the new distribution was applied to Norwegian fire claims data.

Ertefaie and Parsian (2005) dealt with Bayes estimation of the parameters of interest under an asymmetric LINEX loss function, using suitable choice of priors when the scale parameter was known and unknown. Hosking and Wallis (1987) explained that the generalized Pareto distribution is a two-parameter distribution that contains uniform, exponential and Pareto distributions as special cases. It has applications in a number of fields, including reliability studies and the analysis of environmental extreme events.

Howlader and Hossain (2002) presented Bayesian estimation of the survival function of the Pareto distribution of the second kind using the methods of Lindley (1980) and Tierney and Kadane (1986). Kifayat et al. (2012) performed Bayesian analysis of the power distribution using two informative (gamma and Rayleigh) priors and two non-informative (Jeffreys and uniform) priors.

Liang (1993) studied the Pareto distribution with known shape parameter and unknown scale parameter. He studied the problem of estimating the scale parameter under a squared-error loss through the nonparametric empirical Bayes approach.

Lindley (1980) concluded that the Bayes estimators are often obtained as a ratio of two integrals which cannot be expressed in closed forms, consequently numerical approximation are needed. He developed an asymptotic approximation to such ratios, which is of paramount importance in finding out the Bayes estimators of parameters and their functions in such situation. Preda and Ciumara (2007) studied the problem of estimating the scale parameter θ for a Pareto distribution under a weighted squared-error loss through the empirical Bayes approach, proposed an empirical Bayes estimator and gave some asymptotic optimality properties.

Setiya and Kumar (2013) obtained the Bayes estimators of the shape parameters of a Pareto type-I model for different priors using Square Error and Asymmetric Precautionary Error Loss Functions through direct method and Lindley's approach. Bayes estimators of reliability and hazard rate functions have also been obtained. Comparison between Squared Error and Asymmetric Precautionary Error Loss Functions have also been shown with the help of a numerical example. Tierney and Kadane (1986) described approximations to posterior means and variances of positive functions of a real or vector-valued parameter and to the marginal posterior densities of arbitrary parameters. Wang (2005) proposed a criterion of choosing which tells us how to choose a loss function in Bayesian analysis.

In this paper we have obtained Bayes estimates of the scale parameter of Pareto type I model under different priors by using direct method (Section 2.2) and Lindley's approach (Section 2.3). The expressions for posterior expected loss under squared error loss function (SELF) and asymmetric precautionary loss function (APLF) have also been obtained. The comparative study reveals that Mukherjee-Islam's prior with increasing value of its shape parameter α is more suitable prior for estimating scale parameter of Pareto type I distribution for a fixed value of its shape parameter under both methods. This has been illustrated with the help of an example in a simpler manner, which may encourage the researchers carrying out the research work in the relevant field in future.

1.1. Moments

r^{th} moment of X about origin is

$$\mu'_r = E\left(X^r\right) = \frac{p\theta^r}{(p-r)}, \qquad (p > r)$$

$$\mu'_1 = E(X) = \frac{p\theta}{(p-1)}. \qquad (p > 1)$$

Central Moments

$$\mu_2 = Var(X) = \frac{p\theta^2}{(p-1)^2(p-2)}, \qquad (p > 2)$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1,$$

$$= 2\theta^3\left[\frac{p(p+1)}{(p-1)^3(p-2)(p-3)}\right], \qquad (p > 3)$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1,$$

$$= \frac{3\theta^4 p}{(p-1)^4(p-2)(p-3)(p-4)}\left(3p^2 + p + 2\right). \qquad (p > 4)$$

1.2. Karl Pearson's coefficients of skewness and kurtosis

Karl Pearson's Coefficients of Skewness (γ_1) and Kurtosis (γ_2) are given by-

$$\gamma_1 = \frac{2(p+1)}{(p-3)}\sqrt{\frac{p-2}{p}}, \qquad (p > 3)$$

$$\gamma_2 = \frac{6\left(p^3 + p^2 - 6p - 2\right)}{p(p-3)(p-4)}. \qquad (p > 4)$$

Here, γ_1 and γ_2 both are positive, showing that the distribution is positively skewed and leptokurtic.

1.3. Loss function

A loss function shows the associated risk with an event. The estimator having the least expected loss is preferable compared to others. There are different forms of loss function. From Bayesian point of view, selection of loss function is very crucial part in the estimation problems, since there is not any definite methodology to identify the appropriate loss function to be used. In most of the cases, in estimation problems authors for convenience consider the underlying loss function to be squared error which is symmetric in nature. Squared error loss function gives equal weightage to over estimation and under estimation. The squared error loss function is most suited when the loss is symmetric. One advantage of SELF is that it is simple to calculate. But in many situations the loss occurred is not always symmetric, in these cases, it is reasonable to use an alternative asymmetric precautionary loss function, which approaches infinity near the origin to prevent underestimation, thus giving conservative estimators especially when underestimation may lead to serious consequence. In this paper, we have used squared error loss function (SELF) and asymmetric precautionary loss function (APLF).

Squared Error Loss Function (SELF)

A commonly used loss function is the squared error loss function (SELF)

$$L(\theta_B,\theta)=(\theta_B-\theta)^2. \tag{2}$$

Asymmetric Precautionary Error Loss Function (APLF)

A very useful and simple asymmetric precautionary loss function is

$$L(\theta_B,\theta)=\frac{(\theta_B-\theta)^2}{\theta_B}. \tag{3}$$

2. Bayes Estimators of θ Given p under Different Priors

In this section, we have obtained Bayes estimators of θ given p under different priors using direct method and Lindley's approach. It is always comfortable to use direct method if the integrals used for finding posterior distribution and Bayes estimators can be evaluated and may be reduced to closed form. In situation, where these integrals cannot be evaluated, Lindley's or some other approximate method is more useful.

3. Estimation of Parameters (Direct Method)

Likelihood function of the above said distribution is given by

$$l(\theta|\underset{\sim}{x})=\prod_{i=1}^{n}\left(\frac{p\theta^p}{x_i^{p+1}}\right)I_{(\theta,\infty)}(x_i).$$

Let $Y_1,Y_2,\ldots,Y_n$ be the order statistics corresponding to $X_1,X_2,\ldots,X_n$

$$0\le\theta\le Y_1\le Y_2\le\ldots\le Y_n\le\infty,$$

$$l(\theta|\underset{\sim}{x})=\prod_{i=1}^{n}\left(\frac{p\theta^p}{x_i^{p+1}}\right)I_{(\theta,Y_1)}(\theta). \tag{4}$$

Then the posterior distribution of θ is

$$f(\theta|\underset{\sim}{x})=\frac{l(\theta|\underset{\sim}{x})g(\theta)}{\int l(\theta|\underset{\sim}{x})g(\theta)d\theta}. \tag{5}$$

Bayes estimator of θ give p is

$$\theta_B=E(\theta)=\int_0^{Y_1}\theta f(\theta|\underset{\sim}{x})d\theta. \tag{6}$$

(i) Jeffrey's prior

Jeffrey's prior for the parameter θ is given by

$$g(\theta)=\frac{1}{\theta}I_{(1,\mathrm{e})}(\theta).$$

Then the posterior distribution of θ is

$$f(\theta|\underset{\sim}{x})=\frac{\theta^{np-1}}{Y_1^{np}}np.$$

Hence the Bayes estimator (θ_B^J) of θ given p under Jeffrey's prior is

$$\theta_B^J=\frac{np}{np+1}Y_1. \tag{7}$$

(ii) Uniform prior

Uniform prior for the parameter θ is given by

$$g(\theta) = I_{(0,1)}(\theta).$$

Then the posterior distribution of θ is

$$f(\theta|\underset{\sim}{x}) = \frac{\theta^{np}}{Y_1^{np+1}}(np+1).$$

Hence the Bayes estimator (θ_B^U) of θ given p under uniform prior is

$$\theta_B^U = \frac{np+1}{np+2} Y_1. \tag{8}$$

(iii) Exponential prior

Exponential prior for the parameter θ is given by

$$g(\theta) = e^{-\theta}; \quad \theta > 0.$$

Then the posterior distribution of θ is

$$f(\theta|\underset{\sim}{x}) = \frac{\theta^{np} e^{-\theta} I_{(0,Y_i)}^{\theta}}{\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{Y_1^{np+i+1}}{(np+i+1)}}.$$

Hence the Bayes estimator (θ_B^E) of θ under exponential prior is given by

$$\theta_B^E = \frac{\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{Y_1^{i+1}}{(np+i+2)}}{\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{Y_1^{i}}{(np+i+1)}}. \tag{9}$$

(iv) Mukherjee-Islam prior

Mukherjee-Islam prior for the parameter θ is given by

$$g(\theta) = \frac{\alpha}{\sigma^{\alpha}} \theta^{\alpha-1}; \quad 0 < \theta < \sigma; \quad \alpha > 0; \quad \sigma > 0,$$

α is shape parameter and σ is scale parameter. This prior appears as inverse distribution of Pareto distribution. It is used for appropriate representation of the lower tail of the distribution of random variable having fixed lower bound.

Then the posterior distribution of θ is

$$f(\theta|\underset{\sim}{x}) = \frac{(np+\alpha)}{Y_1^{np+\alpha}} \theta^{np+\alpha-1}.$$

Hence the Bayes estimator (θ_B^M) of θ under Mukherjee-Islam prior is

$$\theta_B^M = \frac{(np+\alpha)}{(np+\alpha+1)} Y_1. \tag{10}$$

(v) Gamma prior

Gamma prior for the parameter θ is given by

$$g(\theta) = \frac{1}{\sigma^{\alpha}\sqrt{\alpha}} \theta^{\alpha-1} e^{-\theta/\sigma}; \quad \alpha, \sigma > 0, \quad \theta > 0.$$

Then the posterior distribution of θ is

$$f\left(\theta|\underset{\sim}{x}\right)=\frac{\theta^{np+\alpha-1}e^{-\theta/\sigma}I_{(0,Y_1)}^{\theta}}{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\sigma^i}\frac{Y_1^{np+\alpha+i}}{(np+\alpha+i)}}.$$

Hence the Bayes estimator (θ_B^G) of θ under gamma prior is

$$\theta_B^G=\frac{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\sigma^i}\frac{Y_1^{i+1}}{(np+\alpha+i+1)}}{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\sigma^i}\frac{Y_1^{i}}{(np+\alpha+i)}}. \tag{11}$$

3.1. Posterior expected loss under SELF

In this section, posterior expected losses of Bayes estimator $\left(\theta_B\right)$ of θ under SELF for different priors using direct method are obtained.

Posterior expected loss of Bayes estimator $\left(\theta_B\right)$ of θ is $E\left(\theta_B-\theta\right)^2$.

(i) Jeffrey's prior

We have

$$g(\theta)=\frac{1}{\theta}I_{(1,e)}(\theta),$$

$$f\left(\theta|\underset{\sim}{x}\right)=\frac{\theta^{np-1}}{Y_1^{np}}np.$$

Then posterior expected loss of θ_B^J under Jeffrey's prior is

$$E\left(\theta_B^J-\theta\right)^2=\theta_B^{J^2}+\frac{np}{np+2}Y_1^2-\frac{2\theta_B^J np}{np+1}Y_1. \tag{12}$$

(ii) Uniform prior

We have

$$g(\theta)=I_{(0,1)}(\theta),$$

$$f\left(\theta|\underset{\sim}{x}\right)=\frac{\theta^{np}}{Y_1^{np+1}}(np+1).$$

Then posterior expected loss of θ_B^U under Uniform prior is

$$E\left(\theta_B^U-\theta\right)^2=\theta_B^{U^2}+\frac{(np+1)}{(np+3)}Y_1^2-\frac{2\theta_B^U(np+1)}{(np+2)}Y_1. \tag{13}$$

(iii) Exponential prior

We have

$$g(\theta)=e^{-\theta};\quad \theta>0,$$

$$f\left(\theta|\underset{\sim}{x}\right)=\frac{\theta^{np}e^{-\theta}I_{(0,Y_1)}^{\theta}}{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\frac{Y_1^{np+i+1}}{(np+i+1)}}.$$

Then posterior expected loss of θ_B^E under exponential prior is

$$E\left(\theta_B^E-\theta\right)^2=\theta_B^{E^2}+\frac{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\frac{Y_1^{i+2}}{(np+i+3)}}{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\frac{Y_1^{i}}{(np+i+1)}}-2\theta_B^E\frac{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\frac{Y_1^{i+1}}{(np+i+2)}}{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\frac{Y_1^{i}}{(np+i+1)}}. \tag{14}$$

(iv) Mukherjee-Islam prior

We have

$$g(\theta)=\frac{\alpha}{\sigma^\alpha}\theta^{\alpha-1};\quad 0<\theta<\sigma;\quad \alpha>0;\quad \sigma>0,$$

$$f\left(\theta|\underset{\sim}{x}\right)=\frac{(np+\alpha)}{Y_1^{np+\alpha}}\theta^{np+\alpha-1}.$$

Then posterior expected loss of θ_B^M under Mukherjee-Islam prior is

$$E\left(\theta_B^M-\theta\right)^2=\theta_B^{M^2}+\frac{np+\alpha}{np+\alpha+2}Y_1^2-\frac{2\theta_B^M(np+\alpha)}{(np+\alpha+1)}Y_1. \tag{15}$$

(v) Gamma prior

We have

$$g(\theta)=\frac{1}{\sigma^\alpha\sqrt{\alpha}}\theta^{\alpha-1}e^{-\theta/\sigma};\quad \alpha,\sigma>0,\quad \theta>0,$$

$$f\left(\theta|\underset{\sim}{x}\right)=\frac{\theta^{np+\alpha-1}e^{-\theta/\sigma}I_{(0,Y_1)}^{\theta}}{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\sigma^i}\frac{Y_1^{np+\alpha+i}}{(np+\alpha+i)}}.$$

Then posterior expected loss of θ_B^G under gamma prior is

$$E\left(\theta_B^G-\theta\right)^2=\theta_B^G+\frac{\sum_{i=0}^{\infty}\frac{(-1)^2}{i!\sigma^i}\frac{Y_1^{i+2}}{(np+\alpha+i+2)}}{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\sigma^i}\frac{Y_1^{i}}{(np+\sigma+i)}}-2\theta_B^G\frac{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\sigma^i}\frac{Y_1^{i+1}}{(np+\alpha+i+1)}}{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\sigma^i}\frac{Y_1^{i}}{(np+\alpha+i)}}. \tag{16}$$

3.2. Posterior expected loss under APLF

In this section, posterior expected losses of Bayes estimator (θ_B) of θ under APLF for different priors using direct method are obtained.

Posterior expected loss of Bayes estimator (θ_B) of θ is

$$E\left[\frac{(\theta_B-\theta)^2}{\theta_B}\right].$$

(i) Jeffrey's prior

We have

$$g(\theta)=\frac{1}{\theta}I_{(1,e)}(\theta),$$

$$f\left(\theta|\underset{\sim}{x}\right)=\frac{\theta^{np-1}}{Y_1^{np}}np.$$

Then posterior expected loss of θ_B^J under Jeffrey's prior is

$$E\left[\frac{\left(\theta_B^J-\theta\right)^2}{\theta_B^J}\right]=\theta_B^J+\left(\frac{np}{np+2}\right)\frac{Y_1^2}{\theta_B^J}-\frac{2np}{np+1}Y_1. \tag{17}$$

(ii) Uniform prior

We have

$$g\left(\theta\right)=I_{(0,1)}\left(\theta\right),$$

$$f\left(\theta|\underset{\sim}{x}\right)=\frac{\theta^{np}}{Y_1^{np+1}}\left(np+1\right).$$

Then posterior expected loss of θ_B^U under Uniform prior is

$$E\left[\frac{\left(\theta_B^U-\theta\right)^2}{\theta_B^U}\right]=\theta_B^U+\left(\frac{np+1}{np+3}\right)\frac{Y_1^2}{\theta_B^U}-\frac{2\left(np+1\right)}{\left(np+2\right)}Y_1. \tag{18}$$

(iii) Exponential prior

We have

$$g\left(\theta\right)=e^{-\theta}\quad \theta>0,$$

$$f\left(\theta|\underset{\sim}{x}\right)=\frac{\theta^{np}e^{-\theta}I_{(0,Y_1)}^{\theta}}{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\frac{Y_1^{np+i+1}}{\left(np+i+1\right)}}.$$

Then posterior expected loss of θ_B^E under exponential prior is

$$E\left[\frac{\left(\theta_B^E-\theta\right)^2}{\theta_B^E}\right]=\frac{1}{\theta_B^E}\left[\theta_B^{E^2}+\frac{\sum_{i=0}^{\infty}\frac{(-1)^2}{i!}\frac{Y_1^{i+2}}{\left(np+i+3\right)}}{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\frac{Y_1^{i}}{\left(np+i+i\right)}}-2\theta_B^E\frac{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\frac{Y_1^{i+1}}{\left(np+i+2\right)}}{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\frac{Y_1^{i}}{\left(np+i+1\right)}}\right]. \tag{19}$$

(iv) Mukherjee-Islam prior

We have

$$g\left(\theta\right)=\frac{\alpha}{\sigma^{\alpha}}\theta^{\alpha-1};\quad 0<\theta<\sigma;\quad \alpha>0;\quad \sigma>0,$$

$$f\left(\theta|\underset{\sim}{x}\right)=\frac{\left(np+\alpha\right)}{Y_1^{np+\alpha}}\theta^{np+\alpha-1}.$$

Then posterior expected loss of θ_B^M under Mukherjee-Islam prior is

$$E\left[\frac{\left(\theta_B^M-\theta\right)^2}{\theta_B^M}\right]=\theta_B^M+\frac{np+\alpha}{np+\alpha+2}\frac{Y_1^2}{\theta_B^M}-\frac{2\left(np+\alpha\right)}{\left(np+\alpha+1\right)}Y_1. \tag{20}$$

(v) Gamma prior

We have

$$g\left(\theta\right)=\frac{1}{\sigma^{\infty}\sqrt{\alpha}}\theta^{\alpha-1}e^{-\theta/\sigma};\quad \alpha,\sigma>0;\quad \theta>0,$$

$$f\left(\theta|\underset{\sim}{x}\right)=\frac{\theta^{np+\alpha-1}e^{-\theta/\sigma}I_{(0,Y_1)}^{\theta}}{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\sigma^i}\frac{Y_1^{np+\alpha+i}}{(np+\alpha+i)}}.$$

Then posterior expected loss of θ_B^G under gamma prior is

$$E\left[\frac{\left(\theta_B^G-\theta\right)^2}{\theta_B^G}\right]=\frac{1}{\theta_B^G}\left[\theta_B^{G^2}+\frac{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\sigma^i}\frac{Y_1^{i+2}}{(np+\alpha+i+2)}}{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\sigma^i}\frac{Y_1^{i}}{(np+\alpha+i)}}-2\theta_B^G\frac{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\sigma^i}\frac{Y_1^{i+1}}{(np+\alpha+i+1)}}{\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\sigma^i}\frac{Y_1^{i}}{(np+\alpha+i)}}\right]. \quad (21)$$

3.3. Numerical illustration

To illustrate the calculations of Bayes estimates of θ given p under different priors, we have generated a random sample of size 100 from Pareto type-I model $(p=5,\theta=2)$ with the help of Easy Fit Professional 5.5 software. The generated data are given below:

Table 1 A random sample of size 100 from Pareto type-I model $(p=5,\theta=2)$

4.156719	2.021115	2.109158	3.066756	2.114712	2.766589
3.473910	2.152002	2.542979	2.096120	2.201036	2.800347
2.015538	3.345991	3.276592	2.523002	2.650161	2.200958
4.848990	2.399167	2.426388	2.765243	2.765476	3.788056
2.028549	3.248302	2.000721	2.853407	2.337706	2.330772
2.641111	2.775974	3.377225	2.377766	2.081900	2.043091
2.965438	2.402384	2.123957	2.037887	2.061429	2.100764
3.483411	2.037199	2.132553	2.021792	2.256790	2.523617
2.065490	2.033151	2.030786	2.252590	2.058568	2.330980
2.357053	2.229088	2.498827	2.610020	2.377580	2.909833
2.391189	2.032895	2.241154	2.107714	2.588603	2.367690
2.176588	2.461960	2.647930	3.594695	2.059533	3.254610
2.896456	3.083592	2.724975	2.093613	2.632570	2.174417
2.173150	2.474922	2.120280	2.913333	3.539938	3.520238
2.476360	3.053889	2.662606	2.453834	2.157306	2.008749
2.645826	2.039337	2.078516	2.234017	3.423581	
2.074963	4.111654	2.289557	3.166326	2.125210	

3.4. Comparison and conclusions

To compare the results numerically, we have calculated the values of posterior expected loss under SELF and APLF by using the estimates of θ by direct method under different priors. The calculations are shown in Table 2. The above said values could not be calculated under exponential and gamma priors due to complexity of corresponding expressions. It is revealed from the Table 2 that the value of APLF is smallest under uniform as well as under Mukherjee-Islam prior and as the value of α increases under Mukherjee-Islam prior the value of APLF decreases.

Table 2 The values of θ_B, and the corresponding posterior expected losses under SELF and APLF by using direct method

Prior	θ_B	Posterior expected loss under	
		Squared Error Loss Function	Asymmetric Precautionary Loss Function
Jeffrey's	1.996727	1.58841E-05	7.95509E-06
Uniform	1.996735	1.58211E-05	7.92346E-06
Exponential	1.999398	1.59899E-05	7.99735E-06
Mukherjee-Islam			
$\alpha = 1$	1.996735	1.58211E-05	7.92346E-06
$\alpha = 2$	1.996743	1.57584E-05	7.89205E-06
$\alpha = 3$	1.996751	1.56962E-05	7.86089E-06
Gamma			
$\alpha = 1, \sigma = 1$	1.999398	1.59899E-05	7.99735E-06
$\alpha = 2, \sigma = 1$	1.999406	1.59263E-05	7.96549E-06
$\alpha = 3, \sigma = 1$	1.999414	1.5863E-05	7.93382E-06

4. Estimation of Parameters (Lindley's Approach)

Lindley (1980) developed an asymptotic approximation to the ratio

$$I = \frac{\int_{\theta} h(\theta) l(\theta | \underset{\sim}{x}) g(\theta) d\theta}{\int_{\theta} l(\theta | \underset{\sim}{x}) g(\theta) d\theta}. \tag{22}$$

According to him

$$I \approx h(\theta^*) + \frac{\sigma^{*2}}{2}\left[h_2(\theta^*) + 2h_1(\theta^*)u_1(\theta^*)\right] + \frac{\sigma^{*4}}{2}\left[L_3(\theta^*)h_1(\theta^*)\right], \tag{23}$$

where θ^* is the MLE of θ.

Also

$$L_k(\theta^*) = \frac{\partial^k}{\partial \theta^k} L(\theta)\bigg|_{\theta=\theta^*}, \tag{24}$$

$$h_k(\theta^*) = \frac{\partial^k}{\partial \theta^k} h(\theta)\bigg|_{\theta=\theta^*}, \tag{25}$$

$$\sigma^{*2} = -L_2^{-1}(\theta)\big|_{\theta=\theta^*}, \tag{26}$$

$$u(\theta^*) = \log g(\theta)\big|_{\theta=\theta^*}.$$

Here, we have

$$L = \prod_{i=1}^{n}\left(\frac{p\theta^{p}}{x_i^{p+1}}\right) I_{(0,Y_1)}^{\theta},$$

$$L(\theta) = \log L,$$

$$L(\theta) = \log\left(\frac{p^{n}\theta^{np}}{\prod_{i=1}^{n} x_i^{p+1}}\right),$$

$$L(\theta) = np\log\theta + \text{constant},$$

$$L_1(\theta) = \frac{np}{\theta},$$

$$L_2(\theta) = \frac{np}{\theta^2},$$

$$L_3(\theta) = \frac{2np}{\theta^3}.$$

M.L.E. θ^* of θ is

$$\theta^* = Y_1\,, \tag{27}$$

$$\sigma^{*2} = \left[-L_2(\theta)\right]^{-1},$$

$$= \frac{\theta^{*2}}{np}. \tag{28}$$

Here $h(\theta) = \theta$.

The Bayes estimator (θ_B) of θ given p is given by

$$\theta_B = E\left[\theta \middle| \underset{\wedge}{x}\right] \cong \theta^* + u_1(\theta^*)\sigma^{*2} + \frac{1}{2}\left[L_3(\theta^*)\right]\sigma^{*4}. \tag{29}$$

(i) Jeffrey's prior

Jeffrey's prior for the parameter θ is

$$g(\theta) = \frac{1}{\theta} I_{(1,e)}(\theta).$$

Hence the Bayes estimator (θ_B^J) of θ given p under Jeffrey's prior is

$$\theta_B^J = \theta^*. \tag{30}$$

(ii) Uniform prior

Uniform prior for the parameter θ is

$$g(\theta) = I_{(0,1)}(\theta).$$

Hence the Bayes estimator (θ_B^U) of θ given p under uniform prior is

$$\theta_B^U = \theta^*\left(1 + \frac{1}{np}\right). \tag{31}$$

(iii) Exponential prior

Exponential prior for the parameter θ is

$$g(\theta) = e^{-\theta}; \quad \theta > 0.$$

Hence the Bayes estimator $\left(\theta_B^E\right)$ of θ given p under exponential prior is

$$\theta_B^E = \theta^* - \frac{\theta^{*2}}{np} + \frac{\theta^*}{np}. \tag{32}$$

(iv) Mukherjee-Islam prior

Mukherjee-Islam prior for the parameter θ is

$$g(\theta) = \frac{\alpha}{\sigma^\alpha}\theta^{\alpha-1}; \quad 0 < \theta < \sigma; \quad \alpha > 0; \quad \sigma > 0. \tag{33}$$

Hence the Bayes estimator $\left(\theta_B^M\right)$ of θ given p under Mukherjee-Islam prior is

$$\theta_B^M = \theta^* + \frac{\alpha\theta^*}{np}. \tag{34}$$

(v) Gamma prior

Gamma prior for the parameter θ is given by

$$g(\theta) = \frac{1}{\sigma^\alpha\sqrt{\alpha}}\theta^{\alpha-1}e^{-\theta/\sigma}; \quad \alpha, \sigma > 0, \quad \theta > 0.$$

Hence the Bayes estimator $\left(\theta_B^G\right)$ of θ given p is given by

$$\theta_B^G = \theta^* + \left(\frac{\alpha-1}{\theta^*} - \frac{1}{\sigma}\right)\frac{\theta^{*2}}{np} + \frac{\theta^*}{np}.$$

4.1. Posterior expected loss under SELF

In this section, posterior expected losses of Bayes estimator $\left(\theta_B\right)$ of θ under SELF for different priors using Lindley's Approach are obtained.

Posterior expected loss of Bayes estimator $\left(\theta_B\right)$ of θ is $E\left(h(\theta)\middle|\underset{\sim}{x}\right)$ where

$$h(\theta) = \left(\theta_B - \theta\right)^2,$$

$$h_1(\theta) = -2\left(\theta_B - \theta\right),$$

$$h_2(\theta) = 2,$$

$$E\left[h(\theta)\middle|\underset{\wedge}{x}\right] = h\left(\theta^*\right) + \frac{1}{2}\left[h_2\left(\theta^*\right) + 2h_1\left(\theta^*\right)u_1\left(\theta^*\right)\right]\sigma^{*2} + \frac{1}{2}\left[L_3\left(\theta^*\right)h_1\left(\theta^*\right)\right]\sigma^{*4}. \tag{35}$$

(i) Jeffrey's prior

Here

$$g(\theta) = \frac{1}{\theta}I_{(1,e)}(\theta).$$

Then posterior expected loss of θ_B^J under Jeffrey's prior is

$$E\left[h(\theta)\middle|\underset{\sim}{x}\right] = \left(\theta_B^J - \theta^*\right)^2 + \frac{\theta^{*2}}{np}. \tag{36}$$

(ii) Uniform prior

Here

$$g(\theta)=I_{(0,1)}(\theta).$$

Then posterior expected loss of θ_B^U under uniform prior is

$$E\left[h(\theta)|\underset{\sim}{x}\right]=\left(\theta_B^U-\theta^*\right)^2+\frac{\theta^{*2}}{np}-2\frac{\theta^*\left(\theta_B^U-\theta^*\right)}{np}. \tag{37}$$

(iii) Exponential prior

Here

$$g(\theta)=e^{-\theta};\quad \theta>0.$$

Then posterior expected loss of θ_B^E under exponential prior is

$$E\left[h(\theta)|\underset{\sim}{x}\right]=\left(\theta_B^E-\theta^*\right)^2+\frac{\theta^{*2}}{np}+2\frac{\theta^*\left(\theta^*-1\right)\left(\theta_B^E-\theta^*\right)}{np}. \tag{38}$$

(iv) Mukherjee-Islam prior

Here

$$g(\theta)=\frac{\alpha}{\sigma^2}\theta^{\alpha-1};\quad 0<\theta<\sigma;\quad \alpha>0;\quad \sigma>0.$$

Then posterior expected loss of θ_B^M under Mukherjee-Islam prior is

$$E\left[h(\theta)|\underset{\sim}{x}\right]=\left(\theta_B^M-\theta^*\right)^2+\frac{\theta^{*2}}{np}-2\frac{\alpha\theta^*\left(\theta_B^M-\theta^*\right)}{np}. \tag{39}$$

(v) Gamma prior

Here

$$g(\theta)=\frac{1}{\sigma^\alpha\sqrt{\alpha}}\theta^{\alpha-1}e^{-\theta/\sigma};\quad \alpha,\sigma>0,\quad \theta>0.$$

Then posterior expected loss of θ_B^G under gamma prior is

$$E\left[h(\theta)|x\right]=\left(\theta_B^G-\theta^*\right)^2+\frac{\theta^{*2}}{np}-2\frac{\theta^*\left(\theta_B^G-\theta^*\right)}{np}\left(\alpha-\frac{\theta^*}{\sigma}\right). \tag{40}$$

4.2. Posterior expected loss of θ_B under APLF

In this section, posterior expected losses of Bayes estimator (θ_B) of θ under APLF for different priors using Lindley's Approach are obtained.

Posterior expected loss of Bayes estimator (θ_B) of θ is $E\left(h(\theta)|x\right)$ where

$$h(\theta)=\frac{(\theta_B-\theta)^2}{\theta_B},$$

$$h_1(\theta)=\frac{2(\theta_B-\theta)}{\theta_B},$$

$$h_2(\theta)=\frac{2}{\theta_B},$$

$$E\left[h(\theta)\middle|\underset{\wedge}{x}\right]=h\left(\theta^*\right)+\frac{1}{2}\left[h_2\left(\theta^*\right)+2h_1\left(\theta^*\right)u_1\left(\theta^*\right)\right]\sigma^{*2}+\frac{1}{2}\left[L_3\left(\theta^*\right)h_1\left(\theta^*\right)\right]\sigma^{*4}.$$

(i) Jeffrey's prior

Here

$$g(\theta)=\frac{1}{\theta}I_{(1,e)}(\theta).$$

Then posterior expected loss of θ_B^J under Jeffrey's prior is

$$E\left[h(\theta)\middle|\underset{\sim}{x}\right]=\frac{\left(\theta_B^J-\theta^*\right)^2}{\theta_B^J}+\frac{\theta^{*2}}{\theta_B^J np}. \qquad (41)$$

(ii) Uniform prior

Here

$$g(\theta)=I_{(0,1)}(\theta).$$

Then posterior expected loss of θ_B^U under uniform prior is

$$E\left[h(\theta)\middle|\underset{\sim}{x}\right]=\frac{\left(\theta_B^U-\theta^*\right)^2}{\theta_B^U}+\frac{\theta^{*2}}{\theta_B^U np}-2\frac{\theta^*\left(\theta_B^U-\theta^*\right)}{\theta_B^U np}. \qquad (42)$$

(iii) Exponential prior

Here

$$g(\theta)=e^{-\theta};\quad \theta>0.$$

Then posterior expected loss of θ_B^E under exponential prior is

$$E\left[h(\theta)\middle|\underset{\sim}{x}\right]=\frac{\left(\theta_B^E-\theta^*\right)^2}{\theta_B^E}+\frac{\theta^{*2}}{\theta_B^E np}-2\frac{\theta^*\left(\theta^*-1\right)\left(\theta_B^E-\theta^*\right)}{\theta_B^E np}. \qquad (43)$$

(iv) Mukherjee-Islam prior

Here

$$g(\theta)=\frac{\alpha}{\sigma^\alpha}\theta^{\alpha-1};\quad 0<\theta<\sigma;\quad \alpha>0;\quad \sigma>0.$$

Then posterior expected loss of θ_B^M under Mukherjee-Islam prior is

$$E\left[h(\theta)\middle|\underset{\sim}{x}\right]=\frac{\left(\theta_B^M-\theta^*\right)^2}{\theta_B^M}+\frac{\theta^{*2}}{\theta_B^M np}-2\frac{\alpha\theta^*\left(\theta_B^M-\theta^*\right)}{\theta_B^M np}. \qquad (44)$$

(v) Gamma prior

Here

$$g(\theta)=\frac{1}{\sigma^\alpha\sqrt{\alpha}}\theta^{\alpha-1}e^{-\theta/\sigma};\quad \alpha,\sigma>0,\quad \theta>0.$$

Then posterior expected loss of θ_B^G under gamma prior is

$$E\left[h(\theta)\middle|\underset{\sim}{x}\right]=\frac{\left(\theta_B^G-\theta^*\right)^2}{\theta_B^G}+\frac{\theta^{*2}}{\theta_B^G np}-2\frac{\theta^*\left(\theta_B^G-\theta^*\right)}{\theta_B^G np}\left(\alpha-\frac{\theta^*}{\sigma}\right). \qquad (45)$$

4.3. Numerical illustration

To illustrate the calculations of Bayes estimators of θ given p under different priors, we have used the data given in Table 1.

4.4. Comparison and conclusions

To compare the results numerically, we have calculated the values of posterior expected loss under SELF and APLF by using the estimates of θ by Lindley's approach under different priors. The calculations are shown in Table 3. It is revealed from the table that the value of APLF is smallest under uniform as well as under Mukherjee-Islam prior.

It can also be concluded that the results obtained by direct method are better than that of Lindley's approach, as the values of posterior expected losses under SELF and APLF are small in direct method.

Table 3 The values of θ_B , and the corresponding posterior expected losses under SELF and APLF by using Lindley's approach

Prior	θ_B	Posterior expected loss under	
		Squared Error Loss Function	Asymmetric Precautionary Loss Function
Uniform	2.004722	0.007990	0.003985
Jeffrey's	2.000721	0.008006	0.004001
Exponential	1.996716	0.007990	0.004001
Mukherjee-Islam			
α=1	2.004722	0.007990	0.003985
α=2	2.008724	0.007958	0.003962
α=3	2.012725	0.007926	0.003938
Gamma			
α=1 σ=1	1.996716	0.007990	0.004001
α=2 σ=1	2.000718	0.008022	0.004009
α=3 σ=1	2.004719	0.008054	0.004017

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