Thailand Statistician
January 2016; 14(1): 93-115
http://statassoc.or.th
Contributed paper

Comparison of Some Confidence Intervals for Estimating the Skewness Parameter of a Distribution

Sergio Perez-Meloand and B. M. Golam Kibria*
Department of Mathematics and Statistics, Florida International University, University Park
Miami FL, 33199, USA.
*Corresponding author; e-mail: kibriag@fiu.edu

Received: 30 January 2015
Accepted: 16 September 2015

Abstract

Several methods have been proposed to calculate interval estimators for estimating the skewness of a distribution. Since they considered in different times and under different simulation conditions, their performance are not comparable as a whole. In this paper an attempt has been made to review some existing estimators and compare them under the same simulation condition. In particularly, we consider and compare both classical (normality assumed) and non-parametric (bias-corrected standard bootstrap, Efron's percentile bootstrap, Hall's percentile bootstrap and bias-corrected percentile bootstrap) interval estimators for estimating the skewness of a distribution. A simulation study has been made to compare the performance of the estimators under normal, right and left skewed distributions. Both average widths and coverage probabilities are considered as a criterion of the good estimators. We have found a significant difference in the performance of classical and bootstrap estimators in all cases. Based on the simulation results we have found that both classical estimators and bootstrap estimators work well in terms of coverage probability when data comes from a normal distribution, although bootstrap methods tend to give smaller intervals in that case. When data comes from skewed distributions, bootstrap methods perform better than classical methods in terms of coverage. Amongst the bootstrap methods, the bias corrected percentile interval had the best coverage consistently for skewed data. One real life data sets are analyzed to illustrate the findings of the paper.

Keywords: Bootstrap methods, skewness, coverage probability, confidence interval, nonparametric methods, parametric methods, skewed distributions.

1. Introduction

Skewness is a numerical measure that helps to summarize a distribution's departure from symmetry about the mean. It is defined as the ratio of the third central moment of the distribution and the cube of the standard deviation ($\gamma_1 = \mu_3 / \sigma^3$). A symmetric distribution has a skewness of zero. Positive values of the skewness parameter indicate a distribution with a longer or heavier tail on the right than on the left, while negative values indicate the opposite. A general guideline to classify a distribution according to the severity of its skewness is given by Zieffler et al. (2011).

According to the aforementioned authors:

- If $\gamma_1 = 0$, the distribution is symmetric
- If $|\gamma_1| < 1$, the skewness of the distribution is slight
- If $1 < |\gamma_1| < 2$, the skewness of the distribution is moderate
- If $|\gamma_1| > 2$, the skewness of the distribution is high

Several sample statistics have been proposed as point estimators for the skewness parameter since Pearson (1895). Three of the most commonly used estimators computed by statistical packages are the following:

$$g_1 = \frac{\frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^3}{\left[\frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2\right]^{\frac{3}{2}}},$$

$$G_1 = \frac{n}{(n-1)(n-2)}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{s}\right)^3 = \frac{\sqrt{n(n-1)}}{n-2}g_1,$$

$$b_1 = g_1\left(\frac{n-1}{n}\right)^{\frac{3}{2}}.$$

The first estimator g_1 is called Fisher-Pearson coefficient of skewness, while G_1 is called adjusted Fisher-Pearson standardized third moment coefficient. Both G_1 and b_1 are modified versions of g_1 that adjust for bias in the estimation of the skewness parameter. For large sample sizes, these three statistics converge to a common value.

There are two major branches in inferential statistics: estimation theory and hypothesis testing. Estimation method includes confidence interval (indicate that the population parameter (e.g. μ, σ, γ_1 etc.) will be within this interval with a certain level of confidence) as estimates for population parameters, while the hypothesis testing focuses on the use of statistical tests to accept or reject hypotheses concerning these parameters. A convenient way to perform hypothesis testing is to compute a confidence interval for the parameter and then reject the null hypothesis if the hypothesized parameter value is not in the confidence interval. The confidence level is the probability that the interval estimate will include the true parameter in repeated sampling. In this article we will focus on comparing the performance of different confidence intervals estimators for the skewness parameter of normal, right skewed and left skewed populations.

Several researchers have studied the properties of various interval and point estimators of skewness through empirical studies. To mention a few, Arnold and Groenveld (1995), Rayner et al. (1995), Joanes and Gill (1998), Tabor (2010), Wright and Herrington (2011), Doane and Seward (2011). Since the aforementioned studies compared at different times and under different simulation conditions, they are not comparable as a whole. The objective of this paper is to compare several confidence intervals for skewness and find some good ones for estimating the skewness of normal, positively skewed and negatively skewed distributions within the classical and bootstrap approaches. Since a theoretical comparison is not possible, a simulation study has been made to compare the performance of the parametric estimators (classical confidence intervals with normality assumption) and bootstrap (bias-corrected standard bootstrap, Efron's percentile, Hall's percentile and bias-corrected percentile). Both average widths and coverage probabilities are considered as a criterion of the good estimators.

The organization of this paper is as follows: In Section 2, we presented various interval estimators. A simulation study has been conducted in Section 3. One real life data are analyzed in Section 4. Finally, some concluding remarks are given in Section 5.

2. Methods for Computing Confidence Interval for γ_1 (Skewness Parameter)

Suppose $X_1, X_2, \ldots, X_n$ be a *iid* random sample from a population with mean μ and standard deviation σ. Our target is to find an interval estimate for γ_1 (skewness parameter) with a specific level of confidence. We will take two main approaches to this problem. These are (a) the parametric approach and (b) the bootstrap approach. The (1-α)100% CI for γ_1 by different approaches are presented below.

2.1. Parametric approach (assuming normality)

Three parametric confidence intervals for skewness are proposed in the literature, one for each of the previously defined point estimators g_1, G_1 and b_1. They follow the general pattern:

$$\text{estimate} \pm \text{critical value} \times \text{SE}(\text{estimate})$$

$$g_1 \pm z_{\alpha/2}\sqrt{\frac{6(n-2)}{(n+1)(n+3)}},$$

$$G_1 \pm z_{\alpha/2}\sqrt{\frac{6n(n-1)}{(n+1)(n+3)(n-2)}},$$

$$b_1 \pm z_{\alpha/2}\sqrt{\frac{6(n-2)}{(n+1)(n+3)}}\left(\frac{n-1}{n}\right)^{\frac{3}{2}},$$

where g_1, G_1 and b_1 are the previously defined point estimators for skewness, n is the sample size, and $z_{\alpha/2}$ is the upper $\frac{\alpha}{2}$ percentile of the standard normal distribution (see Joanes and Gill 1998).

2.2. Bootstrap approach

Bootstrap is a commonly used computer-based non-parametric tool (introduced by Efron 1979), which requires no assumptions regarding the underlying population and can be applied to a variety of situations. The accuracy of the bootstrap CI depends on the number of bootstrap samples. If the number of bootstrap samples is large enough, CI may be very accurate. An extensive array of different bootstrap methods is summarized as follows: Let $X^{(*)} = X_1^{(*)}, X_2^{(*)}, \ldots, X_n^{(*)}$, where the i^{th} sample is denoted $X^{(i)}$ for i=1,2 ,…,B, and B is the number of bootstrap samples. Efron (1979) showed reducing B to 400 causes the conditional CV to become too large so he recommended using larger values such as 1,000. The number of bootstrap samples is typically between 1,000 and 2,000; because, the accuracy of the confidence interval depends on the size of the samples (Efron and Tibshirani 1993).

2.2.1. Bias-corrected standard bootstrap approach

Let T be any of the three point estimators for skewness previously defined. The corresponding bias-corrected standard bootstrap confidence interval is

$$T - Bias\left(T\right) \pm z_{\alpha/2}\hat{\sigma}_B$$

where $\hat{\sigma}_B = \sqrt{\frac{1}{B-1}\sum_{i=1}^{B}\left(T_i^* - \bar{T}\right)^2}$ is the bootstrap standard deviation, $\bar{T} = \frac{1}{B}\sum_{i=1}^{B}T_i^*$ is the bootstrap mean and $Bias\left(T\right) = \bar{T} - T$ is the estimated bias (see Manly 1997).

2.2.2. Efron's percentile bootstrap approach

This method was introduced by Efron (1987). Order the sample skewness of each bootstrap samples as follows:

$$T_{(1)}^* \le T_{(2)}^* \le T_{(3)}^* \le \ldots \le T_{(B)}^*$$

Then, the confidence interval will be given by

$$L = T*_{[(\alpha/2)\times B]} \text{ and } U = T*_{[(1-(\alpha/2))\times B]}$$

Percentile bootstrap approach is simpler and easy to implement as it does not require to compute $\hat{\sigma}_B$ compare to Bias-corrected standard Bootstrap approach.

2.2.3. Hall's percentile bootstrap approach

This method was proposed by Hall (1992). Order the errors $\varepsilon_i^* = T_i^* - T$ as follows:

$$\varepsilon_{(1)}^* \le \varepsilon_{(2)}^* \le \varepsilon_{(3)}^* \le \ldots \le \varepsilon_{(B)}^*$$

Then, the confidence interval will be given by:

$$L = T - \varepsilon^*_{\left[\left(1-\frac{\alpha}{2}\right)\times B\right]} = 2T - T^*_{\left[\left(1-\frac{\alpha}{2}\right)\times B\right]} \text{ and } U = T - \varepsilon^*_{\left[\left(\frac{\alpha}{2}\right)\times B\right]} = 2T - T^*_{\left[\left(\frac{\alpha}{2}\right)\times B\right]}.$$

2.2.4. Bias-corrected percentile bootstrap approach

This method was introduced by Efron (1987). The calculations are slightly more complicated than for the previous methods, but can be summarized as follows:

(a) Generate values T_i^*. Find the proportion of times that T_i^* exceeds T. This proportion can be computed as $p = \frac{\#(T_i^* > T)}{B}$

(b) Find z_0 such that $1 - \Phi\left(z_0\right) = p$ where $\Phi\left(.\right)$ is the cumulative distribution function of the standard normal random variable.

(c) Then the lower and upper confidence limits will be given by:

$$L = T^*_{\left[\Phi\left(2z_0 - z_{1-\alpha/2}\right)\times B\right]} \text{ and } U = T^*_{\left[\Phi\left(2z_0 + z_{1-\alpha/2}\right)\times B\right]}$$

Bias-corrected percentile bootstrap performed better than both bias-corrected standard Bootstrap and percentile bootstrap Approaches. For an overview of the calculations of different type of bootstrap type confidence intervals we refer our readers to Thomas and Joseph (1998) and Carpenter and Bithell (2000) among others.

3. Simulation Study

A theoretical comparison of the estimators is not possible, therefore a simulation study has been conducted to compare the performance of the estimators. Our goal is to compare them in terms of coverage probability and mean width for normal, right skewed and left skewed distributions.

3.1. Simulation technique

The main objective of this paper is to compare the performance of the interval estimators, which have been considered in Section 2. The flowchart of our simulation is as follows:

(i) We used n = 30, 50, 100, 300, 700 and 1,500.

(ii) Random samples are generated:

a) Normal distribution with mean 0 and SD 1 (γ_1 = 0) (Figure 1.1)

b) Gamma distribution with shape parameters 4, 1 and 0.25 and scale parameter 1 (γ_1 = 1, 2, 4) (Figure 1.2)

c) Beta distribution with alpha parameters 1, 1 and 3 and beta parameters 0.35181, 0.15470 and 0.10958841 respectively ($\gamma_1 = -1, -2, -4$) (Figure 1.3)

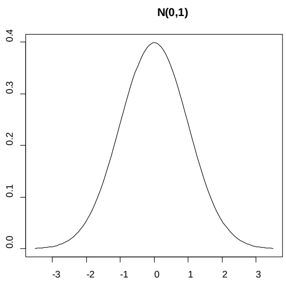

Figure 1.1

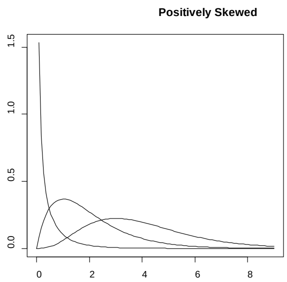

Figure 1.2

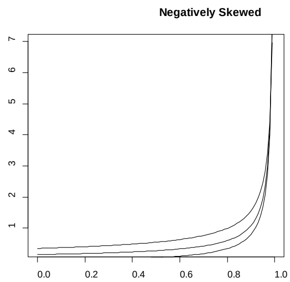

Figure 1.3

3,000 simulation replications are used for each case and 1,000 bootstrap samples for each n. For more on simulation techniques we refer our readers to Kibria and Banik (2013) and Banik and Kibria (2011) among others. The most common 95% confidence interval ($\alpha = 0.05$) for the confidence coefficient is used. It is well known that if the data are from a symmetric distribution (or n is large), the coverage probability will be exact or close to ($1-\alpha$). So the coverage probability is a useful criterion for evaluating the confidence interval. Another criterion is the width of the confidence interval. A smaller width gives a better confidence interval. It is obvious that when coverage probability is the same, a smaller width indicates that the method is appropriate for the specific sample. In order to compare the performance of the various intervals, the following criteria are considered: coverage probabilities and mean widths of the resulting confidence intervals. The coverage probability is found as the proportion of times in the 3,000 simulations that the true value of the skewness parameter falls within the confidence interval. The mean width of the intervals is computed as the average of the widths over the 3,000 simulations. The results are shown in Tables 1-7.

3.2. Results discussion

Our main objective is to compare the performance of the previously discussed interval estimators under different distributions: normal, right skewed and left skewed with different values of skewness ranging from slight to severely skewed. From the results we will be able to find out which interval estimators work best in terms of coverage and width under the different conditions.

To compare the performance of the estimators under the normal distribution, we generated random samples from the normal distribution with mean 0 and SD 1 i.e. N(0,1) and the simulated results are reported in Table 1. It is observed that the average width decreases as the sample size increases in all instances. Also the parametric confidence intervals tend to be longer in width than the bootstrap interval estimators, although the difference seems to decrease with sample size. As for the coverage probability, the classical parametric confidence intervals attained the nominal level 0.95 fairly quickly ($50 < n < 100$). Among the bootstrap estimators, Efron's percentile showed a coverage of 95% starting at $n = 30$. All the other bootstrap estimators increased their coverage probability steadily with sample size, starting around 89%-90% ($n = 30$) until achieving 95% at large sample sizes ($n = 700$ and 1,500). From these results we can conclude that if the normality of the data at hand is not in question, the classical parametric confidence intervals are a good choice. Efron's percentile method also converges fast to the nominal 95% level and it has the advantage of having a smaller width compared to the parametric confidence intervals.

To see the performance of the estimators under the positively skewed distributions, we generated random samples from three different positively skewed Gamma distributions with a range of skewness, ranging from 1 to 4. The results are reported in Table 2 to 4. In all cases the widths were similar amongst all the bootstrap confidence intervals, while the parametric estimators tended to be shorter. Also all intervals tended to become longer as we moved from slightly skewed to larger values of skewness, for the same sample sizes. As for the coverage probability the parametric confidence intervals performed poorly compared to the bootstrap estimators since their coverage probability was not only significantly lower in most cases, but also tended to become smaller as sample size increased. In the case of the slightly skewed distribution, the classical methods were still robust, even though their coverage deteriorates with increasing sample size. On the other hand, bootstrap estimators showed a steady, but not so fast, convergence towards the marginal 95% level. For large sample sizes ($n = 1{,}500$) the best coverage probabilities were about 91%, 87 % and 80% for skewness values of 1, 2 and 4 respectively. Comparing amongst all the estimators the Bias corrected percentile bootstrap method achieved the best coverage probability, followed closely by the Bias corrected standard bootstrap method. Amongst the point estimators used, G_1 produced a better coverage in all instances. Therefore, when dealing with positively skewed data the best pick for confidence interval estimation would be the Bias corrected percentile method using G_1. Still we should keep in mind that the coverage of such method is significantly lower than the marginal 95% level, even for sample sizes as big as $n = 300$.

To assess the performance of the estimators under the negatively skewed distributions, we generated random samples from three different negatively skewed Beta distributions with a range of skewness, ranging from -1 to -4. The results are reported in Table 5 to 7. In all cases the widths were similar amongst all the bootstrap confidence intervals, while the parametric estimators tended to be shorter. Also, intervals tended to become longer as we moved from slightly skewed to larger values of skewness, for the same sample sizes. As for the coverage probability the parametric confidence intervals performed poorly compared to the bootstrap estimators since their coverage probability was either higher than the 95% nominal level, for slightly skewed, or significantly lower for moderately and severely skewed (as low as 37 % for highly skewed and large sample sizes) . On the other hand bootstrap estimators showed a steady convergence towards the marginal 95% level, faster than for the positively skewed distributions. For large sample sizes the best coverage probabilities were about

95%, 94 % and 93% for skewness values of $-1, -2$ and -4 respectively. Comparing amongst all the estimators, the Bias corrected percentile bootstrap method again achieved the best coverage probability. Therefore, when dealing with negatively skewed data, the best pick for confidence interval estimation would be the Bias corrected percentile method, although all the other bootstrap methods discussed perform rather well too in terms of coverage and width compared to the classical parametric intervals.

Table 1 Coverage and mean width properties for N(0, 1) with skewness = 0 and for different sample sizes

Method	Estimator	n = 30		n = 50		n = 100	
		Coverage Probability	Mean width	Coverage Probability	Mean width	Coverage Probability	Mean width
Classical	g_1	0.962	1.589	0.964	1.280	0.953	0.932
	G_1	0.962	1.673	0.964	1.319	0.953	0.946
	b_1	0.962	1.510	0.964	1.241	0.953	0.918
Bias-corrected standard bootstrap	g_1	0.898	1.234	0.900	1.039	0.910	0.811
	G_1	0.896	1.299	0.898	1.071	0.912	0.822
	b_1	0.893	1.174	0.899	1.006	0.907	0.799
Efron's percentile bootstrap	g_1	0.950	1.223	0.941	1.026	0.934	0.803
	G_1	0.947	1.288	0.945	1.059	0.936	0.815
	b_1	0.949	1.163	0.943	0.996	0.933	0.790
Hall's percentile bootstrap	g_1	0.880	1.223	0.886	1.026	0.904	0.803
	G_1	0.879	1.288	0.884	1.059	0.906	0.815
	b_1	0.882	1.162	0.889	0.997	0.907	0.790
Bias-corrected percentile bootstrap	g_1	0.901	1.222	0.913	1.025	0.916	0.804
	G_1	0.902	1.287	0.915	1.059	0.917	0.816
	b_1	0.905	1.162	0.917	0.996	0.917	0.792

Table 1 (Continued)

Method	Estimator	n = 300		n = 700		n = 1,500	
		Coverage Probability	Mean width	Coverage Probability	Mean width	Coverage Probability	Mean width
Classical	g_1	0.953	0.549	0.949	0.361	0.957	0.247
	G_1	0.953	0.552	0.949	0.362	0.957	0.248
	b_1	0.953	0.546	0.949	0.361	0.957	0.247
Bias-corrected standard bootstrap	g_1	0.929	0.515	0.938	0.353	0.949	0.242
	G_1	0.931	0.519	0.938	0.354	0.950	0.243
	b_1	0.931	0.514	0.935	0.352	0.948	0.242
Efron's percentile bootstrap	g_1	0.936	0.513	0.938	0.351	0.950	0.242
	G_1	0.937	0.516	0.939	0.352	0.949	0.242
	b_1	0.937	0.510	0.940	0.351	0.950	0.242
Hall's percentile bootstrap	g_1	0.927	0.513	0.942	0.352	0.951	0.242
	G_1	0.925	0.515	0.939	0.351	0.951	0.242
	b_1	0.929	0.511	0.940	0.351	0.947	0.241
Bias-corrected percentile bootstrap	g_1	0.926	0.514	0.932	0.352	0.945	0.242
	G_1	0.926	0.517	0.934	0.352	0.945	0.242
	b_1	0.925	0.512	0.932	0.351	0.948	0.242

Table 2 Coverage and mean width properties for Gamma(4, 1) with skewness = 1 and for different sample sizes

Method	Estimator	n = 30		n = 50		n = 100	
		Coverage Probability	Mean width	Coverage Probability	Mean width	Coverage Probability	Mean width
Classical	g_1	0.875	1.589	0.854	1.280	0.831	0.932
	G_1	0.899	1.673	0.872	1.319	0.841	0.946
	b_1	0.851	1.510	0.836	1.241	0.821	0.918
Bias-corrected standard bootstrap	g_1	0.708	1.347	0.739	1.170	0.798	0.986
	G_1	0.744	1.420	0.757	1.209	0.809	0.998
	b_1	0.677	1.281	0.712	1.138	0.784	0.969
Efron's percentile bootstrap	g_1	0.682	1.322	0.708	1.140	0.781	0.953
	G_1	0.726	1.395	0.736	1.175	0.790	0.966
	b_1	0.636	1.258	0.680	1.106	0.764	0.939
Hall's percentile bootstrap	g_1	0.696	1.323	0.715	1.139	0.771	0.954
	G_1	0.727	1.394	0.735	1.176	0.780	0.967
	b_1	0.668	1.258	0.694	1.106	0.759	0.938
Bias-corrected percentile bootstrap	g_1	0.723	1.323	0.749	1.138	0.811	0.951
	G_1	0.762	1.393	0.774	1.174	0.818	0.965
	b_1	0.696	1.257	0.728	1.103	0.799	0.937

Table 2 (Continued)

Method	Estimator	n = 300		n = 700		n = 1,500	
		Coverage Probability	Mean width	Coverage Probability	Mean width	Coverage Probability	Mean width
Classical	g_1	0.796	0.549	0.797	0.361	0.777	0.247
	G_1	0.798	0.552	0.800	0.362	0.778	0.248
	b_1	0.788	0.546	0.795	0.361	0.776	0.247
Bias-corrected standard bootstrap	g_1	0.844	0.706	0.898	0.519	0.912	0.375
	G_1	0.846	0.710	0.899	0.519	0.911	0.376
	b_1	0.839	0.702	0.893	0.517	0.907	0.375
Efron's percentile bootstrap	g_1	0.836	0.689	0.891	0.510	0.907	0.371
	G_1	0.841	0.692	0.895	0.511	0.907	0.372
	b_1	0.830	0.684	0.889	0.509	0.904	0.371
Hall's percentile bootstrap	g_1	0.835	0.688	0.897	0.510	0.906	0.371
	G_1	0.842	0.692	0.897	0.511	0.909	0.371
	b_1	0.832	0.685	0.893	0.509	0.905	0.371
Bias-corrected percentile bootstrap	g_1	0.851	0.692	0.901	0.515	0.909	0.375
	G_1	0.853	0.696	0.901	0.517	0.910	0.376
	b_1	0.848	0.689	0.898	0.515	0.910	0.375

Table 3 Coverage and mean width properties for Gamma (1, 1) with skewness = 2 and for different sample sizes

Method	Estimator	n = 30		n = 50		n = 100	
		Coverage Probability	Mean width	Coverage Probability	Mean width	Coverage Probability	Mean width
Classical	g_1	0.508	1.589	0.520	1.280	0.508	0.932
	G_1	0.585	1.673	0.567	1.319	0.539	0.946
	b_1	0.435	1.510	0.471	1.241	0.490	0.918
Bias-corrected standard bootstrap	g_1	0.533	1.710	0.590	1.590	0.653	1.389
	G_1	0.581	1.801	0.621	1.641	0.671	1.409
	b_1	0.487	1.626	0.561	1.543	0.641	1.370
Efron's percentile bootstrap	g_1	0.467	1.671	0.537	1.530	0.619	1.323
	G_1	0.535	1.760	0.573	1.577	0.642	1.343
	b_1	0.408	1.589	0.495	1.485	0.600	1.303
Hall's percentile bootstrap	g_1	0.494	1.670	0.563	1.532	0.626	1.322
	G_1	0.539	1.760	0.591	1.578	0.640	1.345
	b_1	0.448	1.588	0.535	1.486	0.610	1.304
Bias-corrected percentile bootstrap	g_1	0.607	1.689	0.635	1.568	0.682	1.338
	G_1	0.653	1.778	0.666	1.613	0.698	1.357
	b_1	0.561	1.603	0.605	1.517	0.665	1.318

Table 3 (Continued)

Method	Estimator	n = 300		n = 700		n = 1,500	
		Coverage Probability	Mean width	Coverage Probability	Mean width	Coverage Probability	Mean width
Classical	g_1	0.484	0.549	0.471	0.361	0.457	0.247
	G_1	0.489	0.552	0.477	0.362	0.459	0.248
	b_1	0.471	0.546	0.468	0.361	0.457	0.247
Bias-corrected standard bootstrap	g_1	0.748	1.093	0.828	0.865	0.862	0.682
	G_1	0.755	1.099	0.833	0.869	0.865	0.684
	b_1	0.743	1.088	0.829	0.863	0.862	0.683
Efron's percentile bootstrap	g_1	0.726	1.044	0.820	0.839	0.853	0.667
	G_1	0.734	1.050	0.822	0.842	0.857	0.669
	b_1	0.715	1.039	0.814	0.837	0.850	0.667
Hall's percentile bootstrap	g_1	0.720	1.045	0.815	0.840	0.854	0.668
	G_1	0.724	1.050	0.819	0.841	0.856	0.669
	b_1	0.711	1.039	0.811	0.838	0.854	0.666
Bias-corrected percentile bootstrap	g_1	0.767	1.051	0.843	0.848	0.867	0.678
	G_1	0.775	1.055	0.845	0.852	0.868	0.678
	b_1	0.758	1.044	0.841	0.847	0.868	0.677

Table 4 Coverage and mean width properties for Gamma (1/4, 1) with skewness = 4 and for different sample sizes

Method	Estimator	n = 30		n = 50		n = 100	
		Coverage Probability	Mean width	Coverage Probability	Mean width	Coverage Probability	Mean width
Classical	g_1	0.110	1.589	0.168	1.280	0.187	0.932
	G_1	0.159	1.673	0.189	1.319	0.203	0.946
	b_1	0.081	1.510	0.141	1.241	0.171	0.918
Bias-corrected standard bootstrap	g_1	0.339	2.478	0.463	2.586	0.542	2.458
	G_1	0.389	2.613	0.492	2.663	0.566	2.500
	b_1	0.295	2.355	0.437	2.506	0.523	2.424
Efron's percentile bootstrap	g_1	0.209	2.401	0.407	2.511	0.515	2.378
	G_1	0.293	2.531	0.451	2.584	0.532	2.414
	b_1	0.119	2.286	0.365	2.434	0.488	2.342
Hall's percentile bootstrap	g_1	0.296	2.402	0.403	2.509	0.497	2.379
	G_1	0.338	2.530	0.425	2.586	0.512	2.415
	b_1	0.256	2.285	0.368	2.435	0.482	2.345
Bias-corrected percentile bootstrap	g_1	0.453	2.346	0.609	2.566	0.636	2.530
	G_1	0.541	2.472	0.639	2.654	0.651	2.567
	b_1	0.338	2.226	0.574	2.487	0.617	2.494

Table 4 (Continued)

Method	Estimator	n = 300		n = 700		n = 1,500	
		Coverage Probability	Mean width	Coverage Probability	Mean width	Coverage Probability	Mean width
Classical	g_1	0.187	0.549	0.177	0.361	0.183	0.247
	G_1	0.195	0.552	0.180	0.362	0.182	0.248
	b_1	0.181	0.546	0.174	0.361	0.184	0.247
Bias-corrected standard bootstrap	g_1	0.645	2.138	0.707	1.795	0.779	1.542
	G_1	0.651	2.148	0.708	1.801	0.782	1.544
	b_1	0.639	2.127	0.704	1.792	0.775	1.544
Efron's percentile bootstrap	g_1	0.621	2.015	0.687	1.699	0.767	1.471
	G_1	0.628	2.024	0.686	1.702	0.771	1.473
	b_1	0.609	2.006	0.681	1.694	0.767	1.470
Hall's percentile bootstrap	g_1	0.606	2.016	0.676	1.698	0.758	1.471
	G_1	0.612	2.025	0.680	1.701	0.762	1.474
	b_1	0.598	2.004	0.672	1.694	0.758	1.471
Bias-corrected percentile bootstrap	g_1	0.688	2.074	0.734	1.725	0.800	1.492
	G_1	0.704	2.089	0.738	1.727	0.800	1.495
	b_1	0.685	2.067	0.729	1.720	0.798	1.491

Table 5 Coverage and mean width properties for Beta (1, 0.35181) with skewness = -1 and for different sample sizes

Method	Estimator	n = 30		n = 50		n = 100	
		Coverage Probability	Mean width	Coverage Probability	Mean width	Coverage Probability	Mean width
Classical	g_1	0.977	1.589	0.983	1.280	0.987	0.932
	G_1	0.979	1.673	0.982	1.319	0.987	0.946
	b_1	0.973	1.510	0.982	1.241	0.987	0.918
Bias-corrected standard bootstrap	g_1	0.921	1.329	0.928	1.048	0.945	0.745
	G_1	0.939	1.400	0.937	1.080	0.944	0.756
	b_1	0.895	1.263	0.915	1.017	0.935	0.732
Efron's percentile bootstrap	g_1	0.936	1.329	0.935	1.047	0.943	0.743
	G_1	0.951	1.399	0.941	1.079	0.943	0.754
	b_1	0.917	1.263	0.924	1.015	0.940	0.732
Hall's percentile bootstrap	g_1	0.908	1.329	0.922	1.047	0.942	0.743
	G_1	0.932	1.401	0.937	1.079	0.950	0.755
	b_1	0.875	1.263	0.905	1.015	0.933	0.732
Bias-corrected percentile bootstrap	g_1	0.934	1.370	0.935	1.066	0.940	0.749
	G_1	0.939	1.446	0.936	1.097	0.942	0.761
	b_1	0.919	1.304	0.925	1.034	0.938	0.739

Table 5 (Continued)

Method	Estimator	n = 300		n = 700		n = 1,500	
		Coverage Probability	Mean width	Coverage Probability	Mean width	Coverage Probability	Mean width
Classical	g_1	0.989	0.549	0.987	0.361	0.986	0.247
	G_1	0.988	0.552	0.987	0.362	0.987	0.248
	b_1	0.990	0.546	0.987	0.361	0.986	0.247
Bias-corrected standard bootstrap	g_1	0.945	0.432	0.957	0.285	0.947	0.195
	G_1	0.951	0.434	0.956	0.285	0.947	0.194
	b_1	0.944	0.430	0.954	0.284	0.949	0.194
Efron's percentile bootstrap	g_1	0.946	0.431	0.956	0.284	0.948	0.193
	G_1	0.944	0.433	0.954	0.284	0.949	0.194
	b_1	0.945	0.428	0.953	0.283	0.950	0.193
Hall's percentile bootstrap	g_1	0.945	0.430	0.955	0.284	0.951	0.194
	G_1	0.946	0.433	0.956	0.284	0.949	0.194
	b_1	0.940	0.429	0.954	0.283	0.946	0.193
Bias-corrected percentile bootstrap	g_1	0.946	0.432	0.954	0.284	0.948	0.194
	G_1	0.946	0.434	0.954	0.284	0.948	0.194
	b_1	0.943	0.430	0.954	0.283	0.949	0.193

Table 6 Coverage and mean width properties for Beta (1, 0.15470) with skewness = -2 and for different sample sizes

Method	Estimator	n = 30		n = 50		n = 100	
		Coverage Probability	Mean width	Coverage Probability	Mean width	Coverage Probability	Mean width
Classical	g_1	0.846	1.589	0.852	1.280	0.855	0.932
	G_1	0.882	1.673	0.869	1.319	0.864	0.946
	b_1	0.792	1.510	0.831	1.241	0.841	0.918
Bias-corrected standard bootstrap	g_1	0.871	2.113	0.909	1.782	0.929	1.266
	G_1	0.900	2.224	0.930	1.839	0.940	1.286
	b_1	0.830	2.007	0.889	1.728	0.914	1.250
Efron's percentile bootstrap	g_1	0.931	2.117	0.951	1.778	0.947	1.263
	G_1	0.955	2.227	0.962	1.834	0.953	1.281
	b_1	0.899	2.011	0.929	1.726	0.935	1.244
Hall's percentile bootstrap	g_1	0.807	2.116	0.882	1.779	0.910	1.263
	G_1	0.856	2.227	0.904	1.834	0.920	1.282
	b_1	0.754	2.009	0.848	1.724	0.891	1.244
Bias-corrected percentile bootstrap	g_1	0.930	2.221	0.943	1.887	0.946	1.304
	G_1	0.942	2.321	0.948	1.948	0.952	1.321
	b_1	0.913	2.209	0.941	1.829	0.943	1.283

Table 6 (Continued)

Method	Estimator	n = 300		n = 700		n = 1,500	
		Coverage Probability	Mean width	Coverage Probability	Mean width	Coverage Probability	Mean width
Classical	g_1	0.864	0.549	0.880	0.361	0.863	0.247
	G_1	0.867	0.552	0.880	0.362	0.862	0.248
	b_1	0.859	0.546	0.879	0.361	0.862	0.247
Bias-corrected standard bootstrap	g_1	0.938	0.725	0.950	0.473	0.942	0.323
	G_1	0.944	0.728	0.956	0.474	0.939	0.323
	b_1	0.935	0.722	0.952	0.473	0.941	0.322
Efron's percentile bootstrap	g_1	0.942	0.723	0.953	0.472	0.943	0.321
	G_1	0.945	0.727	0.952	0.472	0.942	0.322
	b_1	0.941	0.718	0.953	0.470	0.941	0.321
Hall's percentile bootstrap	g_1	0.932	0.723	0.948	0.472	0.943	0.321
	G_1	0.937	0.725	0.951	0.473	0.939	0.322
	b_1	0.925	0.719	0.948	0.471	0.942	0.321
Bias-corrected percentile bootstrap	g_1	0.942	0.729	0.955	0.473	0.942	0.322
	G_1	0.943	0.733	0.954	0.474	0.940	0.322
	b_1	0.945	0.725	0.951	0.473	0.940	0.322

Table 7 Coverage and mean width properties for Beta (3, 0.10958841) with skewness = -4 and for different sample sizes

Method	Estimator	n = 30		n = 50		n = 100	
		Coverage Probability	Mean width	Coverage Probability	Mean width	Coverage Probability	Mean width
Classical	g_1	0.241	1.589	0.297	1.280	0.335	0.932
	G_1	0.299	1.673	0.343	1.319	0.346	0.946
	b_1	0.184	1.510	0.257	1.241	0.318	0.918
Bias-corrected standard bootstrap	g_1	0.558	2.914	0.682	3.019	0.772	2.756
	G_1	0.637	3.074	0.718	3.110	0.792	2.798
	b_1	0.487	2.769	0.651	2.925	0.752	2.719
Efron's percentile bootstrap	g_1	0.591	2.782	0.753	2.971	0.828	2.727
	G_1	0.677	2.928	0.795	3.066	0.846	2.772
	b_1	0.466	2.643	0.710	2.882	0.806	2.687
Hall's percentile bootstrap	g_1	0.443	2.780	0.581	2.971	0.703	2.725
	G_1	0.492	2.930	0.622	3.064	0.725	2.774
	b_1	0.394	2.643	0.542	2.879	0.681	2.688
Bias-corrected percentile bootstrap	g_1	0.774	2.610	0.862	3.048	0.878	3.026
	G_1	0.838	2.747	0.882	3.147	0.884	3.072
	b_1	0.693	2.476	0.835	2.961	0.868	2.987

Table 7 (Continued)

Method	Estimator	n = 300		n = 700		n = 1,500	
		Coverage Probability	Mean width	Coverage Probability	Mean width	Coverage Probability	Mean width
Classical	g_1	0.369	0.549	0.384	0.361	0.368	0.247
	G_1	0.371	0.552	0.384	0.362	0.368	0.248
	b_1	0.366	0.546	0.377	0.361	0.367	0.247
Bias-corrected standard bootstrap	g_1	0.883	1.900	0.917	1.318	0.922	0.924
	G_1	0.887	1.907	0.920	1.323	0.921	0.926
	b_1	0.876	1.890	0.912	1.315	0.918	0.923
Efron's percentile bootstrap	g_1	0.903	1.883	0.928	1.310	0.925	0.918
	G_1	0.908	1.890	0.932	1.313	0.926	0.919
	b_1	0.894	1.871	0.924	1.308	0.922	0.918
Hall's percentile bootstrap	g_1	0.853	1.881	0.902	1.311	0.910	0.918
	G_1	0.865	1.891	0.905	1.314	0.909	0.920
	b_1	0.847	1.873	0.897	1.308	0.907	0.920
Bias-corrected percentile bootstrap	g_1	0.920	1.997	0.934	1.350	0.932	0.934
	G_1	0.924	2.004	0.937	1.352	0.930	0.934
	b_1	0.916	1.983	0.932	1.345	0.931	0.933

4. Application

To illustrate the performance of the proposed confidence intervals for population skewness, a real life data (Postmortem interval) are analyzed in this Section. The postmortem interval (PMI) is defined as the elapsed time between death and an autopsy. Knowledge of PMI is considered essential when conducting medical research on human cadavers. The following data are PMIs of 22 human brain specimens obtained at autopsy in a recent study (Data Source: Hayes and Lewis 1995). We want to find the skewness of the PMI.

5.5, 14.5, 6.0, 5.5, 5.3, 5.8, 11.0, 6.1, 7.0, 14.5, 10.4,
4.6, 4.3, 7.2, 10.5, 6.5, 3.3, 7.0, 4.1, 6.2, 10.4, 4.9

The mean and SD of the data are 7.3 and 3.18 respectively. The sample skewness is 1.06 and the following histogram (Figure 2) that the data are right skewed and are not normally distributed. We assume that PMI data are from a gamma distribution with shape parameter, $\alpha = 5.25$, and scale parameter, $\beta = 1.39$. Using a Kolmogorov-Smirnov (*ks*) goodness of fit test, we obtain *ks* = 0.18 with *p-value* = 0.41, which indicates that PMI data are from a gamma distribution with shape parameter, $\alpha = 5.25$, and scale parameter, $\beta = 1.39$. The population $skewness = \frac{2}{\sqrt{\alpha}} = 0.88$. The 95% resulting confidence intervals and the corresponding confidence width are given in Table 8. From this Table, we see that all proposed estimators cover the hypothesized true skewness 0.88. However, Efron's percentile bootstrap has the shortest width followed by bias corrected standard bootstrap and classical (g_1) interval has the widest width.

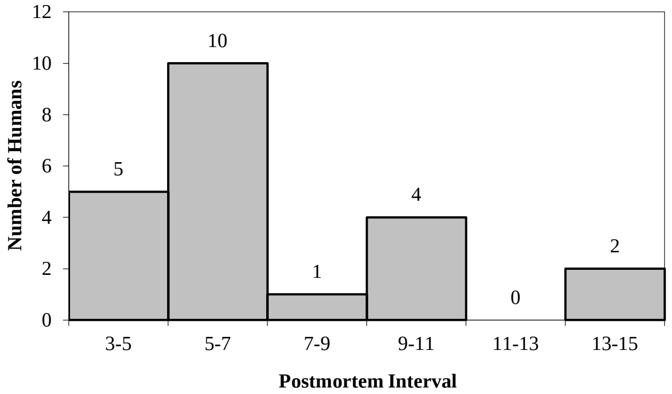

Figure 2 Histogram

Table 8 95% confidence intervals and widths for postmortem interval data

Methods	Confidence Interval	Width
Classical (g_1)	(0.027, 1.818)	1.791
Classical (G_1)	(0.029, 1.696)	1.667
Classical (b_1)	(0.026, 1.696)	1.670
Bias-corrected standard bootstrap (g_1)	(0.168, 1.672)	1.504
Bias-corrected standard bootstrap (G_1)	(0.209, 1.790)	1.581
Bias-corrected standard bootstrap (b_1)	(0.249, 1.597)	1.348
Efron's percentile bootstrap (g_1)	(0.255, 1.707)	1.452
Efron's percentile bootstrap (G_1)	(0.261, 1.816)	1.555
Efron's percentile bootstrap (b_1)	(0.254, 1.484)	1.230
Hall's percentile bootstrap (g_1)	(0.145, 1.577)	1.432
Hall's percentile bootstrap (G_1)	(0.101, 1.722)	1.621
Hall's percentile bootstrap (b_1)	(0.103, 1.485)	1.382
Bias-corrected percentile bootstrap (g_1)	(0.333, 1.818)	1.485
Bias-corrected percentile bootstrap (G_1)	(0.414, 2.022)	1.608
Bias-corrected percentile bootstrap (b_1)	(0.325, 1.805)	1.480

5. Conclusions

This paper reviews various interval estimators for estimating the skewness parameter of a distribution. Since a theoretical comparison is not possible, a simulation study has been conducted to compare the performance of the estimators. We have compared both classical (normality assumed) and non-parametric (bias-corrected standard bootstrap, Efron's percentile bootstrap, Hall's percentile bootstrap and bias-corrected percentile bootstrap) intervals, where the data are generated from various distributions such as normal (symmetric), gamma (right skewed) and beta (left skewed) distributions. Coverage probability and average width are considered as a criterion of a good estimator. Our simulation results indicate that the performance of the classical estimators and non-parametric estimators differs significantly across sample sizes and type of skewness. For normal data, the classical estimators perform well in terms of coverage. For small sample size, the classical methods have better coverage than most bootstrap methods, although the later have a smaller width for small sample size. For large sample sizes, the coverage and widths of both types of methods don't differ greatly when data comes from a normal distribution. For positively skewed distributions, bootstrap methods had better coverage than classical, being the best the Bias Corrected Percentile Bootstrap estimator. Classical methods were still robust for slightly right skewed data, although their coverage deteriorated with increasing sample size. It should also be noted that no method achieved the 95% nominal level even for a big sample size of $n = 1{,}500$. For negatively skewed distributions, bootstrap methods worked better than classical estimators in all instances in terms of coverage and shorter width for all left skewed distributions considered. The best coverage amongst the bootstrap methods was achieved by the Bias Corrected Percentile Bootstrap. Also the rate of convergence towards the 95% nominal level for bootstrap estimators was faster than for positively skewed distributions. To illustrate the findings of the paper, a real data set are studied and the results supported the simulation study to some extent. We believe that the findings of this paper will be helpful for different applied researchers/practitioners in the field of science and social sciences.

Acknowledgements

The second author, B. M. Golam Kibria dedicates this paper to his elementary school teachers, Late Mr. Abdur Rahim Bhuiyan (Father), Late Shree Dino Bondhu Mitra, Late Mr. Abdul Hannan Moulovi and Late Mr. Abdul Mohhamad for their fundamental education, inspiration and affection that motivated him to go to High School, College, University and finally to achieve this present position.

References

Arnold BC, Groeneveld RA. Measuring skewness with respect to the mode. Am Stat. 1995; 49: 34-38.

Banik S, Kibria BMG. Estimating the population coefficient of variation by confidence intervals. Commun Stat-Simul C. 2011; 40: 1236–1261.

Carpenter J, Bithell J. Bootstrap confidence intervals: when, which, what? A practical guide for medical statisticians. Stat Med. 2000; 19: 1141-1164.

Doane DP, Seward LE. Measuring skewness: A forgotten statistic?. J Stat Educ. 2011; 19: 1-18.

Efron B. Better bootstrap confidence intervals. J Am Stat Assoc. 1987; 82: 171-200.

Efron B. Bootstrap methods: another look at the jack knife. Ann Stat. 1979; 7: 1-26.

Efron B, Tibshirani RJ. An introduction to the bootstrap. New York: Chapman & Hall; 1993.

Hall P. The bootstrap and edgeworth expansion. New York: Springer-Verlag; 1992.

Hayes TL, Lewis DA. Anatomical specialization of the anterior motor speech area: hemispheric differences in magnopyramids neurons. Brain Lang. 1995; 49: 292.

Joanes DN, Gill CA. Comparing measures of sample skewness and kurtosis. The Statistician. 1998; 47: 183-189.

Kibria BMG, Banik S. Parametric and nonparametric confidence intervals for estimating the difference of means of two skewed populations. J Appl Stat. 2013; 40: 2617-2636.

Manly BFJ. Randomization, bootstrap and monte carlo methods in biology. New York: Chapman & Hall; 1997.

Pearson K. Contributions to the mathematical theory of evolution, II: Skew variation in homogeneous material. Transactions of the Royal Philosophical Society, Series A, 1895; 186: 343-414.

Rayner JCW, Best DJ, Mathews KL. Interpreting the skewness coefficient. Commun Stat Theory. 1995; 24: 593-600.

Tabor J. Investigating the investigative task: Testing for skewness - an investigation of different test statistics and their power to detect skewness. J Stat Educ. 2010; 18: 1-13.

Thomas JD, Joseph OR. A review of bootstrap confidence intervals. J R Stat Soc Series B Stat Methodol. 1998; 50: 338-354.

Wright DB, Herrington JA. Problematic standard errors and confidence intervals for skewness and kurtosis. Behav Res Methods. 2011; 43: 8-17.

Zieffler AS, Harring JR, Long JD. Comparing groups: Randomization and bootstrap methods using R. New Jersey: Wiley; 2011.