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Reliability Analysis of the Beta Exponentiated Weibull Poisson Distribution

Tipagorn Insuk [a], Winai Bodhisuwan [b] and Uraiwan Jaroengeratikun* [a]
[a] Department of Applied Statistics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bang Sue, Bangkok 10800, Thailand.
[b] Department of Statistics, Faculty of Science, Kasetsart University, Bangkok 10900, Thailand.
*Corresponding author; e-mail: uraiwan.j@sci.kmutnb.ac.th

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Abstract

The beta exponentiated Weibull Poisson (BEWP) distribution is introduced by Insuk et al. in 2015, some properties of this distribution are discussed therein. In this paper, we applied the BEWP in reliability analysis. Derivation of reliability functions, e.g. survival function, hazard function, moment, mean and variance of residual life function are presented. Finally, we apply the BEWP's mean residual life (MRL) function to find the optimal burn-in time for real data with the bathtub failure shape.

Keywords: Beta exponentiated Weibull Poisson distribution, reliability, hazard function, mean residual life.

1. Introduction

Modeling the lifetime data with failure distribution is intrinsic component of lifetime study. The Weibull distribution was one of the earliest and most accepted model for failure time. The Weibull distribution will also be the initial choice for modeling lifetime data. However, many researchers have utilized Weibull distribution as the baseline for developing the new distribution that can provide greater flexibility and accommodate more complicated data. Almalki and Nadarajah (2014) reviewed the modifications of the Weibull distribution for both discrete and continuous Weibull distributions encompassing discrete modified Weibull (Nooghabi et al. 2011), discrete additive Weibull (Bebbington et al. 2012), Inverse Weibull (Drapella 1993), exponentiated Weibull (Mudholkar and Srivastava 1993), Weibull geometric (Barreto-Souza 2011), Weibull Poisson (Lu and Shi 2012), Weibull Power Series (Morais and Barreto-Souza 2011), Flexible Weibull extension (Bebbington 2007), beta Weibull (Lee et al. 2007), Kumaraswamy Weibull (Cordeiro et al. 2010) and etc.

More recently, Insuk et al. (2015) introduced the generalized class of lifetime distribution, namely beta exponentiated Weibull Poisson (BEWP) distribution. The BEWP distribution has been developed by mixing exponentiated Weibull Poisson (Mahmoudi and Sepahdar 2013) distribution and the beta distribution.

Let X be a random variable of the BEWP distribution. Then the cumulative distribution function (cdf) and probability density function (pdf) of X respectively are given by

$$F(x) = I_{\left(e^{\lambda(1-u)^{\alpha}}\right)/\left(e^{\lambda}-1\right)}(a,b), \quad x>0,$$

where $\alpha,\beta,\theta,\lambda,a,b>0$, $u=e^{-(\theta x)^{\beta}}$, $I_{(\bullet)}(a,b)$ is the regularized incomplete beta function and

$$f(x)=\frac{\lambda\alpha\beta\theta^{\beta}x^{\beta-1}u(1-u)^{\alpha-1}e^{\lambda(1-u)^{\alpha}}}{\left(e^{\lambda}-1\right)B(a,b)}\left(\frac{e^{\lambda(1-u)^{\alpha}}-1}{e^{\lambda}-1}\right)^{a-1}\left(1-\frac{e^{\lambda(1-u)^{\alpha}}-1}{e^{\lambda}-1}\right)^{b-1}, \quad x>0.$$

The intriguing feature of this class is it is the generalized class of the modified Weibull distributions and also contains the sub-model up to 32 distributions including Weibull distribution. One advantage of generalized distribution is it can enhance the flexibility of probability distribution and increases their applicability in distribution fitting.

In addition to the probability density function for lifetime distributions, most distributions are characterized with respect to aging by the behavior of their hazard ($h(t)$) or mean residual life (MRL or $m(t)$). Both functions have an inverse relationship (Ahmed et al. 2011), that is

$$h(t)=\frac{m'(t)+1}{m(t)}$$

where $m'(t)\geq -1$ and $m'(t)$ is the first derivative of the MRL function.

Furthermore, the hazard function which is the traditional reliability measurement. The MRL also plays the crucial role in various applications, for instance in engineering study where MRL is a useful tool when conducting burn-in analysis (Lai and Xie 2006). For insurance, it is use for setting the life insurance rates and benefits and, etc. Tang et al. (1999) analyzed the characteristic of the general behaviors of the MRL for both continuous & discrete lifetime distributions, with respect to their failure rates.

In this paper, we extend the structural properties of a new class of Beta-G named beta exponentiated Weibull Poisson (BEWP) distribution. Beta-G distribution was introduced by Eugene et al. (2002) which is the class of generalized beta distribution where G be the continuous cumulative distribution function. The purpose of this study is to introduce more structural properties of BEWP distribution by focusing on the reliability application based on its hazard and mean residual life function.

The remainder of this paper has been arranged in the following sequence. Section 2 discusses about the survival (reliability) and hazard (failure rate) function and of BEWP distribution. Its mean residual life is introduced in Section 3, and the application to real data set is presented in Section 4. Some concluding remarks are given in Section 5.

2. Survival (Reliability) and Hazard (Failure rate) Function

In the field of reliability studies, the survival function and hazard function plays an important role. In particular, the shape of the hazard plot provides meaningful guidance in recognizing the status of failure rate. There are three particular types of patterns; a decreasing failure rate pattern indicates influence of poor design, mainly manufacturing or assembly variables, a constant failure rate pattern indicates effect of random causes, and increasing failure rate pattern indicates wearing-out failures (Rai and Singh 2009). When all three patterns aforementioned patterns combine there will be a bathtub pattern. Nadarajah (2009) reviewed the known distributions that exhibit this shape.

Moreover, to describe the characteristics of distribution, the hazard function is a more precise tool when identifying individuals from each distribution, rather than either cdf or pdf. In this section, we present the survival and hazard function of BEWP distribution.

Let X be a random variable of the BEWP distribution. Then the survival function and hazard (failure rate) function of X respectively are given by

$$S(x) = 1 - I_{\left(e^{\lambda(1-u)^{\alpha}}-1\right)/\left(e^{\lambda}-1\right)}(a,b), \quad x > 0,$$

where $\alpha, \beta, \theta, \lambda, a, b > 0$, $u = e^{-(\theta x)^{\beta}}$ and

$$h(x) = \frac{\lambda\alpha\beta\theta^{\beta}x^{\beta-1}u(1-u)^{\alpha-1}e^{\lambda(1-u)^{\alpha}}}{\left(e^{\lambda}-1\right)B(a,b)\left(1-I_{(e^{\lambda(1-u)^{\alpha}}-1)/(e^{\lambda}-1)}(a,b)\right)}\left(\frac{e^{\lambda(1-u)^{\alpha}}-1}{e^{\lambda}-1}\right)^{a-1}\left(1-\frac{e^{\lambda(1-u)^{\alpha}}-1}{e^{\lambda}-1}\right)^{b-1}, \quad x > 0.$$

We obtain the cumulative hazard (failure rate) function of X, that is

$$\begin{aligned} H(x) &= \int_0^x h(t)dt \\ &= -\log R(x) \\ &= -\log\left(1-I_{(e^{\lambda(1-u)^{\alpha}}-1)/(e^{\lambda}-1)}(a,b)\right), \end{aligned}$$

where $u = e^{-(\theta x)^{\beta}}$.

To identify the hazard shape of BEWP by using Glaser's approach (Glaser 1980) we find that

$$\eta(x) = -\frac{f'(x)}{f(x)},$$

$g(x)$ and $G(x)$ denote the pdf and cdf of the generalized class of distribution respectively. We regard them to be the baseline for creating the Beta-G distribution. The pdf of Beta-G distribution is given by

$$f(x) = \frac{g(x)G(x)^{a-1}\left(1-G(x)\right)^{b-1}}{B(a,b)}$$

and

$$f'(x) = f(x)\left(\frac{g'(x)}{g(x)} + \frac{(a-1)g(x)}{G(x)} - \frac{(b-1)g(x)}{1-G(x)}\right)$$

then

$$\eta(x) = -\left(\frac{g'(x)}{g(x)} + \frac{(a-1)g(x)}{G(x)} - \frac{(b-1)g(x)}{1-G(x)}\right).$$

That deal with only baseline distribution. For BEWP that $g(x)$ and $G(x)$ are

$$g(x)=\frac{\lambda\alpha\beta\theta^{\beta}}{\left(e^{\lambda}-1\right)}x^{\beta-1}u\left(1-u\right)^{\alpha-1}e^{\lambda(1-u)^{\alpha}}, \quad x>0$$

$$G(x)=\frac{e^{\lambda(1-u)^{\alpha}}-1}{e^{\lambda}-1},$$

where $u=e^{-(\theta x)^{\beta}}$. We can rewrite $g(x)$ and $G(x)$ in term of pdf $q(x)$ and cdf $Q(x)$ of Weibull distribution

$$q(x)=\beta\theta^{\beta}x^{\beta-1}e^{-(\theta x)^{\beta}}, \quad x>0$$

$$Q(x)=1-e^{-(\theta x)^{\beta}},$$

that are easier for derivation. Hence, we obtain

$$g(x)=\frac{\lambda\alpha}{\left(e^{\lambda}-1\right)}q(x)Q(x)^{\alpha-1}e^{\lambda Q(x)^{\alpha}}, \quad x>0$$

$$G(x)=\frac{e^{\lambda Q(x)^{\alpha}}-1}{e^{\lambda}-1}.$$

Then

$$\eta'(x)=-\frac{d}{dx}\left(\frac{g'(x)}{g(x)}+\frac{(a-1)g(x)}{G(x)}-\frac{(b-1)g(x)}{1-G(x)}\right),$$

where

$$\frac{d}{dx}\left(\frac{g'(x)}{g(x)}\right)=-\frac{(\beta-1)\left(\beta(\theta x)^{\beta}+1\right)}{x^{2}}+(\alpha-1)\left(\frac{Aq(x)}{Q(x)}-\frac{\left[q(x)\right]^{2}}{\left[Q(x)\right]^{2}}\right)+\lambda\alpha\left(\frac{Aq(x)}{\left[Q(x)\right]^{1-\alpha}}+\frac{(\alpha-1)\left[q(x)\right]^{2}}{\left[Q(x)\right]^{2-\alpha}}\right)$$

$$\frac{d}{dx}\left(\frac{(a-1)g(x)}{G(x)}\right)=(a-1)\left(\frac{Bg(x)G(x)-\left[g(x)\right]^{2}}{\left[G(x)\right]^{2}}\right)$$

$$\frac{d}{dx}\left(\frac{(b-1)g(x)}{1-G(x)}\right)=(b-1)\left(\frac{Bg(x)\left[1-G(x)\right]-\left[g(x)\right]^{2}}{\left[1-G(x)\right]^{2}}\right),$$

where $A=\frac{1}{x}\left(\beta-1-\beta(\theta x)^{\beta}\right)$ and $B=A+\left[\frac{(\alpha-1)q(x)}{\left(Q(x)\right)}\right]+\left[\frac{\lambda\alpha q(x)}{\left(Q(x)\right)^{1-\alpha}}\right].$

For $\eta'(x)$ equation, it is rather complex to completely characterize for possible values of each parameter in this study, however we can present the overall of shapes of the survival $S(x)$ and hazard functions $h(x)$ in Figure 1 that can be increasing, decreasing, upside-down and bathtub shape depending on its parameter values.

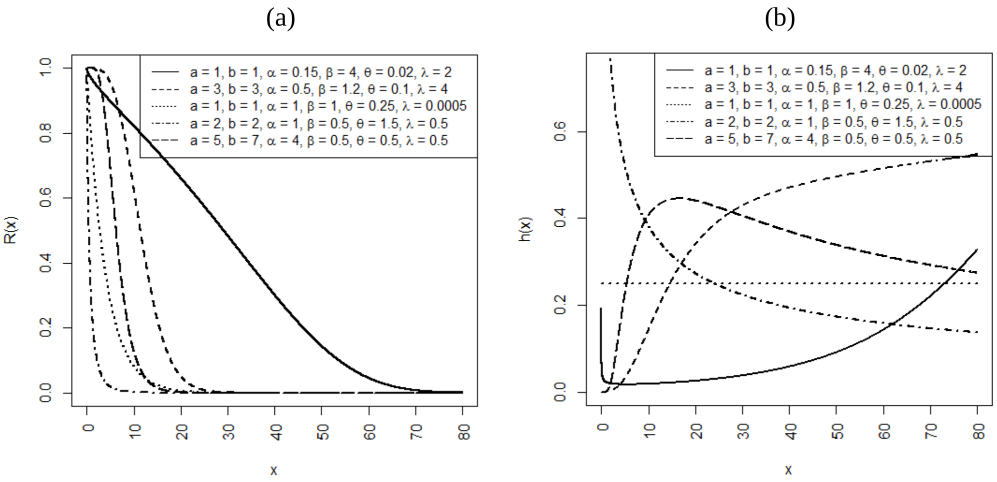

Figure 1 (a) The survival $S(x)$ (b) hazard functions $h(x)$ function of the BEWP distribution

For general case of Beta-G distribution,

$$S(x) = 1 - F(x)$$

$$= 1 - \frac{B_{G(x)}(a,b)}{B(a,b)}$$

$$= \frac{B_{1-G(x)}(b,a)}{B(a,b)}$$

$$h(x) = \frac{g(x)G(x)^{a-1}\left[1-G(x)\right]^{b-1}}{B_{1-G(x)}(b,a)},$$

where $B(a,b)$ and $B_{(\bullet)}(a,b)$ are the beta function and the incomplete beta function respectively.

From above equation, we can see that hazard function depend on the form of baseline distribution function that are $g(x)$, $G(x)$ and the value of a and b so we propose $S(x)$, $h(x)$ and $\eta(x)$ of BEWP distribution to 3 categories in Table 1 that are when $a > 0$ and $b > 0$, this category is Beta-G distribution, for $a > 0$ and $b = 1$, this category is Exponentiated-G distribution and for $a = 1$ and $b = 1$ in the last category, this is the base line distribution.

Table 1 The survival, hazard and $\eta(x)$ functions of BEWP distribution

Category	$S(x)$	$h(x)$	$\eta(x)$
1. $a>0, b>0$	$\frac{B_{1-G(x)}(b,a)}{B(a,b)}$	$\frac{g(x)G(x)^{a-1}\left[1-G(x)\right]^{b-1}}{B_{1-G(x)}(b,a)}$	$-\left[\frac{g'(x)}{g(x)}+\frac{(a-1)g(x)}{G(x)}-\frac{(b-1)g(x)}{\left[1-G(x)\right]}\right]$
2. $a>0, b=1$	$1-G(x)^{a}$	$\frac{ag(x)G(x)^{a-1}}{1-G(x)^{a}}$	$-\left[\frac{g'(x)}{g(x)}+\frac{(a-1)G'(x)}{G(x)}\right]$
3. $a=1, b=1$	$1-G(x)$	$\frac{g(x)}{1-G(x)}$	$-\frac{g'(x)}{g(x)}$

We apply these 3 categories in Table 1 to BEWP's sub-model in Table 2. For example, in case of beta Weibull distribution, when $a=1$ and $b=1$, beta Weibull distribution will be transformed to Weibull distribution or baseline distribution (category 3). $S(x)$ and $h(x)$ of Weibull distribution are respectively given by

$$S(x)=e^{-(\theta x)^{\beta}},$$

$$h(x)=\frac{\beta\theta^{\beta}x^{\beta-1}e^{-(\theta x)^{\beta}}}{e^{-(\theta x)^{\beta}}}.$$

When $a>0$ and $b=1$, beta Weibull distribution will be transformed to exponentiated Weibull distribution or Exponentiated-G distribution (category 2). $S(x)$ and $h(x)$ of exponentiated Weibull distribution are respectively given by

$$S(x)=1-\left(1-e^{-(\theta x)^{\beta}}\right)^{a},$$

$$h(x)=\frac{a\left(\beta\theta^{\beta}x^{\beta-1}e^{-(\theta x)^{\beta}}\right)\left(1-e^{-(\theta x)^{\beta}}\right)^{a-1}}{1-\left(1-e^{-(\theta x)^{\beta}}\right)^{a}}.$$

For $a>0$ and $b>0$, it is the beta Weibull distribution or Beta-G distribution (category 1). $S(x)$ and $h(x)$ of beta Weibull distribution are respectively given by

$$S(x)=\frac{B_{e^{-(\theta x)^{\beta}}}(b,a)}{B(a,b)},$$

$$h(x)=\frac{\left(\beta\theta^{\beta}x^{\beta-1}e^{-(\theta x)^{\beta}}\right)\left(1-e^{-(\theta x)^{\beta}}\right)^{a-1}\left(e^{-(\theta x)^{\beta}}\right)^{b-1}}{B_{e^{-(\theta x)^{\beta}}}(b,a)}.$$

Hence in reliability analysis, we can create the $S(x)$, $h(x)$ and $\eta(x)$ of Exponentiated-G and Beta-G distribution from any base line distribution without starting at its pdf or cdf.

Table 2 The sub-model table of BEWP distribution

Category	Distribution	Parameters				Base line distribution	
		α	β	θ	λ	$G(x)$	$g(x)$
1 3	beta exponentiated Weibull Poisson (BEWP) exponentiated Weibull Poisson (EWP)	α	β	θ	λ	$\frac{e^{\lambda(1-u)^{\alpha}}-1}{e^{\lambda}-1}$, $u=e^{-(\theta x)^{\beta}}$	$\frac{\lambda\alpha\beta\theta^{\beta}}{e^{\lambda}-1}x^{\beta-1}u(1-u)^{\alpha-1}e^{\lambda(1-u)^{\alpha}}$
1 3	beta exponentiated Rayleigh Poisson (BERP) exponentiated Rayleigh Poisson (ERP)	α	2	θ	λ	$\frac{e^{\lambda(1-u)^{\alpha}}-1}{e^{\lambda}-1}$, $u=e^{-(\theta x)^{2}}$	$\frac{2\lambda\alpha\theta^{2}}{e^{\lambda}-1}xu(1-u)^{\alpha-1}e^{\lambda(1-u)^{\alpha}}$
1 3	beta exponentiated exponential Poisson (BEEP) exponentiated exponential Poisson (EEP)	α	1	θ	λ	$\frac{e^{\lambda(1-u)^{\alpha}}-1}{e^{\lambda}-1}$, $u=e^{-(\theta x)}$	$\frac{\lambda\alpha\theta}{e^{\lambda}-1}u(1-u)^{\alpha-1}e^{\lambda(1-u)^{\alpha}}$
1 2 3	beta Weibull Poisson (BWP) generalized Weibull Poisson (GWP) Weibull Poisson (WP)	1	β	θ	λ	$\frac{e^{\lambda(1-u)}-1}{e^{\lambda}-1}$, $u=e^{-(\theta x)^{\beta}}$	$\frac{\lambda\beta\theta^{\beta}}{e^{\lambda}-1}x^{\beta-1}ue^{\lambda(1-u)}$
1 2 3	beta Rayleigh Poisson (BRP) generalized Rayleigh Poisson (GRP) Rayleigh Poisson (RP)	1	2	θ	λ	$\frac{e^{\lambda(1-u)}-1}{e^{\lambda}-1}$, $u=e^{-(\theta x)^{2}}$	$\frac{2\lambda\theta^{2}}{e^{\lambda}-1}xue^{\lambda(1-u)}$
1 2 3	beta exponential Poisson (BEP) generalized exponential Poisson (GEP) exponential Poisson (EP)	1	1	θ	λ	$\frac{e^{\lambda(1-u)}-1}{e^{\lambda}-1}$, $u=e^{-(\theta x)}$	$\frac{\lambda\theta}{e^{\lambda}-1}ue^{\lambda(1-u)}$

Table 2 (Continued)

Category	Distribution	Parameters				Base line distribution	
		α	β	θ	λ	$G(x)$	$g(x)$
1	beta exponentiated Weibull (BEW)	α	β	θ	$\to 0^{+}$	$(1-u)^{\alpha}$, $u=e^{-(\theta x)^{\beta}}$	$\alpha\beta\theta^{\beta}x^{\beta-1}u(1-u)^{\alpha-1}$
2	generalized exponentiated Weibull (GEW)						
3	exponentiated Weibull (EW)						
1	beta exponentiated Rayleigh (BER)	α	2	θ	$\to 0^{+}$	$(1-u)^{\alpha}$, $u=e^{-(\theta x)^{2}}$	$2\alpha\theta^{\beta}xu(1-u)^{\alpha-1}$
2	generalized exponentiated Rayleigh (GER)						
3	exponentiated Rayleigh (ER)						
1	beta exponentiated exponential (BEE)	α	1	θ	$\to 0^{+}$	$(1-u)^{\alpha}$, $u=e^{-(\theta x)}$	$\alpha\theta u(1-u)^{\alpha-1}$
2	generalized exponentiated exponential (GEE)						
3	exponentiated exponential (EE)						
1	beta Weibull (BW)	1	β	θ	$\to 0^{+}$	$1-e^{-(\theta x)^{\beta}}$	$\beta\theta^{\beta}x^{\beta-1}e^{-(\theta x)^{\beta}}$
2	generalized Weibull (GW)						
3	Weibull (W)						
1	beta Rayleigh (BR)	1	2	θ	$\to 0^{+}$	$1-e^{-(\theta x)^{2}}$	$2\theta^{2}xe^{-(\theta x)^{2}}$
2	generalized Rayleigh (GR)						
3	Rayleigh (R)						
1	beta exponential (BE)	1	1	θ	$\to 0^{+}$	$1-e^{-(\theta x)}$	$\theta e^{-(\theta x)}$
2	generalized exponential (GE)						
3	exponential (E)						

By definition of hazard function, it can be defined based on the concept of conditional probability. In addition to the hazard function, this concept has also been applied to define the other interesting reliability measurements, for instance, the MRL function. This function characterizes the random variable in the total remaining interval of time after a fixed time point t (Jeong 2014), whereas the failure rate provides a description of it in an infinitesimal interval of time. Both are complementary of each other functions. The MRL function will be presented in the next section.

3. Mean Residual Life Function of the BEWP Distribution

Guess and Proschan (as cited in (Krishnaiah and Rao 1988)) defined the MRL function is like the density function, the moment generating function, or the characteristic function. It greatly impacts the functionality of many applications such as engineering, insurance and etc. We introduced the interpretation of MRL in the previous section, by definition, given that a unit is of age t, the remaining life after time t is random so that the mean residual life at time t is the expected value of this random residual life or $E\left[X-t\middle|X>t\right]$. In this section we present the r^{th} moment, mean and second moment of residual life function.

Theorem 1 *Let X be a random variable of the BEWP distribution with parameters $\alpha,\beta,\theta,\lambda,a$ and b. The r^{th} moment of residual life function of X is*

$$m_r(t)=\frac{1}{S(t)}\sum_{j=0}^{\infty}\sum_{i=0}^{j}\frac{s_{j,i}\lambda_{j,i}\alpha}{\theta^k\left(e^{\lambda_{j,i}}-1\right)}\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}t^{r-k}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\lambda_{j,i}^m}{m!}(-1)^n\binom{\alpha(m+1)-1}{n}(n+1)^{-\left(\frac{k}{\beta}+1\right)}\Gamma\left(\frac{k}{\beta}+1,(n+1)(\theta t)^\beta\right)$$

Proof:

$$m_r(t)=E\left[(X-t)^r\,\middle|X>t\right]$$

$$=\frac{1}{S(t)}\int_t^\infty (x-t)^r\, f(x)dx\,.$$

We refer $f(x)=\sum_{j=0}^{\infty}\sum_{i=o}^{j}s_{j,i}g(x;\alpha,\beta,\theta,\lambda_{j,i})$ from Insuk et al. (2015) where

$$g(x;\alpha,\beta,\theta,\lambda_{j,i})=\frac{\lambda_{j,i}\alpha\beta\theta^\beta x^{\beta-1}u(1-u)^{\alpha-1}e^{\lambda_{j,i}(1-u)^\alpha}}{\left(e^{\lambda_{j,i}}-1\right)},\quad s_{j,i}=\frac{s_j(j+1)(-1)^i\binom{j}{i}\left(e^{\lambda_{j,i}}-1\right)}{(j-i+1)\left(e^{\lambda}-1\right)^{j+1}},$$

$$\lambda_{j,i}=\lambda(j-i+1),\quad s_j=\frac{r_{j+1}}{B(a,b)},\quad r_j(a,b)=\sum_{i=0}^{\infty}c_i(a,b)d_j(a+i),\quad d_j(\alpha)=\sum_{j=i}^{\infty}\frac{(-1)^{i+j}\Gamma(\alpha+1)}{\Gamma(\alpha-i+1)(i-j)!\,j!}$$

and $c_i(a,b)=\dfrac{(-1)^i\binom{b-1}{i}}{(a+i)}$,

then we obtain

$$m_r(t)=\frac{1}{S(t)}\int_t^\infty (x-t)^r\sum_{j=0}^{\infty}\sum_{i=0}^{j}s_{j,i}\frac{\lambda_{j,i}\alpha\beta\theta^\beta x^{\beta-1}u(1-u)^{\alpha-1}e^{\lambda_{j,i}(1-u)^\alpha}}{\left(e^{\lambda_{j,i}}-1\right)}dx.$$

Since $(x-t)^r=\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}x^k t^{r-k}$, hence

$$m_r(t)=\frac{1}{S(t)}\int_t^\infty\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}x^k t^{r-k}\sum_{j=0}^{\infty}\sum_{i=0}^{j}s_{j,i}\frac{\lambda_{j,i}\alpha\beta\theta^\beta x^{\beta-1}u(1-u)^{\alpha-1}e^{\lambda_{j,i}(1-u)^\alpha}}{\left(e^{\lambda_{j,i}}-1\right)}dx$$

$$=\frac{1}{S(t)}\int_t^\infty\sum_{j=0}^{\infty}\sum_{i=0}^{j}\frac{s_{j,i}\lambda_{j,i}\alpha\beta\theta^\beta}{\left(e^{\lambda_{j,i}}-1\right)}\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}t^{r-k}x^{k+\beta-1}u(1-u)^{\alpha-1}e^{\lambda_{j,i}(1-u)^\alpha}dx$$

$$=\frac{1}{S(t)}\int_t^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{j}\frac{s_{j,i}\lambda_{j,i}\alpha\beta\theta^{\beta}}{\left(e^{\lambda_{j,i}}-1\right)}\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}t^{r-k}x^{k+\beta-1}u\sum_{m=0}^{\infty}\frac{\lambda_{j,i}^{m}(1-u)^{\alpha m+\alpha-1}}{m!}dx$$

$$=\frac{1}{S(t)}\sum_{j=0}^{\infty}\sum_{i=0}^{j}\frac{s_{j,i}\lambda_{j,i}\alpha\beta\theta^{\beta}}{\left(e^{\lambda_{j,i}}-1\right)}\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}t^{r-k}\int_t^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\lambda_{j,i}^{m}}{m!}(-1)^{n}\binom{\alpha(m+1)-1}{n}x^{k+\beta-1}u^{n+1}dx.$$

Since $\int_t^{\infty}x^{k+\beta-1}u^{n+1}dx=\int_t^{\infty}x^{k+\beta-1}e^{-(n+1)(\theta x)^{\beta}}dx$ and

$$\int_t^{\infty}x^{k+\beta-1}u^{n+1}dx=\int_{(n+1)(\theta t)^{\beta}}^{\infty}\frac{z^{\frac{k}{\beta}}e^{-z}}{\beta(n+1)^{\frac{k}{\beta}+1}\theta^{\beta+k}}dz$$

$$=\frac{\Gamma\left(\frac{k}{\beta}+1,(n+1)(\theta t)^{\beta}\right)}{\beta(n+1)^{\frac{k}{\beta}+1}\theta^{\beta+k}}.$$

We obtain

$$m_r(t)=\frac{1}{S(t)}\sum_{j=0}^{\infty}\sum_{i=0}^{j}\frac{s_{j,i}\lambda_{j,i}\alpha}{\theta^{k}\left(e^{\lambda_{j,i}}-1\right)}\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}t^{r-k}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\lambda_{j,i}^{m}}{m!}(-1)^{n}\binom{\alpha(m+1)-1}{n}(n+1)^{-\left(\frac{k}{\beta}+1\right)}\Gamma\left(\frac{k}{\beta}+1,(n+1)(\theta t)^{\beta}\right).$$

Theorem 2 *Let* X *be a random variable of the BEWP distribution with parameters* $\alpha,\beta,\theta,\lambda,a$ *and* b*. The MRL function of* X *is*

$$m(t)=\sum_{j=0}^{\infty}\sum_{i=0}^{j}\frac{s_{j,i}\lambda_{j,i}\alpha}{S(t)\theta\left(e^{\lambda_{j,i}}-1\right)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\lambda_{j,i}^{m}}{m!}(-1)^{n}\binom{\alpha(m+1)-1}{n}(n+1)^{-\left(\frac{1}{\beta}+1\right)}\Gamma\left(\frac{1}{\beta}+1,(n+1)(\theta t)^{\beta}\right)-t$$

Proof:

$$m(t)=E\left[X-t\,|\,X>t\right]$$

$$=\frac{1}{S(t)}\int_t^{\infty}(x-t)f(x)dx$$

$$=\frac{1}{S(t)}\int_t^{\infty}xf(x)dx-t.$$

We refer $f(x)=\sum_{j=0}^{\infty}\sum_{i=o}^{j}s_{j,i}g(x;\alpha,\beta,\theta,\lambda_{j,i})$ from Insuk et al. (2015) where

$$g(x;\alpha,\beta,\theta,\lambda_{j,i})=\frac{\lambda_{j,i}\alpha\beta\theta^{\beta}x^{\beta-1}u(1-u)^{\alpha-1}e^{\lambda_{j,i}(1-u)^{\alpha}}}{\left(e^{\lambda_{j,i}}-1\right)},\ s_{j,i}=\frac{s_j(j+1)(-1)^{i}\binom{j}{i}\left(e^{\lambda_{j,i}}-1\right)}{(j-i+1)\left(e^{\lambda}-1\right)^{j+1}},\ \lambda_{j,i}=\lambda(j-i+1),$$

$$s_j=\frac{r_{j+1}}{B(a,b)},\ \ r_j(a,b)=\sum_{i=0}^{\infty}c_i(a,b)d_j(a+i),\ \ d_j(\alpha)=\sum_{j=i}^{\infty}\frac{(-1)^{i+j}\Gamma(\alpha+1)}{\Gamma(\alpha-i+1)(i-j)!\,j!}\ \text{and}\ c_i(a,b)=\frac{(-1)^{i}\binom{b-1}{i}}{(a+i)},$$

then we obtain

$$\int_t^\infty xf(x)dx = \int_t^\infty x\sum_{j=0}^\infty\sum_{i=0}^j s_{j,i}\frac{\lambda_{j,i}\alpha\beta\theta^\beta x^{\beta-1}u(1-u)^{\alpha-1}e^{\lambda_{j,i}(1-u)^\alpha}}{\left(e^{\lambda_{j,i}}-1\right)}dx$$

$$=\sum_{j=0}^\infty\sum_{i=0}^j s_{j,i}\frac{\lambda_{j,i}\alpha\beta\theta^\beta}{\left(e^{\lambda_{j,i}}-1\right)}\int_t^\infty x^\beta u(1-u)^{\alpha-1}\sum_{m=0}^\infty\frac{\left[\lambda_{j,i}(1-u)^\alpha\right]^m}{m!}dx$$

$$=\sum_{j=0}^\infty\sum_{i=0}^j s_{j,i}\frac{\lambda_{j,i}\alpha\beta\theta^\beta}{\left(e^{\lambda_{j,i}}-1\right)}\int_t^\infty\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{\lambda_{j,i}^m}{m!}(-1)^n\binom{\alpha(m+1)-1}{n}x^\beta u^{n+1}dx.$$

We consider $\int_t^\infty x^\beta u^{n+1}dx = \int_t^\infty x^\beta e^{-(n+1)(\theta x)^\beta}dx$ and

$$\int_t^\infty x^\beta u^{n+1}dx = \int_{(n+1)(\theta t)^\beta}^\infty\frac{z^{\frac{1}{\beta}}e^{-z}}{\beta(n+1)^{\frac{1}{\beta}+1}\theta^{\beta+1}}dz$$

$$=\frac{\Gamma\left(\frac{1}{\beta}+1,(n+1)(\theta t)^\beta\right)}{\beta(n+1)^{\frac{1}{\beta}+1}\theta^{\beta+1}}.$$

Then we obtain

$$\int_t^\infty x\sum_{j=0}^\infty\sum_{i=0}^j s_{j,i}g\left(x;\alpha,\beta,\theta,\lambda_{j,i}\right)dx$$

$$=\sum_{j=0}^\infty\sum_{i=0}^j\frac{s_{j,i}\lambda_{j,i}\alpha}{\theta\left(e^{\lambda_{j,i}}-1\right)}\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{\lambda_{j,i}^m}{m!}(-1)^n\binom{\alpha(m+1)-1}{n}(n+1)^{-\left(\frac{1}{\beta}+1\right)}\Gamma\left(\frac{1}{\beta}+1,(n+1)(\theta t)^\beta\right)$$

and

$$m(t)=\sum_{j=0}^\infty\sum_{i=0}^j\frac{s_{j,i}\lambda_{j,i}\alpha}{S(t)\theta\left(e^{\lambda_{j,i}}-1\right)}\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{\lambda_{j,i}^m}{m!}(-1)^n\binom{\alpha(m+1)-1}{n}(n+1)^{-\left(\frac{1}{\beta}+1\right)}\Gamma\left(\frac{1}{\beta}+1,(n+1)(\theta t)^\beta\right)-t.$$

To show more variety of the hazard functions and MRL function shapes, some specified parameters of the BEWP distribution are provided in Figure 2.

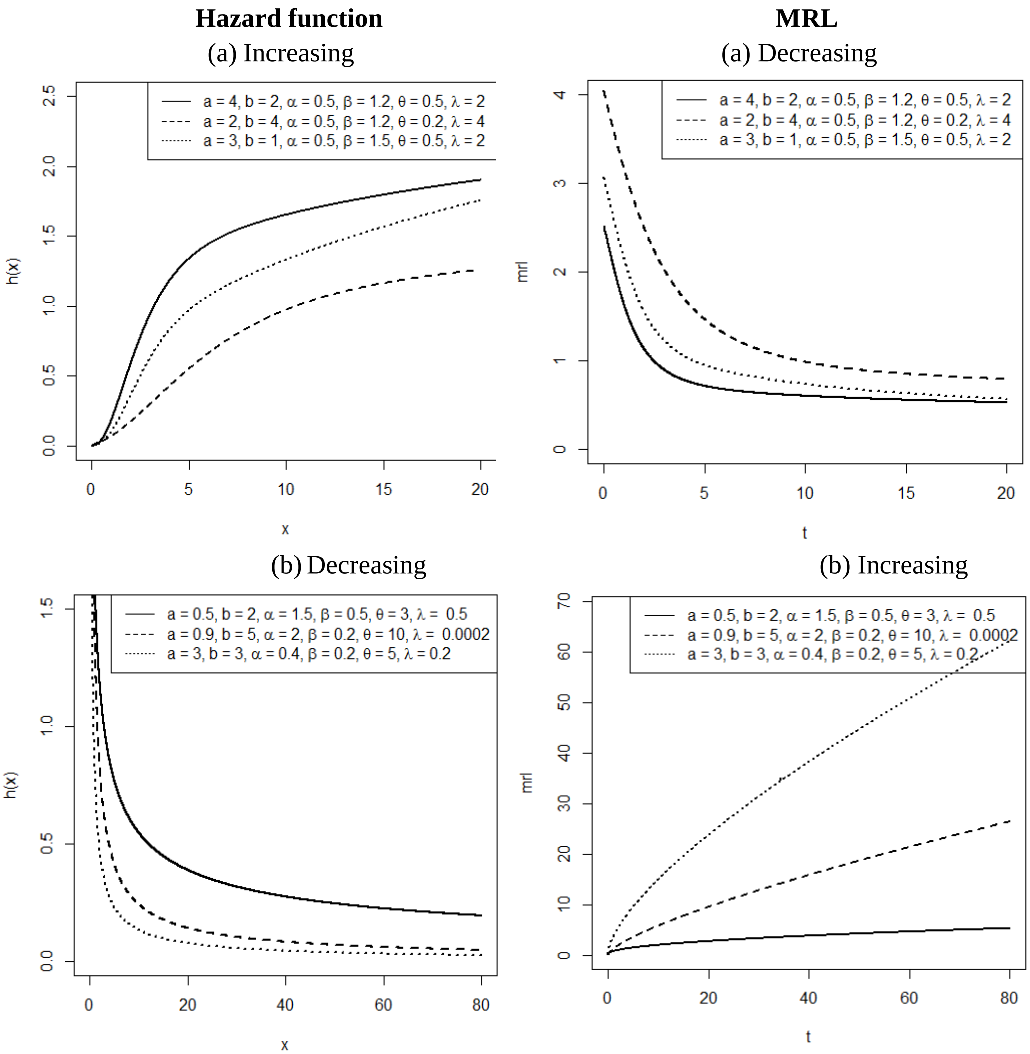

Figure 2 Hazard function and mean residual life function shapes of the BEWP distribution

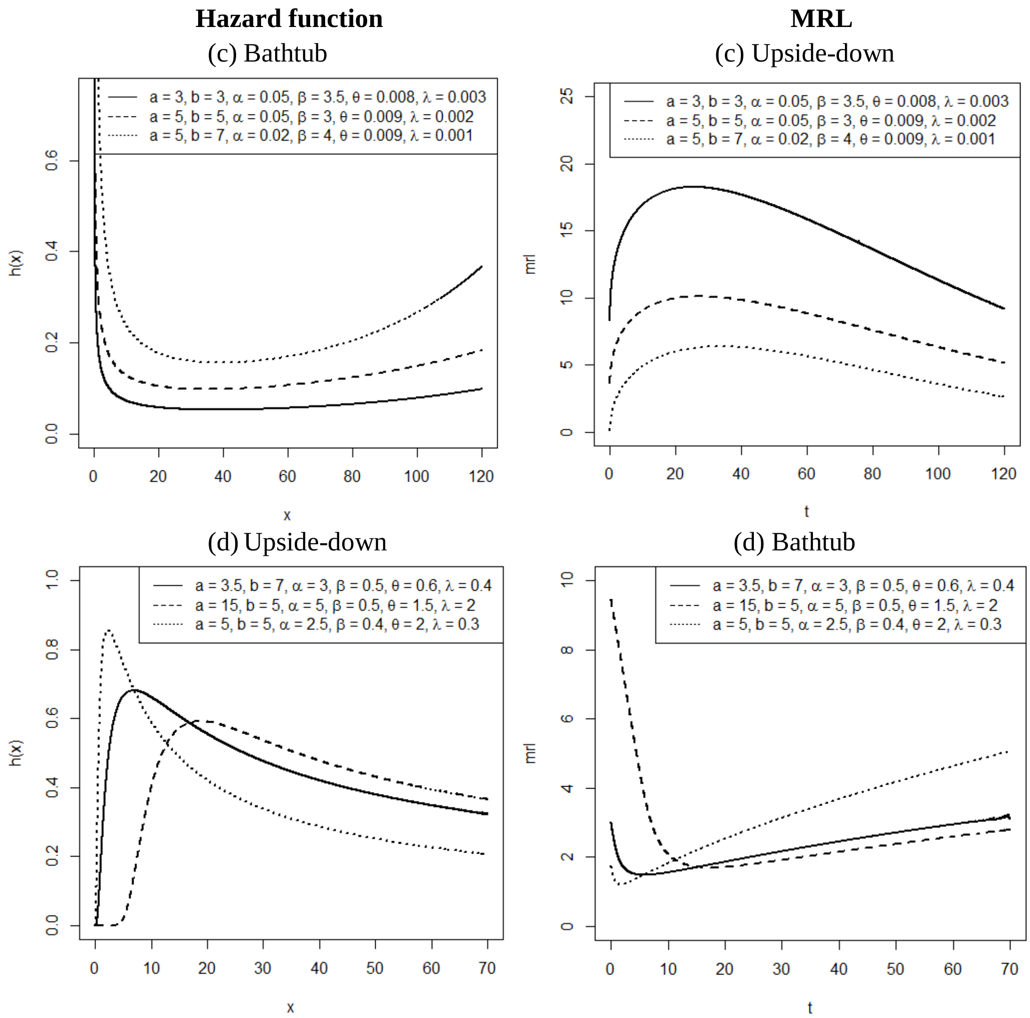

Figure 2 (Continued)

Theorem 3 *Let X be a random variable of the BEWP distribution with parameters* $\alpha,\beta,\theta,\lambda,a$ *and b. The second moment of the residual life function of* X *is*

$$m_2(t)=\sum_{j=0}^{\infty}\sum_{i=0}^{j}\frac{s_{j,i}\lambda_{j,i}\alpha}{S(t)\theta^2\left(e^{\lambda_{j,i}}-1\right)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\lambda_{j,i}^m}{m!}(-1)^n\binom{\alpha(m+1)-1}{n}(n+1)^{-\left(\frac{2}{\beta}+1\right)}\Gamma\left(\frac{2}{\beta}+1,(n+1)(\theta t)^{\beta}\right)$$

$$-2t\sum_{j=0}^{\infty}\sum_{i=0}^{j}\frac{s_{j,i}\lambda_{j,i}\alpha}{S(t)\theta\left(e^{\lambda_{j,i}}-1\right)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\lambda_{j,i}^m}{m!}(-1)^n\binom{\alpha(m+1)-1}{n}(n+1)^{-\left(\frac{1}{\beta}+1\right)}\Gamma\left(\frac{1}{\beta}+1,(n+1)(\theta t)^{\beta}\right)+t^2$$

Proof:

$$\begin{aligned}m_2(t)&=E\left[(X-t)^2\middle|X>t\right]\\&=\frac{1}{S(t)}\int_t^{\infty}(x-t)^2\, f(x)dx\\&=\frac{1}{S(t)}\left(\int_t^{\infty}x^2 f(x)dx-2t\int_t^{\infty}xf(x)dx+t^2\right).\end{aligned}$$

We refer $f(x)=\sum_{j=0}^{\infty}\sum_{i=o}^{j}s_{j,i}g(x;\alpha,\beta,\theta,\lambda_{j,i})$ from Insuk et al. (2015) where

$$g(x;\alpha,\beta,\theta,\lambda_{j,i})=\frac{\lambda_{j,i}\alpha\beta\theta^{\beta}x^{\beta-1}u(1-u)^{\alpha-1}e^{\lambda_{j,i}(1-u)^{\alpha}}}{\left(e^{\lambda_{j,i}}-1\right)},\quad s_{j,i}=\frac{s_j(j+1)(-1)^i\binom{j}{i}\left(e^{\lambda_{j,i}}-1\right)}{(j-i+1)\left(e^{\lambda}-1\right)^{j+1}},$$

$$\lambda_{j,i}=\lambda(j-i+1),\quad s_j=\frac{r_{j+1}}{B(a,b)},\quad r_j(a,b)=\sum_{i=0}^{\infty}c_i(a,b)d_j(a+i),\quad d_j(\alpha)=\sum_{j=i}^{\infty}\frac{(-1)^{i+j}\Gamma(\alpha+1)}{\Gamma(\alpha-i+1)(i-j)!\,j!}$$

and $c_i(a,b)=\dfrac{(-1)^i\binom{b-1}{i}}{(a+i)}$,

then we obtain

$$\begin{aligned}\int_t^{\infty}x^2\sum_{j=0}^{\infty}\sum_{i=0}^{j}s_{j,i}g(x;\alpha,\beta,\theta,\lambda_{j,i})dx&=\int_t^{\infty}x^2\sum_{j=0}^{\infty}\sum_{i=0}^{j}s_{j,i}\frac{\lambda_{j,i}\alpha\beta\theta^{\beta}x^{\beta-1}u(1-u)^{\alpha-1}e^{\lambda_{j,i}(1-u)^{\alpha}}}{\left(e^{\lambda_{j,i}}-1\right)}dx\\&=\sum_{j=0}^{\infty}\sum_{i=0}^{j}s_{j,i}\frac{\lambda_{j,i}\alpha\beta\theta^{\beta}}{\left(e^{\lambda_{j,i}}-1\right)}\int_t^{\infty}x^{\beta+1}u(1-u)^{\alpha-1}\sum_{m=0}^{\infty}\frac{\left[\lambda_{j,i}(1-u)^{\alpha}\right]^m}{m!}dx\\&=\sum_{j=0}^{\infty}\sum_{i=0}^{j}s_{j,i}\frac{\lambda_{j,i}\alpha\beta\theta^{\beta}}{\left(e^{\lambda_{j,i}}-1\right)}\int_t^{\infty}x^{\beta}u\sum_{m=0}^{\infty}\frac{\lambda_{j,i}^m}{m!}\sum_{n=0}^{\infty}(-1)^n\binom{\alpha(m+1)-1}{n}u^n dx\\&=\sum_{j=0}^{\infty}\sum_{i=0}^{j}s_{j,i}\frac{\lambda_{j,i}\alpha\beta\theta^{\beta}}{\left(e^{\lambda_{j,i}}-1\right)}\int_t^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\lambda_{j,i}^m}{m!}(-1)^n\binom{\alpha(m+1)-1}{n}x^{\beta+1}u^{n+1}dx.\end{aligned}$$

We consider $\int_t^\infty x^{\beta+1}u^{n+1}dx = \int_t^\infty x^{\beta+1}e^{-(n+1)(\theta x)^\beta}dx$ and

$$\int_t^\infty x^{\beta+1}u^{n+1}dx = \int_{(n+1)(\theta t)^\beta}^\infty \frac{z^{\frac{2}{\beta}}e^{-z}}{\beta(n+1)^{\frac{2}{\beta}+1}\theta^{\beta+2}}dz$$

$$= \frac{\Gamma\left(\frac{2}{\beta}+1,(n+1)(\theta t)^\beta\right)}{\beta(n+1)^{\frac{2}{\beta}+1}\theta^{\beta+2}},$$

then we obtain

$$\int_t^\infty x^2\sum_{j=0}^\infty\sum_{i=0}^j s_{j,i}g(x;\alpha,\beta,\theta,\lambda_{j,i})dx = \sum_{j=0}^\infty\sum_{i=0}^j \frac{s_{j,i}\lambda_{j,i}\alpha\beta\theta^\beta}{\left(e^{\lambda_{j,i}}-1\right)}\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{\lambda_{j,i}^m}{m!}(-1)^n\binom{\alpha(m+1)-1}{n}\frac{\Gamma\left(\frac{2}{\beta}+1,(n+1)(\theta t)^\beta\right)}{\beta(n+1)^{\frac{2}{\beta}+1}\theta^{\beta+2}}$$

$$= \sum_{j=0}^\infty\sum_{i=0}^j \frac{s_{j,i}\lambda_{j,i}\alpha}{\theta^2\left(e^{\lambda_{j,i}}-1\right)}\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{\lambda_{j,i}^m}{m!}(-1)^n\binom{\alpha(m+1)-1}{n}(n+1)^{-\left(\frac{2}{\beta}+1\right)}\Gamma\left(\frac{2}{\beta}+1,(n+1)(\theta t)^\beta\right).$$

We use the resulting of $\int_t^\infty xf(x)dx$ term from Theorem 2, then

$$m_2(t) = \sum_{j=0}^\infty\sum_{i=0}^j \frac{s_{j,i}\lambda_{j,i}\alpha}{S(t)\theta^2\left(e^{\lambda_{j,i}}-1\right)}\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{\lambda_{j,i}^m}{m!}(-1)^n\binom{\alpha(m+1)-1}{n}(n+1)^{-\left(\frac{2}{\beta}+1\right)}\Gamma\left(\frac{2}{\beta}+1,(n+1)(\theta t)^\beta\right)$$

$$-2t\sum_{j=0}^\infty\sum_{i=0}^j \frac{s_{j,i}\lambda_{j,i}\alpha}{S(t)\theta\left(e^{\lambda_{j,i}}-1\right)}\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{\lambda_{j,i}^m}{m!}(-1)^n\binom{\alpha(m+1)-1}{n}(n+1)^{-\left(\frac{1}{\beta}+1\right)}\Gamma\left(\frac{1}{\beta}+1,(n+1)(\theta t)^\beta\right)+t^2$$

We can find variance of the residual life of the BEWP distribution by using the relationship of each moment in Theorems 2 and 3.

4. Application

In this section, we illustrate the application of BEWP's failure rate and MRL function as a means for undertaking the burn-in analysis. Because of the high failure rate in the infant mortality period of bathtub model and to reduce the product population failure rate, burn-in is the desired method for detecting and eliminating the failure in this period before the product is released to customer. Cheng (2006) also examined data to identify and critically discuss the practical methods to quantify and eliminate failure in each part of the bathtub curve. In this study, to eliminate failures in the first period of bathtub model, MRL function is the useful tool to determine the optimal burn-in time. Mi (1995) applied the simple idea to reveal maximized mean life without cost constrain by using the MRL function to find the optimal burn-in time. It means that Mi (1995) determined *t* that causes the maximized MRL.

For application with real data, we provide the lifetime failure data of an electronic from Wang (2000) as displayed in Table 3.

Table 3 Time to failure of 18 electronic devices

Time to failure of 18 electronic devices								
5	11	21	31	46	75	98	122	145
165	195	224	245	293	321	330	350	420

To examine that this data set is bathtub shape failure rate we can show by the total time on test (TTT) plot (Aarset 1987) in Figure 3 (a) with p-value of Barlow-Proschan's test is 0.32. We fit the BEWP distribution to this data set by using the maximum likelihood (ML) method that Insuk et al. (2015) have shown in details. The ML estimates of the parameters are $\hat{\alpha} = 1.6053$, $\hat{\beta} =$ 4.0597, $\hat{\theta} =$ 0.0029, $\hat{\lambda} =$ 3.5905, $\hat{a} =$ 0.1093 and $\hat{b} = 1.9903$ the Kolmogorov-Smirnov (K-S) statistic and the corresponding p-value for the fitted models are 0.0708 and 0.9999 respectively. To show the graphical goodness of the fit by probability plot. We also plot the data against the BEWP distribution. In Figure 3 (b), the data closely forms a straight line. It indicates this data set follows the BEWP distribution.

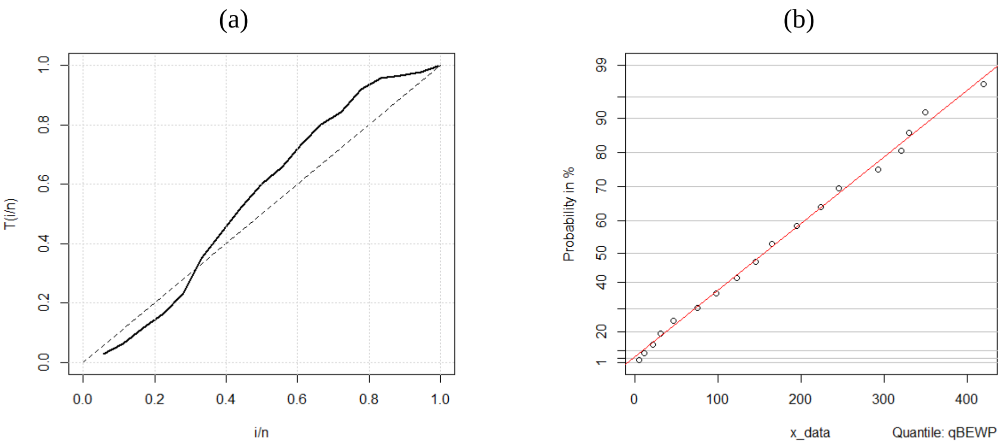

Figure 3 TTT plot and probability plot of the time to failure of 18 electronic devices

In order to make decisions to determine the optimal burn-in time under Mi (1995) approach, a burn-in test can be terminated at the time point of t^* = 11 with $m(t^*)$ = 177.3229. Shen et al. (2009) illustrated the optimum time by the other criterion.

5. Conclusions

For this paper, we present the structural properties in term of reliability a Beta class distribution, namely BEWP. We introduce its basic reliability properties such as reliability function, hazard function including its MRL function. For the application, we apply the MRL function of BEWP distribution for the purpose of discovering the optimal burn-in time for real failure data with the bathtub failure shape. Next research will discuss the details of the parameter that related to failure shape.

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