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Analytic and Numerical Solutions of ARL of CUSUM Procedure for Exponentially Distributed Observations

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Abstract

This paper aims at deriving the explicit expressions of the Average Run Length (ARL) for a negative Cumulative Sum (CUSUM) chart for a lower-sided case when observations are from exponential distribution. ARL is found using two approaches; Integral Equation (IE) and Numerical Integral Equation (NI). The comparison for accuracy of results for explicit expression have been solved with the Integral Equation approach, while, the numerical approximations have been solved with the Numerical Integral equation, which both tend to an acceptable agreement. The computational time obtained from the NI approach is significantly longer than that obtained from the IE approach.

Keywords: Average Run Length, CUSUM, exponential distribution, integral equation.

1. Introduction

Statistical process control (SPC) is a statistical procedure aiming at improving the quality and productivity in areas such as industry, manufacturing, health care and epidemiology, clinical chemistry, finance and economics, environment science, and computer intrusion detection. The cornerstone of SPC is control charts. Clearly, there are various things to detect in a, say, production process, and there are various statistical ways to build control charts for detection. For detecting small shifts in a process mean, the CUSUM, first proposed by Page (1954), has produced many follow-up works. In particular, Ducan (1974), Hawkins and Olwell (1998) and Vargas et al. (2004) showed that CUSUM is much more efficient than the usual Shewhart control chart, as far as small variations in the mean are concerned.

The quality characteristic which is widely used to measure the performance of the CUSUM chart is the ARL, which is the expected number of runs to an alarm and is context dependent. The ARL is classified according to some stopping time scheme: the ARL_0 value (measuring the time before a process that is on target is falsely signaled as being out of control), and the ARL_1 (measuring the time, before a process that has gone out of control, which is signaled as being out of control).

In the literature, there are many methods able to calculate the ARL of CUSUM charts, primarily: the 'Monte Carlo Simulation' (Fu et al. 2002), 'Markov Chain Approach' (MCA) by Brook and Evans (1972), the 'Martingale Approach' (Sukparungsee and Novikov 2006, 2008), and the 'Integral Equations Approach' (IE) by Champ and Rigdon (1991)). The three former approaches have limitations and are complicated. They are able only to approximate ARL, and cannot achieve explicit expressions. It turns out that the IE approach can provide explicit expressions.

Recently, Busaba et al., (2012) derived an explicit expression of ARL for CUSUM chart using observations from negative exponential distribution for the upper-sided case. The contributions of our present study consist of deriving and analyzing explicit expressions of ARL for negative CUSUM charts when observations are exponentially distributed, for the lower-sided case.

This paper is organized as follows. In the next section, we evaluate the explicit expression of ARL for a negative CUSUM chart when observations are exponentially distributed, for the lower-sided case, using the IE approach. In Section 3, ARLs for CUSUM charts are approximated using numerical integral approximations based on the Gauss-Legrendre quadrature rule. In Section 4, we compare ARLs using displayed error between explicit expressions and numerical approximations. Subsequently, we compare the computational times for both approaches. Finally, the conclusion is addressed in Section 5.

2. Theoretical Results

We follow Vardeman (2001) setting. Let ξ_n be a sequence of independent and identically distributed (i.i.d.) random variables with common density function $f(.)$ on the real line. For the lower-sided case of negative CUSUM chart, consider

$j(x) =$ the ARL of the lower sided CUSUM scheme using a head start of x.

The CUSUM process starts at x, there are three possibilities of an observed variable, ξ_1. If ξ_1 is large $(-\xi_1 + a \geq -b - x)$, then there will be an immediate signal and the run length will be 1. If ξ_1 is small ($-\xi_1 + a \leq -x$), the CUSUM will "zero out", one observation will have been "spent", and on average $j(0)$ more observations are to be faced in order to produce a signal. Finally, if ξ_1 is moderate $(-x < -\xi_1 + a < -b - x)$, then one observation will have been spent and the CUSUM will continue from $x - \xi_1 + a$, requiring on average additional $j(x - \xi_1 + a)$ observations to produce a signal. This reasoning leads to the equation for $j(x)$,

$$j(x) = 1.P\left[-\xi_1 + a \geq -b - x\right] + \left(1 + j(0)\right)P\left[-\xi_1 + a \leq -x\right] + \int_{-x-a}^{-b-x-a} \left(1 + j(x - y + a)\right) f(-y)\, d(-y)$$

$$j(x) = 1 + \left(1 - F\left[x + a\right]\right) j(0) + \int_{-b}^{0} j(y) f(y - (x + a))\, dy. \tag{1}$$

We defined the CUSUM statistic X_n for an independently and identically distributed (i.i.d.) observed sequence of non-negative CUSUM chart with exponential distribution random variables. The recursive equation is

$$X_n = \max\left(X_{n-1} - \xi_n + a, 0\right), \quad n = 1, 2, \ldots, \; X_0 = x, \tag{2}$$

where $y = \max\left[0, y\right]$ and $\tau_b = \inf\left(k \geq 0 : X_k \leq -b\right)$ is the stopping time where a is reference value and b is control limit.

$$j(x) = 1 + \left(1 - F\left[x + a\right]\right) j(0) + \int_{-b}^{0} j(y) f(y - (x + a))\, dy. \tag{3}$$

We consider now the case where $f(t)=\lambda e^{-\lambda z}1_{(t\geq 0)}$ and hence $F(t)=\left(1-e^{-\lambda t}\right)1_{(t\geq 0)}$. Thus, the equation (3) becomes:

$$j(x)=1+j(0)\left(e^{-\lambda(x+a)}\right)+\ \lambda e^{-\lambda(x+a)}\int_{-b}^{0} j(y)e^{\lambda y}dy,$$

noting that the solution $j(.)$ of this integral equation is continuous since its right hand side is a continuous function of x.

Now, consider the complete metric space $(C(I),d)$ where I is a compact subset of the real line $\mathbb{R}$ (e.g., a closed and bounded interval), $C(I)$ denoting the space of all continuous functions $g:I\to\mathbb{R}$, and the metric d is generated by the sup-norm $\|g\|=\sup_{x\in I}|g(x)|$, i.e., $d(g,h)=\|g-h\|$. Recall that, a mapping $T:C(I)\to C(I)$ is called a contraction if there is $q\in(0,1)$ such that $d(T(g),T(h))\leq qd(g,h)$, for all $g,h\in C(I)$. An element $g\in C(I)$ is a fixed point of $T(.)$ if $T(g)=g$. A useful device in numerical analysis is the Banach fixed point theorem which says that if $T(.):(X,d)\to(X,d)$ is a contraction, where (X,d) is a complete metric space, then $T(.)$ has a unique fixed point x which is the limit (as $n\to\infty$) of $x_n=T(x_{n-1})$, where, for any arbitrary starting point $x_0\in X$, $x_1=T(x_0)$, $x_2=T(x_1),\ldots,$ $x_n=T(x_{n-1})$, and with error bound given by $d(x,x_n)\leq\frac{q^n}{1-q}d(x_0,x_1)$. See, e.g., Venkateshwara et al. (2001).

Now, in view of

$$j(x)=1+j(0)\left(e^{-\lambda(x+a)}\right)+\ \lambda e^{-\lambda(x+a)}\int_{-b}^{0} j(y)e^{\lambda y}dy,$$

we define an operator $T(.)$ on $C(I)$, for $x\in I$, as

$$T(j)(x)=1+j(0)\left(e^{-\lambda(x+a)}\right)+\ \lambda e^{-\lambda(x+a)}\int_{-b}^{0} j(y)e^{\lambda y}dy.$$

Theorem 2.1 *The operator $T(.)$ so define on $(C(I),d)$ is a contraction.*

Proof:

First, to prove that T is a contraction we may check that for any $x\in I$, and $j_1,j_2\in C(I)$, we have the inequality $\|T(j_1)-T(j_2)\|_1\leq q\|j_1-j_2\|$, where q is a positive constant, $q<1$

$$\begin{aligned}\|T(j_1)-T(j_2)\|_\infty &\leq \underset{x\in[0,a]}{Sup}\left\{\left|\left(j_1(0)-j_2(0)\right)\right|\left(e^{-\lambda(x+a)}\right)+\lambda e^{-\lambda(x+a)}\int_{-b}^{0}\left(j_1(y)-j_2(y)\right)e^{\lambda y}dy\right\}\\ &\leq\|j_1-j_2\|_\infty\underset{x\in[0,a]}{Sup}\left\{\left(e^{-\lambda(x+a)}\right)+\lambda e^{-\lambda(x+a)}\int_{-b}^{0}e^{\lambda y}dy\right\}\\ &=\|j_1-j_2\|_\infty\underset{x\in[0,a]}{Sup}\left\{e^{-\lambda(x+a)}+\ \lambda e^{-\lambda(x+a)}\int_{-b}^{0}e^{\lambda y}dy\right\}\\ &=\left(2e^{-\lambda(x+a)}-e^{-\lambda a-\lambda x-\lambda b}\right)\|j_1-j_2\|_\infty\\ &=q_1\|j_1-j_2\|_\infty,\end{aligned}$$

where $0 < q_1 = \left(2e^{-\lambda(x+a)} - e^{-\lambda a-\lambda x-\lambda b}\right) < 1.$

In the proof, we have used the triangle inequality for norms and the fact that

$$\left|j_1(0)-j_2(0)\right| \le \underset{x\in[0,a)}{Sup}\left|j_1(x)-j_2(x)\right| = \left\|j_1-j_2\right\|_\infty .$$

In Theorem 2.2, we derive explicit expression of the integral equations Eq.(3). The uniqueness of solutions is guaranteed by Theorem 2.1. Next, we derive the explicit expression of the Fredholm integral equation Eq.(3). The solution is given in Theorem 2.2.

Theorem 2.2 *The solution of* $T\left(j(x)\right)=j(x)$ *is*

$$j(x)=1+\frac{e^{-\lambda(x+a)}}{1-e^{-\lambda a}}\left(1+\frac{\left(1-e^{-\lambda a}\right)\left(1-e^{-\lambda b}\right)+\lambda b e^{-\lambda a}}{1-e^{-\lambda a}-\lambda b e^{-\lambda a}}\right),\ x\le a. \tag{5}$$

Proof: From Eq. (5), we have for $x\in[0,a)$ that

$$j(x)=1+j(0)\left(e^{-\lambda(x+a)}\right)+\lambda e^{-\lambda(x+a)}\int_{-b}^{0} j(y)e^{\lambda y}dy,\quad x\in[0,a),\quad a>b.$$

Define $\quad d=\int_{-b}^{0} j(y)e^{\lambda y}dy.$

So that, we have

$$j(x)=1+\ \lambda e^{-\lambda(x+a)}d+j(0)\left(e^{-\lambda(x+a)}\right). \tag{6}$$

If $x=0$ then $j(0)=\dfrac{1+\lambda e^{-\lambda a}d}{1-e^{-\lambda a}}$. Substitute $j(0)$ into Eq. (6), then

$$j(x)=\frac{1-e^{-\lambda a}+e^{-\lambda(x+a)}+\lambda e^{-\lambda(x+a)}d}{1-e^{-\lambda a}}. \tag{7}$$

If $x=a$ then,

$$j(a)=\frac{1-e^{-\lambda a}+e^{-2\lambda a}+\lambda e^{-2\lambda a}d}{1-e^{-\lambda a}}.$$

Next, we find the constant d is following

$$d=\int_{-b}^{0}\left(\frac{1-e^{-\lambda a}+e^{-\lambda(y+a)}+\lambda e^{-\lambda(y+a)}d}{1-e^{-\lambda a}}\right)e^{\lambda y}dy$$

$$=\frac{1}{\left(1-e^{-\lambda a}\right)}\int_{-b}^{0}\left(e^{\lambda y}-e^{-\lambda a+\lambda y}+e^{-\lambda a}+\lambda e^{-\lambda a}d\right)dy$$

$$=\frac{1}{\left(1-e^{-\lambda a}\right)}\left(\frac{1}{\lambda}\left|e^{\lambda y}\right|_{-b}^{0}-\frac{e^{-\lambda a}}{\lambda}\left|e^{\lambda y}\right|_{-b}^{0}+e^{-\lambda a}\left|y\right|_{-b}^{0}+\lambda d e^{-\lambda a}\left|y\right|_{-b}^{0}\right)$$

$$d=\frac{1}{\left(1-e^{-\lambda a}\right)}\left(\frac{1}{\lambda}\left[1-e^{-\lambda b}\right]-\frac{e^{-\lambda a}}{\lambda}\left[1-e^{-\lambda b}\right]+b e^{-\lambda a}+\lambda b d e^{-\lambda a}\right)$$

$$d=\frac{\left(1-e^{-\lambda a}\right)\left(1-e^{-\lambda b}\right)+\lambda b e^{-\lambda a}}{\lambda\left(1-e^{-\lambda a}-\lambda b e^{-\lambda a}\right)}.$$

Then, the constant d is into Eq. (7)

$$j(x)=1+\frac{e^{-\lambda(x+a)}}{1-e^{-\lambda a}}\left(1+\frac{\left(1-e^{-\lambda a}\right)\left(1-e^{-\lambda b}\right)+\lambda b e^{-\lambda a}}{1-e^{-\lambda a}-\lambda b e^{-\lambda a}}\right),\ x\in[0,a],\ a<b. \quad (8)$$

Thus, the explicit expressions for ARL_0 and ARL_1 follow as

$$ARL_0 = j_{L_0}^{-}(x)=1+\frac{e^{-\lambda_0(x+a)}}{1-e^{-\lambda_0 a}}\left(1+\frac{\left(1-e^{-\lambda_0 a}\right)\left(1-e^{-\lambda_0 b}\right)+\lambda_0 b e^{-\lambda_0 a}}{1-e^{-\lambda_0 a}-\lambda_0 b e^{-\lambda_0 a}}\right), x\in[\infty,a]$$
$$,a>b, 1-e^{-a}\neq 0 \ \ and \ \ 1-e^{-a}-be^{-a}\neq 0 \quad (9)$$

and

$$ARL_1 = j_{L_1}^{-}(x)=1+\frac{e^{-\lambda(x+a)}}{1-e^{-\lambda a}}\left(1+\frac{\left(1-e^{-\lambda a}\right)\left(1-e^{-\lambda b}\right)+\lambda b e^{-\lambda a}}{1-e^{-\lambda a}-\lambda b e^{-\lambda a}}\right), x\in[\infty,a], a>b$$
$$,1-e^{-\lambda a}\neq 0 \ \ and \ \ 1-e^{-\lambda a}-\lambda be^{-\lambda a}\neq 0. \quad (10)$$

3. The Numerical Integral Approximation

A numerical scheme evaluates solutions of the integral equations described in Section 2. Next, we evaluate this equation using the Gauss-Legendre quadrature rule, which is one of the Nystrom methods (Mititelu et al. 2010 and Busaba et al. 2011). Firstly, recall Eq. (1) as

$$j(x)=1+\left(1-F[x+a]\right)j(0)\ +\int_{-b}^{0} j(y)f(y-(x+a))dy,$$

where $F(x)=1-e^{-\lambda x}$ and $f(x)=\dfrac{dF(x)}{dx}=\lambda e^{-\lambda x}$.

We approximate the integral by a sum of areas of rectangles with bases b/m with heights is chosen as the values of f at the midpoints of intervals of length b/m beginning at $-b$, i.e., on the interval $[-b,0]$ with the division points $-b\le a_1\le a_2\le\ldots\le a_m\le 0$ and weights w_k. The integral of $j(y)$ can be approximated by summation following;

$$\int_{-b}^{0} j(y)dy\approx\sum_{k=1}^{m} w_k f(a_k) \ \text{ with } \ a_k=-b+\frac{b}{m}\left(k-\frac{1}{2}\right),\ k=1,2,\ldots,m.$$

Represent to Eq. (1) so that,

$$j(a_i)\approx 1+j(a_m)\left[1-F(a+a_i)\right]+\sum_{k=1}^{m} w_k j(a_k) f(a_k-a-a_i),\ i=1,2,\ldots,m.$$

Therefore

$$j(a_1)=1+w_1 f(a_1-a-a_1)j(a_1)+w_2 f(a_2-a-a_1)j(a_2)+\ldots+w_{m-1}f(a_{m-1}-a-a_1)j(a_{m-1})$$
$$+\left[1-F(a+a_1)+w_m f(a_m-a-a_1)\right]j(a_m)$$

$$j(a_2)=1+w_1 f(a_1-a-a_2)\,j(a_1)+w_2 f(a_2-a-a_2)\,j(a_2)+\ldots+w_{m-1} f(a_{m-1}-a-a_2)\,j(a_{m-1})$$
$$+\left[1-F(a+a_2)+w_m f(a_m-a-a_2)\right] j(a_m)$$
$$\vdots$$
$$j(a_m)=1+w_1 f(a_1-a-a_m)\,j(a_1)+w_2 f(a_2-a-a_m)\,j(a_2)+\ldots+w_{m-1} f(a_{m-1}-a-a_m)\,j(a_{m-1})$$
$$+\left[1-F(a+a_m)+w_m f(a_m-a-a_m)\right] j(a_m)$$

where j is a system of m linear equations in the m unknown $j(a_1), j(a_2),\ldots, j(a_m)$, which can be written in the matrix form as

$$J_{m\times 1}=1_{m\times 1}+R_{m\times n}J_{m\times 1} \quad or \quad (I_m-R_{m\times m})J_{m\times 1}=1_{m\times 1}, \tag{11}$$

where

$$R_{m\times m}=$$
$$\begin{bmatrix} w_1 f(a) & w_2 f(a_2-a-a_1) & \ldots & w_{m-1} f(a_{m-1}-a-a_1) & 1-F[a+a_1]+w_m f(a_m-a-a_1) \\ w_1 f(a_1-a-a_2) & w_1 f(a) & & w_{m-1} f(a_{m-1}-a-a_2) & 1-F[a+a_2]+w_m f(a_m-a-a_2) \\ \vdots & & & & \\ w_1 f(a_1-a-a_m) & w_2 f(a_2-a-a_m) & & w_{m-1} f(a_{m-1}-a-a_m) & 1-F[a+a_1]+w_m f(a) \end{bmatrix},$$

$$J_{m\times 1}=\begin{bmatrix} j(a_1) \\ j(a_2) \\ \vdots \\ j(a_m) \end{bmatrix}, \qquad 1_{m\times 1}=\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

and $I_m=diag(1,1,\ldots,1)$ is the unit matrix of order *m*.

From Eq. (11), we rearrange Eq. (10) as

$$J_{m\times 1}=(I_m-R_{m\times m})^{-1}1_{m\times 1}. \tag{12}$$

If Eq. (11) exists $(I_m-R_{m\times m})^{-1}$, then the solution of the Eq. (11) is Eq. (12). To solve this set of equations for the approximate values of $j(a_1), j(a_2),\ldots, j(a_m)$ can approximate the function $j(x)$ as

$$j^{IE}(x)\approx 1+j(a_m)\left[1-F(a+x)\right]+\sum_{k=1}^{m} w_k\, j(a_k)\, f(a_k-a-x),\ i=1,2,\ldots,m, \tag{13}$$

where $\int_{-b}^{0} j(y)dy \approx \sum_{k=1}^{m} w_k f(a_k)$ with $a_k=-b+\frac{b}{m}\left(k-\frac{1}{2}\right)$, $k=1,2,\ldots,m.$

4. Comparisons with the Analytical Results

In this section, the ARL of two methods are compared using the relative error and computational times. The relative error is a comparison of the explicit expression and the numerical approximation. We will implement real data in future research. Define the relative error as:

$$Diff\,(\%)=\frac{\left|j(x)-j^{IE}(x)\right|}{j(x)}\times 100\%.$$

Table 1 Comparisons of the ARL using the relative error between the results obtained from $j(x)$ and $j^{IE}(x)$ when given $a = 0.0007$, $\lambda = 1$ and $m = 800$

x	b = –0.0093		b = –0.002		b = –0.0013	
	$j(x)$	Diff (%)	$j(x)$	Diff (%)	$j(x)$	Diff (%)
	$j^{IE}(x)$	(Time Used)	$j^{IE}(x)$	(Time Used)	$j^{IE}(x)$	(Time Used)
0	100.063	0.056964	370.595	0.046142	500.288	0.040377
	100.006	(2,193.75)	370.424	(2,138.24)	500.086	(2,098.35)
0.5	61.0849	0.057134	225.171	0.046187	303.834	0.040483
	61.050	(2,193.75)	225.067	(2,138.24)	303.711	(2,098.35)
1.0	37.443	0.056352	136.967	0.045996	184.678	0.040611
	37.422	(2,193.75)	136.904	(2,138.24)	184.603	(2,098.35)
1.5	23.104	0.055402	83.468	0.045766	112.406	0.040033
	23.091	(2,193.75)	83.430	(2,138.24)	112.361	(2,098.35)
2	14.407	0.054141	51.019	0.045473	68.571	0.039958
	14.399	(2,193.75)	50.996	(2,138.24)	68.544	(2,098.35)

Tables 1 and 2 show the relative error has less than 0.1% accuracy between the numerical approximation with for a typical run of approximately 800 iterations, and the explicit expression. The computational time for numerical approximation takes about 40 minutes, while our analytical explicit expression solution for computational time is less than 1 second. The two methods are in excellent agreement with the results of ARL.

Table 2 Comparisons of the ARL using the relative error between the results obtained from $j(x)$ and $j^{IE}(x)$ when given $b = 0.002$, $\lambda = 1$ and $m = 800$

x	a = 0.012		a = 0.004703		a = 0.004	
	$j(x)$	Diff (%)	$j(x)$	Diff (%)	$j(x)$	Diff (%)
	$j^{IE}(x)$	(Time Used)	$j^{IE}(x)$	(Time Used)	$j^{IE}(x)$	(Time Used)
0	100.481	0.011943	370.186	0.045923	500.000	0.0622
	100.493	(2,132.97)	370.356	(2,123.61)	500.311	(2,433.8)
0.5	61.338	0.012227	224.922	0.046238	303.659	0.062241
	61.346	(2,132.97)	225.026	(2,123.61)	303.848	(2,433.8)
1.0	37.597	0.011969	136.816	0.045316	184.572	0.061765
	37.601	(2,132.97)	136.878	(2,123.61)	184.686	(2,433.8)
1.5	23.197	0.012070	83.376	0.045576	112.342	0.06142
	23.200	(2,132.97)	83.414	(2,123.61)	112.411	(2,433.8)
2	14.463	0.011754	50.964	0.045326	68.532	0.061285
	14.465	(2,132.97)	50.987	(2,123.61)	68.574	(2,433.8)

Table 3 Comparisons of the ARL using the relative error between the results obtained from $j(x)$ and $j^{IE}(x)$ when given $a = 0.0007,\ b = -0.002$ and $m = 800$

λ shifts	x = 0		x = 0.5	
	$j(x)$ / $j^{IE}(x)$	Diff (%) (Time Used)	$j(x)$ / $j^{IE}(x)$	Diff (%) (Time Used)
1.00	370.595	0.046142	225.171	0.046187
	370.424	(2,138.24)	225.067	(2,138.24)
1.01	366.921	0.044151	221.835	0.043726
	366.759	(2,664.54)	221.738	(2,664.54)
1.03	359.786	0.040024	215.375	0.039930
	359.642	(2,649.71)	215.289	(2,649.71)
1.05	352.923	0.035985	209.182	0.035854
	352.796	(2,652.53)	209.107	(2,652.53)
1.10	336.858	0.026124	194.773	0.026184
	336.770	(2,656.84)	194.722	(2,656.84)
2.00	185.039	0.154022	68.7043	0.152538
	185.324	(2,647.12)	68.8091	(2,647.12)

Table 3 shows the relative error between the numerical approximation and explicit values of ARL. For fixed $ARL_0 = 370,\ a = 0.0007,\ b = -0.002$ in a typical run of approximately 800 iterations. Note that $\lambda = 1$ is the value assumed for in the control parameter, so that first row is the values of ARL_0. For $\lambda > 1$ corresponds to values out of the control parameter, therefore these rows are the values of ARL_1. The numerical approximation and explicit values give the good agreement results which the relative error accuracy is less than 0.2%. The computational time for numerical approximation about 40 minutes, while the result obtained from the explicit expression solution takes less than 1 second.

5. Conclusions

We present ARL for CUSUM chart when observations are from negative CUSUM with exponential distribution for the lower-sided case by two methods based on the Integral Equation approach. First method, the explicit expression has been solved using the Integral Equation approach method. Second method, the numerical approximations have been solved using the Numerical Integral equation (NI approach) method based on Gauss-Legendre integration rules for approximation. Moreover, we compare the accuracy of the numerical results obtained from the integral equation and numerical integral equation. We conclude that the numerical results obtained from the two methods are in good agreement. However, computational time for numerical approximation (about 40 minutes) is significantly longer than for explicit expression (less than 1 second).

In our present work, we have specified equation (3) to the case where f is an exponential density function and carried out the analysis. The same method of analysis could be carried out for other types of distributions of the observations. This will clearly enlarge the domain of applications of out techniques. Such general setting will be our future research.

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