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Bayesian Estimation on the Generalized Logistic Distribution under Left Type-II Censoring

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Abstract

In this paper, given a left type II censored sample from a generalized logistic distribution, we obtain the Bayes estimators and corresponding risks of the unknown parameter under different asymmetric loss functions, assuming different informative and non-informative priors. Elicitation of hyperparameter through prior predictive approach is also discussed. Also we derive the expression for posterior predictive distributions and the credible Intervals. As an illustration, comparisons of these estimators are made through simulation study as well as real life data example along graphical results. The findings of the study indicate that the Bayes estimation under the gamma prior can be preferred.

Keywords: Left censoring, loss functions, posterior predictive distributions, credible intervals.

1. Introduction

Balakrishnan and Leung (1988) defined the type I generalized logistic distribution (Type I GLD) as one of the three generalized forms of the standard logistic distribution. Type I generalized logistic distribution has got additional attention in estimating its parameters for practical usage (see Balakrishnan (1992)). The skew logistic distribution with the skewness parameter θ has been studied by many others (see for example, Wahed and Ali (2001), Gupta et al. (2002), Nadrajah and Kotz (2006 and 2007), Nadrajah (2009), Gupta and Kundu (2010), and Chakraborty et al. (2012)). The cumulative distribution function (cdf), and the probability density function (pdf) of the type I generalized logistic distribution with shape or skewness parameter $\theta > 0$ are respectively as follows,

$$F(x \mid \theta) = \frac{1}{\{1+\exp(-x)\}^{\theta}}, \quad \theta > 0, \ -\infty < x < \infty, \tag{1}$$

$$f(x \mid \theta) = \frac{\theta \exp(-x)}{\{1+\exp(-x)\}^{\theta+1}}, \quad \theta > 0, \ -\infty < x < \infty. \tag{2}$$

The use of a Bayesian approach allows both sample and prior information to be incorporated into the statistical analysis, which will improve the quality of the inferences and permit a reduction in sample size. The decision-theoretic viewpoint takes into account additional information concerning the possible consequences of our decisions (quantified by a loss function). The main aim of this is to consider the statistical analysis of the unknown parameters when the data are left censored from the generalized logistic distribution. There is a widespread application and use of left-censoring or left-censored data in survival analysis and reliability theory. For example, in medical studies patients are subject to regular examinations. Discovery of a condition only tells us that the onset of sickness fell in the period since the previous examination and nothing about the exact date of the attack. Thus the time elapsed since onset has been left censored. Similarly, we have to handle left-censored data when estimating functions of exact policy duration without knowing the exact date of policy entry; or when estimating functions of exact age without knowing the exact date of birth. A study on the "Patterns of Health Insurance Coverage among Rural and Urban Children" (Coburn, McBride and Ziller 2001) faces this problem due to the incidence of a higher proportion of rural children whose spells were "left censored" in the sample (i.e., those children who entered the sample uninsured), and who remained uninsured throughout the sample. A Job duration might be incomplete because the beginning of the job spells is not observed, which is an incidence of left censoring (Bagger 2005).

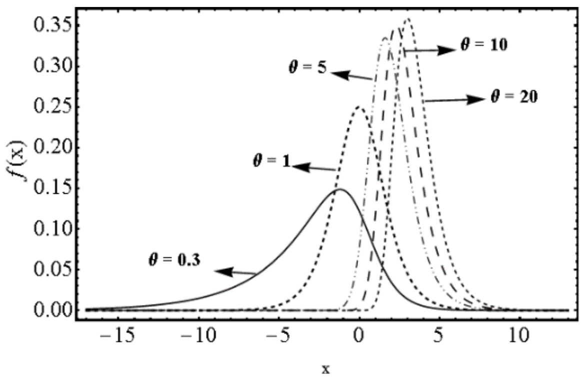

Figure 1 Density functions of type I generalized logistic distribution for different values of θ

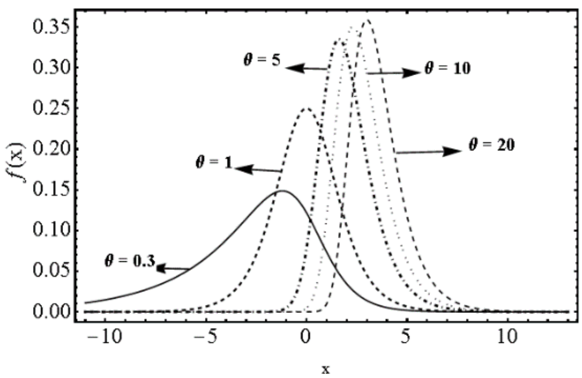

Figure 2 Density functions of type I generalized logistic distribution for different values of θ under left type-II censoring

The shapes of the density functions of type I generalized logistic distribution for different values of θ are given in Figure 1 and shapes of the density functions of type I generalized logistic distribution for different values of θ under left type-II censoring are presented in Figure 2. It is obvious from the figures that the shapes of the density function s of type I generalized logistic distribution are quite different for different values of θ. It is positively skewed for $\theta > 1$ and negatively skewed $0 < \theta < 1$. Because of this reason, the parameter θ can also be termed as the skewness parameter. For $\theta = 1$ the type I generalized logistic distribution coincides the standard logistic distribution and is symmetric. If the random variable X follows type I generalized logistic distribution, then the moment generating function of X is

$$M_X(t) = E\left(e^{tX}\right) = \theta \int_{-\infty}^{\infty} e^{-(1-t)x}\left(1+e^{-x}\right)^{-(\theta+1)} dx = \frac{\Gamma(1-t)(\theta+t)}{\Gamma(\theta)}.$$

Hence, the mean variance and different moments can be easily obtained. The mean and variance of X can be obtained as

$$E(X) = \psi(\theta) - \psi(1), \quad Var(X) = \psi'(\theta) - \psi'(1),$$

where $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$ and $\psi'(x) = \frac{d}{dx}\psi(x)$, known as digamma and polygamma functions respectively. Third and fourth order moment are $\psi''(\theta) - \psi''(1)$ and $\psi'''(\theta) - \psi'''(1)$, respectively. The coefficient of variation is $\frac{\sqrt{\psi'(\theta) - \psi'(1)}}{\psi(\theta) - \psi(1)}$. The skewness and kurtosis are $\frac{\psi''(\theta) - \psi''(1)}{\left(\psi'(\theta) - \psi'(1)\right)^{\frac{3}{2}}}$ and $\frac{\psi'''(\theta) - \psi'''(1)}{\left(\psi'(\theta) - \psi'(1)\right)^{2}}$, respectively.

This paper is devoted to obtain and compare Bayesian estimation based on different loss functions. The rest of the paper is organized as follows. In Section 2, we derive posterior distribution under informative and non-informative priors in the presence of left censoring. In Section 3, we provide the Bayes estimator and corresponding posterior risks under different loss functions. Credible intervals are discussed in Section 4. Method of Elicitation of the hyper-parameters via prior predictive approach is discussed in Section 5. Posterior predictive distributions are derived in Section 6. Simulation study is conducted in Section 7. Data analysis with graphical results is discussions in Section 8. Section 9 presents the conclusion of the study.

2. Likelihood Function and Posterior Distribution

Let $X_{(r+1)}, \ldots, X_{(n)}$ be the last $n-r$ order statistics from a random sample of size n following Type I generalized logistic distribution. Then the joint probability density function of $X_{(r+1)}, \ldots, X_{(n)}$ is given by

$$\begin{aligned} f\left(x_{(r+1)}, \ldots, x_{(n)}; \theta\right) &= \frac{n!}{r!}\left(F(x_{(r+1)})\right)^r f\left(x_{(r+1)}\right) \ldots f\left(x_{(n)}\right) \\ &= \frac{n!}{r!}\left[\left\{1+\exp\left(-x_{(r+1)}\right)\right\}^{-\theta}\right]^r \prod_{i=r+1}^{n} \theta \exp\left(-x_{(i)}\right)\left\{1+\exp\left(-x_{(i)}\right)\right\}^{-(\theta+1)}, \\ &\propto \exp\left[-r\theta \ln\left\{1+\exp\left(-x_{(r+1)}\right)\right\}\right]\theta^{n-r} \exp\left[-(\theta+1)\left(\sum_{i=r+1}^{n} \ln\left\{1+\exp\left(-x_{(i)}\right)\right\}\right)\right], \\ &\propto \theta^{n-r} \exp\left[-\theta\left\{\sum_{i=r+1}^{n} \ln\left\{1+\exp\left(-x_{(i)}\right)\right\} + r\ln\left\{1+\exp\left(-x_{(r+1)}\right)\right\}\right\}\right], \end{aligned} \tag{3}$$

$$\propto \theta^R \exp\left[-\theta\zeta\left(x_{(i)}\right)\right],$$

where $R = n - r$, and $\zeta\left(x_{(i)}\right) = \sum_{i=r+1}^{n} \ln\left\{1+\exp\left(-x_{(i)}\right)\right\} + r\ln\left\{1+\exp\left(-x_{(r+1)}\right)\right\}.$

2.1. Prior and posterior distributions

Uniform prior reflects the lack of prior information and the Bayesian methodology may still work. Uniform prior may be proper or improper. Even if uniform prior is improper, we can still have a proper posterior. Equation (4) presents an improper prior while the posterior given in Equation (5) is proper one having total area under the curve equals to unity. The uniform prior for θ is defined as:

$$p(\theta) \propto k, \quad \theta > 0. \tag{4}$$

The posterior distribution under the uniform prior for the left censored data is:

$$p(\theta \mid x) = \frac{\left\{\zeta\left(x_{(i)}\right)\right\}^{R+1} \theta^R \exp\left\{-\theta\zeta\left(x_{(i)}\right)\right\}}{\Gamma(R+1)}, \quad \theta > 0. \tag{5}$$

Jeffreys prior is perhaps the most widely used non-informative prior in Bayesian analysis. The only requirement is a likelihood function from which the prior is then derived using Jeffreys' rule, which is to take the prior distribution to be the determinant of the square root of the Fisher information matrix.

$$p(\theta) \propto \frac{1}{\theta}, \quad \theta > 0. \tag{6}$$

The posterior distribution under the Jeffreys prior for the left censored data is:

$$p(\theta \mid x) = \frac{\left\{\zeta\left(x_{(i)}\right)\right\}^{R} \theta^{R-1} \exp\left\{-\theta\zeta\left(x_{(i)}\right)\right\}}{\Gamma(R)}, \quad \theta > 0. \tag{7}$$

The informative prior for the parameter θ is assumed to be exponential distribution:

$$p(\theta) = ce^{-c\theta}, \quad c > 0, \ \theta > 0. \tag{8}$$

The posterior distribution under the assumption of exponential prior is:

$$p(\theta \mid x) = \frac{\left\{c+\zeta\left(x_{(i)}\right)\right\}^{R+1} \theta^R \exp\left[-\theta\left\{c+\zeta\left(x_{(i)}\right)\right\}\right]}{\Gamma(R+1)}, \quad \theta > 0. \tag{9}$$

The informative prior for the parameter θ is assumed to be gamma distribution:

$$p(\theta) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}, \quad a, b, \theta > 0. \tag{10}$$

The posterior distribution under the assumption of gamma prior for the left censored data is:

$$p(\theta \mid x) = \frac{\left\{b+\zeta\left(x_{(i)}\right)\right\}^{R+a} \theta^{R+a-1} \exp\left[-\theta\left\{b+\zeta\left(x_{(i)}\right)\right\}\right]}{\Gamma(R+a)}, \quad \theta > 0. \tag{11}$$

The informative prior for the parameter θ is assumed to be inverse Levy distribution:

$$p(\theta) = \sqrt{\frac{h}{2\pi}}\theta^{-\frac{1}{2}}e^{-\left(\frac{h\theta}{2}\right)}, \quad h, \ \theta > 0. \tag{12}$$

The posterior distribution under the inverse Levy prior for the left censored data is:

$$p(\theta \mid \mathbf{x}) = \frac{\left\{\frac{h}{2}+\zeta\left(x_{(i)}\right)\right\}^{R+\frac{1}{2}} \theta^{R+\frac{1}{2}-1} \exp\left[-\theta\left\{\frac{h}{2}+\zeta\left(x_{(i)}\right)\right\}\right]}{\Gamma\left(R+\frac{1}{2}\right)}, \quad \theta > 0. \tag{13}$$

3. Bayes Estimators and Posterior Risks under Different Loss Functions

This section enlightens the derivation of the Bayes Estimator (BE) and corresponding Posterior Risks (PR) under different loss functions. The Bayes estimators are evaluated under Squared Error Loss Function (SELF), Precautionary Loss Function (PLF), Weighted Squared Error Loss Function (WSELF), Quasi-Quadratic Loss Function (QQLF), and Squared-Log Error Loss Function (SLELF). The Bayes Estimator (BE) and corresponding Posterior Risks (PR) under different loss functions are given in the following Table.

Table 1 Bayes estimator and posterior risks under different loss functions

Loss Function= $L\left(\theta,\hat{\theta}\right)$	Bayes Estimator	Posterior Risk
SELF: $\left(\theta-\hat{\theta}\right)^2$	$E\left(\theta \mid \mathbf{x}\right)$	$Var\left(\theta \mid \mathbf{x}\right)$
PLF: $\frac{\left(\theta-\hat{\theta}\right)^2}{\hat{\theta}}$	$\sqrt{E\left(\theta^2 \mid \mathbf{x}\right)}$	$2\left\{\sqrt{E\left(\theta^2 \mid \mathbf{x}\right)}-E\left(\theta \mid \mathbf{x}\right)\right\}$
WSELF: $\frac{\left(\theta-\hat{\theta}\right)^2}{\theta}$	$\left\{E\left(\theta \mid \mathbf{x}\right)\right\}^{-1}$	$E\left(\theta \mid \mathbf{x}\right)-\left\{E\left(\theta \mid \mathbf{x}\right)\right\}^{-1}$
QQLF: $\left(e^{-c\hat{\theta}}-e^{-c\theta}\right)^2$	$\frac{-1}{c}\ln\left\{E\left(e^{-c\theta} \mid \mathbf{x}\right)\right\}$	$E\left(e^{-c\theta}\right)-\left\{E\left(e^{-c\theta}\right)\right\}^2$
SLELF: $\left(\ln\hat{\theta}-\ln\theta\right)^2$	$\exp\left\{\mathrm{E}\left(\ln\theta \mid \mathbf{x}\right)\right\}$	$\mathrm{E}\left\{\left(\ln\theta \mid \mathbf{x}\right)\right\}^2-\left\{\mathrm{E}\left(\ln\theta \mid \mathbf{x}\right)\right\}^2$

The Bayes estimators and posterior risks under uniform prior are:

$$\hat{\theta}_{SELF} = \frac{n-r+1}{\zeta\left(x_{(i)}\right)}, \quad \rho\left(\hat{\theta}_{SELF}\right) = \frac{n-r+1}{\left\{\zeta\left(x_{(i)}\right)\right\}^2},$$

$$\hat{\theta}_{PLF} = \frac{\sqrt{(n-r+1)(n-r+2)}}{\zeta\left(x_{(i)}\right)}, \quad \rho\left(\hat{\theta}_{PLF}\right) = 2\left\{\frac{\sqrt{(n-r+1)(n-r+2)}}{\zeta\left(x_{(i)}\right)} - \frac{(n-r+1)}{\zeta\left(x_{(i)}\right)}\right\},$$

$$\hat{\theta}_{WSELF} = \frac{(n-r)}{\zeta\left(x_{(i)}\right)}, \quad \rho\left(\hat{\theta}_{WSELF}\right) = \frac{1}{\zeta\left(x_{(i)}\right)},$$

$$\hat{\theta}_{QQLF} = Log\left(\frac{\zeta\left(x_{(i)}\right)}{1+\zeta\left(x_{(i)}\right)}\right)^{-(n-r+1)}, \quad \rho\left(\hat{\theta}_{QQLF}\right) = \left(\frac{\zeta\left(x_{(i)}\right)}{2+\zeta\left(x_{(i)}\right)}\right)^{(n-r+1)} - \left(\frac{\zeta\left(x_{(i)}\right)}{1+\zeta\left(x_{(i)}\right)}\right)^{2(n-r+1)},$$

$$\hat{\theta}_{SLELF}=\frac{\exp\left(\psi\left(n-r+1\right)\right)}{\zeta\left(x_{(i)}\right)}, \qquad \rho\left(\hat{\theta}_{SLELF}\right)=\left\{\psi'\left(n-r+1\right)\right\}.$$

The Bayes estimators and posterior risks under the rest of priors can be obtained in a similar manner.

4. Bayes Credible Interval for the Left Censored Data

The Bayesian credible intervals for type II left censored data under informative and non-informative priors, as discussed by Saleem and Aslam (2009) are presented in the following. The credible intervals for type II left censored data under all priors are:

$$\frac{\chi^2_{2(n-r+1)\left(\frac{\alpha}{2}\right)}}{2\zeta\left(x_{(i)}\right)}<\theta_{Uniform}<\frac{\chi^2_{2(n-r+1)\left(1-\frac{\alpha}{2}\right)}}{2\zeta\left(x_{(i)}\right)}, \qquad \frac{\chi^2_{2(n-r)\left(\frac{\alpha}{2}\right)}}{2\zeta\left(x_{(i)}\right)}<\theta_{Jeffreys}<\frac{\chi^2_{2(n-r)\left(1-\frac{\alpha}{2}\right)}}{2\zeta\left(x_{(i)}\right)},$$

$$\frac{\chi^2_{2(n-r+1)\left(\frac{\alpha}{2}\right)}}{2\left\{c+\zeta\left(x_{(i)}\right)\right\}}<\theta_{Exponential}<\frac{\chi^2_{2(n-r+1)\left(1-\frac{\alpha}{2}\right)}}{2\left\{c+\zeta\left(x_{(i)}\right)\right\}}, \qquad \frac{\chi^2_{2(n-r+a)\left(\frac{\alpha}{2}\right)}}{2\left\{b+\zeta\left(x_{(i)}\right)\right\}}<\theta_{Gamma}<\frac{\chi^2_{2(n-r+a)\left(1-\frac{\alpha}{2}\right)}}{2\left\{b+\zeta\left(x_{(i)}\right)\right\}},$$

$$\frac{\chi^2_{2(n-r+1/2)\left(\frac{\alpha}{2}\right)}}{2\left\{h/2+\zeta\left(x_{(i)}\right)\right\}}<\theta_{In-Levy}<\frac{\chi^2_{2(n-r+1/2)\left(1-\frac{\alpha}{2}\right)}}{2\left\{h/2+\zeta\left(x_{(i)}\right)\right\}}.$$

5. Elicitation

Bayesian analysis elicitation of opinion is a crucial step. It helps make it easy for us to understand what the experts believe in and what their opinions are. In statistical inference the characteristics of a certain predictive distribution proposed by an expert determine the hyperparameters of a prior distribution.

In this article, we focus on a probability elicitation method known as prior predictive elicitation. Predictive elicitation is a method for estimating hyperparameters of prior distributions by inverting corresponding prior predictive distributions. Elicitation of hyperparameter from the prior $p(\theta)$ is conceptually difficult task because we first have to identify prior distribution and then its hyperparameters. The prior predictive distribution is used for the elicitation of the hyperparameters which is compared with the experts' judgment about this distribution and then the hyperparameters are chosen in such a way so as to make the judgment agree closely as possible with the given distribution (reader desires more detail see Grimshaw et al. (2001), O'Hagan et al. (2006), Kadane et al. (1996), Jenkinson (2005) and Leon et al. (2003)). According to Aslam (2003), the method of assessment is to compare the predictive distribution with experts' assessment about this distribution and then to choose the hyperparameters that make the assessment agree closely with the member of the family. He discusses three important methods to elicit the hyperparameters: (i) Via the prior predictive probabilities (ii) Via elicitation of the confidence levels (iii) Via the predictive mode and confidence level. We will use the prior predictive approach by Aslam (2003).

5.1. Prior predictive distribution

The prior predictive distribution is:

$$p(y)=\int_0^\infty p\left(y\mid\theta\right)p\left(\theta\right)d\theta. \tag{14}$$

The predictive distribution under exponential prior is:

$$p(y)=\int_0^\infty \theta\exp(-y)\{1+\exp(-y)\}^{-(\theta+1)}c\exp\{-\theta c\}d\theta. \quad (15)$$

After some simplification it reduces as

$$p(y)=\frac{c\exp(-y)\{1+\exp(-y)\}^{-1}}{\left[c+\ln\{1+\exp(-y)\}\right]^{2}}, \quad y>0. \quad (16)$$

The predictive distribution under gamma prior is:

$$p(y)=\frac{ab^{a}\exp(-y)\{1+\exp(-y)\}^{-1}}{\left[b+\ln\{1+\exp(-y)\}\right]^{(a+1)}}, \quad y>0, \quad (17)$$

$$p(y)=\frac{\sqrt{h}\exp(-y)\{1+\exp(-y)\}^{-1}}{2^{3/2}\left[\frac{h}{2}+\ln\{1+\exp(-y)\}\right]^{3/2}}, \quad y>0. \quad (18)$$

By using the method of elicitation defined by Aslam (2003), we obtain the following hyper-parameters c = 0.398658, a = 14.67459, b = 3.85996 and h =0.004175.

6. Posterior Predictive Distribution

The predictive distribution contains the information about the independent future random observation given preceding observations. The reader desire more details can see Bansal (2007). The posterior predictive distribution of the future observation $y=x_{n+1}$ is

$$p(y\mid x)=\int_0^\infty p(\theta\mid x)p(y\mid\theta)d\theta, \quad (19)$$

where $p(y)=\theta\exp(-y)\{1+\exp(-y)\}^{-(\theta+1)}$ is the future observation density and $p(\theta\mid\mathbf{x})$ is the posterior distribution obtained by incorporating the likelihood with the respective prior distributions.

The posterior predictive distribution of the future observation $y=x_{n+1}$ under uniform prior is

$$p(y\mid\mathbf{x})=\frac{(R+1)\left\{\zeta\left(x_{(i)}\right)\right\}^{(R+1)}\exp(-y)\{1+\exp(-y)\}^{-1}}{\left[\zeta\left(x_{(i)}\right)+\ln\{1+\exp(-y)\}\right]^{(R+2)}}, \quad y>0. \quad (20)$$

The posterior predictive distribution of the future observation $y=x_{n+1}$ under Jeffreys prior is

$$p(y\mid\mathbf{x})=\frac{(R)\left\{\zeta\left(x_{(i)}\right)\right\}^{(R)}\exp(-y)\{1+\exp(-y)\}^{-1}}{\left[\zeta\left(x_{(i)}\right)+\ln\{1+\exp(-y)\}\right]^{(R+1)}}, \quad y>0. \quad (21)$$

The posterior predictive distribution of the future observation $y=x_{n+1}$ under exponential prior is

$$p(y\mid\mathbf{x})=\frac{(R+1)\left\{c+\zeta\left(x_{(i)}\right)\right\}^{(R+1)}\exp(-y)\{1+\exp(-y)\}^{-1}}{\left[c+\zeta\left(x_{(i)}\right)+\ln\{1+\exp(-y)\}\right]^{(R+2)}}, \quad y>0. \quad (22)$$

The posterior predictive distribution of the future observation $y = x_{n+1}$ under gamma prior is

$$p(y \mid \mathbf{x}) = \frac{(R+a)\left\{b+\zeta\left(x_{(i)}\right)\right\}^{(R+a)} \exp(-y)\left\{1+\exp(-y)\right\}^{-1}}{\left[b+\zeta\left(x_{(i)}\right)+\ln\left\{1+\exp(-y)\right\}\right]^{(R+a+1)}}, \quad y > 0. \tag{23}$$

The posterior predictive distribution of the future observation $y = x_{n+1}$ under In-Levy prior is

$$p(y \mid \mathbf{x}) = \frac{\sum_{k=0}^{r}(-1)^k \binom{r}{k} \frac{s+1/2}{y^{(\upsilon+1)}\left\{c/2+\varsigma\left(x_{(i)}\right)+y^{-\upsilon}\right\}^{(s+3/2)}}}{\sum_{k=0}^{r}(-1)^k \binom{r}{k} \frac{1}{\left\{c/2+\varsigma\left(x_{(i)}\right)\right\}^{(s+1/2)}}}, \quad y > 0. \tag{24}$$

7. Simulation Study

This section shows how simulation can be helpful and illuminating way to approach problems in Bayesian analysis. Bayesian problems of updating estimates can be handled easily and straight forwardly with simulation. Since we can express the distribution function of the generalized logistic distribution as well as its inverse in closed form, the inversion method of simulation is straightforward to implement. The study has been carried out for different values of (n, r) using $\theta \in 2.5$ and 5. Censoring rates are assumed to be 20%. Sample size is varied to observe the effect of small and large samples on the estimators. Changes in the estimators and their risks have been determined when changing the loss function and the prior distribution of θ while keeping the sample size fixed. All these results are based on 5,000 repetitions. Tables 2 – 17 gives the estimated value of the parameter, posterior risks (PR) and 95% confidence limits (LCL & UCL) for the parameter. The results are summarized in the following tables. The first entry is the simulated Bayes estimator. The second entry is the simulate posterior risk.

Table 2 Bayes Estimates and the corresponding posterior risks under SELF

n, r	Uniform		Jeffreys		Exponential	
	$\theta = 2.5$	$\theta = 5$	$\theta = 2.5$	$\theta = 5$	$\theta = 2.5$	$\theta = 5$
30, 6	2.68033	5.33681	2.56662	5.13278	2.58650	4.89382
	(0.294739)	(1.16860)	(0.281384)	(1.12327)	(0.274677)	(0.978552)
60, 12	2.57973	5.16179	2.53812	5.10567	2.53024	4.92549
	(0.137739)	(0.550064)	(0.135883)	(0.549801)	(0.132474)	(0.500662)
90, 18	2.55791	5.12951	2.52782	5.06304	2.52777	4.96577
	(0.090404)	(0.363487)	(0.089456)	(0.344648)	(0.088270)	(0.34055)
120, 24	2.53853	5.07669	2.51627	5.03331	2.51245	4.97006
	(0.066877)	(0.267273)	(0.066351)	(0.265573)	(0.065469)	(0.25618)
150, 30	2.51844	5.06797	2.50929	5.02912	2.50663	4.98969
	(0.052675)	(0.21332)	(0.052027)	(0.211764)	(0.052181)	(0.206795)

Table 3 Bayes estimates and the corresponding posterior risks under PLF

n, *r*	Uniform		Jeffreys		Exponential	
	$\theta = 2.5$	$\theta = 5$	$\theta = 2.5$	$\theta = 5$	$\theta = 2.5$	$\theta = 5$
30, 6	2.71984	5.45424	2.61727	5.21259	2.61915	5.05590
	(0.105621)	(0.211835)	(0.105459)	(0.210631)	(0.101724)	(0.196364)
60, 12	2.62469	5.27570	2.55022	5.19151	2.55711	5.03876
	(0.052759)	(0.106047)	(0.052314)	(0.104444)	(0.051400)	(0.101284)
90, 18	2.58408	5.18152	2.52382	5.07511	2.52528	5.03819
	(0.035039)	(0.070259)	(0.034692)	(0.069762)	(0.034242)	(0.068315)
120, 24	2.54835	5.13372	2.51621	5.04698	2.51956	5.01761
	(0.026070)	(0.052519)	(0.026008)	(0.052166)	(0.025776)	(0.051331)
150, 30	2.53032	5.06359	2.51008	5.01676	2.51161	5.00815
	(0.020783)	(0.041590)	(0.020780)	(0.041547)	(0.020629)	(0.041135)

Table 4 Bayes estimates and the corresponding posterior risks under WSELF

n, *r*	Uniform		Jeffreys		Exponential	
	$\theta = 2.5$	$\theta = 5$	$\theta = 2.5$	$\theta = 5$	$\theta = 2.5$	$\theta = 5$
30, 6	2.57694	5.21852	2.43025	4.86256	2.46226	4.72470
	(0.107372)	(0.217438)	(0.105663)	(0.211416)	(0.102594)	(0.196862)
60, 12	2.53358	5.10645	2.48527	4.92696	2.47465	4.81522
	(0.052783)	(0.106384)	(0.052878)	(0.104829)	(0.051551)	(0.100317)
90, 18	2.52611	5.06052	2.49103	4.95018	2.47536	4.91283
	(0.035085)	(0.070285)	(0.035085)	(0.069721)	(0.034380)	(0.068234)
120, 24	2.51301	5.02666	2.49397	4.99753	2.49320	4.93399
	(0.026177)	(0.052361)	(0.026252)	(0.052606)	(0.025971)	(0.051396)
150, 30	2.50370	5.01409	2.49866	5.00320	2.49482	4.94446
	(0.020864)	(0.041784)	(0.020997)	(0.042044)	(0.020790)	(0.041204)

Table 5 Bayes Estimates and the corresponding posterior risks under QQLF

n, *r*	Uniform		Jeffreys		Exponential	
	$\theta = 2.5$	$\theta = 5$	$\theta = 2.5$	$\theta = 5$	$\theta = 2.5$	$\theta = 5$
30, 6	2.55257	4.82952	2.43703	4.62747	2.41163	4.44132
	(0.001736)	(0.000117)	(0.002043)	(0.000157)	(0.002005)	(0.000179)
60, 12	2.53239	4.93617	2.47281	4.79326	2.46107	4.71617
	(0.000881)	(0.000040)	(0.000950)	(0.000049)	(0.000941)	(0.000053)
90, 18	2.52018	4.93974	2.49487	4.90322	2.48216	4.79527
	(0.000575)	(0.000023)	(0.000608)	(0.000024)	(0.000608)	(0.000028)
120, 24	2.51229	4.95924	2.49567	4.91882	2.48561	4.86820
	(0.000439)	(0.000016)	(0.000453)	(0.000017)	(0.000453)	(0.000018)
150, 30	2.50994	4.97115	2.50503	4.93258	2.49119	4.88738
	(0.000351)	(0.000012)	(0.000356)	(0.000013)	(0.000359	(0.000013)

Table 6 Bayes estimates and the corresponding posterior risks under SLELF

n, r	Uniform		Jeffreys		Exponential	
	$\theta = 2.5$	$\theta = 5$	$\theta = 2.5$	$\theta = 5$	$\theta = 2.5$	$\theta = 5$
30, 6	2.64073	5.22982	2.53572	5.08856	2.53144	4.81258
	(0.040811)	(0.040811)	(0.042547)	(0.042547)	(0.040811)	(0.040811)
60, 12	2.55853	5.11589	2.52731	5.04852	2.51374	4.91917
	(0.020618)	(0.020618)	(0.021052)	(0.021052)	(0.020618)	(0.020618)
90, 18	2.54426	5.07246	2.51544	5.02094	2.50443	4.93295
	(0.013793)	(0.013793)	(0.013986)	(0.013985)	(0.013793)	(0.013793)
120, 24	2.52333	5.06746	2.50648	5.01515	2.50406	4.95648
	(0.010363)	(0.010363)	(0.010471)	(0.010471)	(0.010363)	(0.010363)
150, 30	2.51737	5.05222	2.49698	5.00952	2.49488	4.98616
	(0.008299)	(0.008299)	(0.008368)	(0.008368)	(0.008299)	(0.008299)

Table 7 Bayes estimates and the corresponding posterior risks under gamma Prior

n, r	SELF		PLF		WSELF	
	$\theta = 2.5$	$\theta = 5$	$\theta = 2.5$	$\theta = 5$	$\theta = 2.5$	$\theta = 5$
30, 6	2.75648	4.26790	2.79646	4.33131	2.69038	4.14008
	(0.209767)	(0.500624)	(0.074726)	(0.115739)	(0.075415)	(0.116051)
60, 12	2.65186	4.50814	2.67247	4.57392	2.61160	4.46962
	(0.116875)	(0.336938)	(0.043509)	(0.074465)	(0.043764)	(0.074899)
90, 18	2.60726	4.644880	2.62874	4.68128	2.58136	4.60071
	(0.080831)	(0.256100)	(0.030773)	(0.054801)	(0.03085)	(0.054983)
120, 24	2.59638	4.74280	2.59909	4.76320	2.55704	4.69345
	(0.062351)	(0.207904)	(0.023753)	(0.043530)	(0.023748)	(0.043589)
150, 30	2.56876	4.79671	2.56235	4.81636	2.54475	4.73286
	(0.049956)	(0.174049)	(0.019253)	(0.036098)	(0.019326)	(0.035944)

Table 8 Bayes estimates and the corresponding posterior risks under inverse Levy prior

n, r	SELF		PLF		WSELF	
	$\theta = 2.5$	$\theta = 5$	$\theta = 2.5$	$\theta = 5$	$\theta = 2.5$	$\theta = 5$
30, 6	2.62816	5.23870	2.66574	5.33688	2.47008	4.96032
	(0.289184)	(1.15059)	(0.105252)	(0.211382)	(0.105110)	(0.211078)
60, 12	2.55766	5.12329	2.59017	5.24265	2.48816	4.97960
	(0.13663)	(0.54808)	(0.052594)	(0.106452)	(0.052593)	(0.104834)
90, 18	2.54785	5.08249	2.54467	5.11444	2.49057	4.98348
	(0.090235)	(0.359609)	(0.034740)	(0.069823)	(0.034833)	(0.069699)
120, 24	2.52797	5.06450	2.52627	5.09098	2.49689	4.99233
	(0.066641)	(0.267473)	(0.025977)	(0.052167)	(0.026146)	(0.052276)
150, 30	2.51758	5.02683	2.52227	5.05723	2.50265	5.02147
	(0.052877)	(0.210726)	(0.020802)	(0.041709)	(0.020943)	(0.042021)

Table 9 Bayes estimates and the corresponding posterior risks under gamma prior

n, r	QQLF		SLELF	
	$\theta = 2.5$	$\theta = 5$	$\theta = 2.5$	$\theta = 5$
30, 6	2.81007	2.26061	2.73732	4.20357
	(0.000803)	(0.000118)	(0.027642)	(0.027642)
60, 12	2.66023	4.51028	2.63967	4.50476
	(0.000579)	(0.000048)	(0.016618)	(0.016618)
90, 18	2.63309	4.62996	2.58029	4.63679
	(0.000428)	(0.000028)	(0.011879)	(0.011879)
120, 24	2.60891	4.71788	2.55821	4.72034
	(0.000342)	(0.000019)	(0.009244)	(0.009244)
150, 30	2.57598	4.77845	2.54761	4.76712
	(0.000291)	(0.000014)	(0.007566)	(0.007566)

Table 10 Bayes estimates and the corresponding posterior risks under inverse Levy prior

n, r	QQLF		SLELF	
	$\theta = 2.5$	$\theta = 5$	$\theta = 2.5$	$\theta = 5$
30, 6	2.48208	4.67539	2.56837	5.18622
	(0.001916)	(0.000144)	(0.041661)	(0.041661)
60, 12	2.48882	4.86899	2.53024	5.08555
	(0.000924)	(0.000043)	(0.020833)	(0.020833)
90, 18	2.49015	4.93031	2.52944	5.04281
	(0.000603)	(0.000024)	(0.013889)	(0.013889)
120, 24	2.49475	4.95492	2.51793	5.01487
	(0.000451)	(0.000016)	(0.010417)	(0.010417)
150, 30	2.50722	4.96276	2.50284	5.01054
	(0.000354)	(0.000012)	(0.008333)	(0.008333)

Table 11 The lower (LL), the upper (UL) and the width of the 95% CI under uniform prior

n, r	$\theta = 2.5$		Width	$\theta = 5$		Width
	LL	UL		LL	UL	
30, 6	1.68321	3.71521	2.03200	3.34187	7.37629	4.03442
60, 12	1.89680	3.33000	1.43320	3.79685	6.66572	2.86887
90, 16	1.98241	3.14136	1.15895	3.98359	6.31245	2.32886
120, 24	2.04919	3.05403	1.00484	4.07764	6.07715	1.99951
150, 30	2.09915	3.00001	0.90086	4.16553	5.95318	1.78765

Table 12 The lower (LL), the upper (UL) and the width of the 95% CI under Jeffreys prior

n, r	$\theta = 2.5$		Width	$\theta = 5$		Width
	LL	UL		LL	UL	
30, 6	1.59982	3.59049	1.99067	3.17633	7.12866	3.95233
60, 12	1.85185	3.27030	1.41845	3.70687	6.54621	2.83934
90, 16	1.95176	3.10272	1.15096	3.92200	6.23481	2.31281
120, 24	2.02574	3.02538	0.99964	4.03097	6.02013	1.98916
150, 30	2.08011	2.97723	0.89712	4.12775	5.90798	1.78023

Table 13 The lower (LL), the upper (UL) and the width of the 95% CI under exponential prior

n, r	$\theta = 2.5$		Width	$\theta = 5$		Width
	LL	UL		LL	UL	
30, 6	1.61617	3.56725	1.95108	3.08762	6.81509	3.72747
60, 12	1.85804	3.26196	1.40392	3.64467	6.39855	2.75388
90, 16	1.95540	3.09856	1.14316	3.87602	6.14199	2.26597
120, 24	2.02813	3.02264	0.99451	3.99508	5.95410	1.95902
150, 30	2.08180	2.97521	0.89341	4.09775	5.85631	1.75856

Table 14 The lower (LL), the upper (UL) and the width of the 95% CI under gamma prior

n, r	$\theta = 2.5$		Width	$\theta = 5$		Width
	LL	UL		LL	UL	
30, 6	1.91352	3.67144	1.75792	2.96267	5.68442	2.72175
60, 12	2.01855	3.34615	1.32730	3.45894	5.73304	2.27410
90, 16	2.06589	3.16703	1.10114	3.70963	5.68689	1.97726
120, 24	2.11157	3.07808	0.96651	3.85326	5.61698	1.76372
150, 30	2.14858	3.02151	0.87293	3.97167	5.58528	1.61361

Table 15 The lower (LL), the upper (UL) and the width of the 95% CI under In-Levy prior

n, r	$\theta = 2.5$		Width	$\theta = 5$		Width
	LL	UL		LL	UL	
30, 6	1.64110	3.65211	2.01101	3.25759	7.24946	3.99187
60, 12	1.87411	3.29980	1.42569	3.75102	6.60454	2.85352
90, 16	1.96694	3.12181	1.15487	3.95221	6.27273	2.32052
120, 24	2.03735	3.03954	1.00219	4.05386	6.04799	1.99413
150, 30	2.08954	2.98849	0.89895	4.14628	5.93007	1.78379

8. Comparison of Bayes Estimators and Posterior Risks for Real Life Data Set

In this section we present the analysis of one real data set for illustrative purposes. It is a strength data originally considered by Badar and Priest (1982). The data represent the strength measured in GPA, for single carbon fibers and impregnated 1000-carbon fiber tows. The data are presented below.

Data Set: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027.

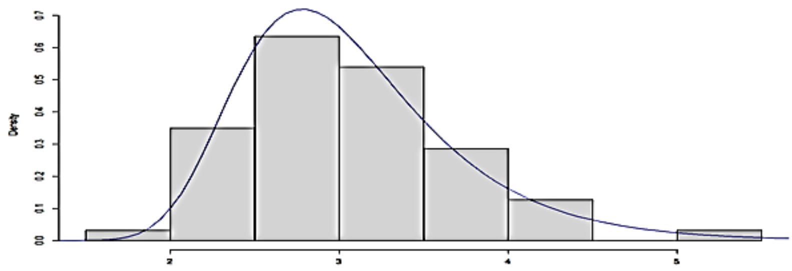

Figure 3 Fitted density function and the histogram of the data

The mean, variance, skewness and kurtosis are 3.461, 0.3855, 0.6328486 and 3.286345, respectively. From the chi-squared value 6.524 and p-value 0.8362, it is observed that data follow the Type I generalized logistic distribution. The sample characteristics required to evaluate the estimates of shape parameter of the generalized logistic distribution are as follows:

$$n = 60,\ r = 12 \text{ and } \sum_{i=r+1}^{n} \ln\left\{1+\exp(-x_{(i)})\right\} = 2.18090 \text{ and } \ln\left\{1+\exp(-x_{(r+1)})\right\} = 0.077536.$$

Table 16 Bayes estimates and the corresponding posterior risks under the real data set

Prior	SELF		PLF		WSELF	
	BEs	PRs	BEs	PRs	BEs	PRs
Uniform	15.74890	5.06178	15.90880	0.31978	15.4275	0.32141
Jeffreys	15.42750	4.95848	15.58740	0.31974	15.1061	0.32141
Exponential	13.96020	3.97726	15.88750	0.31935	13.6753	0.28490
Gamma	8.70349	1.24848	8.774920	0.14286	8.56005	0.14345
Inverse-Levy	15.57770	5.00341	15.73750	0.31955	15.2565	0.32119

Table 17 Bayes estimates and the corresponding posterior risks under the real data set

Prior	QQLF		SLELF	
	BEs	PRs	BEs	PRs
Uniform	13.6561	2.59269×10^{-11}	15.5885	0.020618
Jeffreys	13.3774	4.24510×10^{-11}	15.2671	0.0210519
Exponential	12.2834	2.31861×10^{-11}	13.8179	0.0206180
Gamma	8.13319	1.39517×10^{-7}	8.63187	0.0166179
Inverse-Levy	13.5089	3.35957×10^{-11}	15.4174	0.0208326

8.1. Graphical results of posterior distribution for real life data set

The below graphs reveal that posterior distributions under different informative and non informative priors. Figure 1 depicts posterior densities under uniform and Jeffreys priors. It is obvious that both the priors yield the approximately the identical posterior inferences. Figure 2 shows that the posterior densities under exponential, gamma and inverse Levy priors are not identical. However, gamma prior may be a better a choice because of its two hyper parameters which ensure better fit.

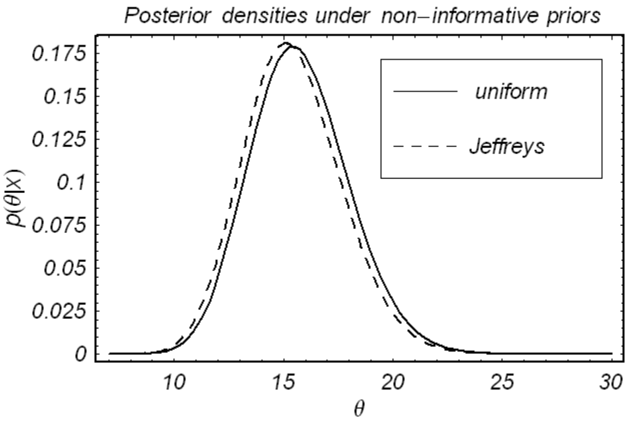

Figure 4 Posterior densities under uniform and Jeffreys priors

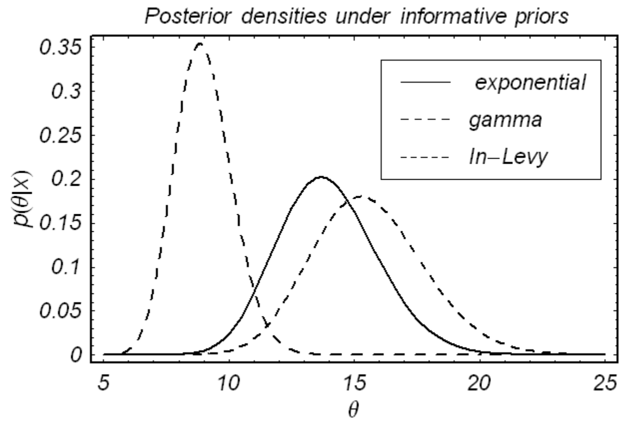

Figure 5 Posterior densities under exponential, gamma and inverse Levy priors

9. Conclusions

The findings of the simulation study are pretty interesting. The parameter has been under-estimated for majority of the cases. The extent of under estimation is more severe under exponential WSELF. To be more specific, larger degrees of parameter results in bigger sizes of underestimation. Then extent of this over or under estimation is directly proportional to parametric value and inversely proportional to the sample size. Further, the increase in sample size reduces the posterior risks of θ. The performance of squared-log error loss function is independent of choice of parametric value. Similarly, the increased true parametric values impose a negative impact on the convergence of the estimates under all priors. However, it can be observed that by increasing the sample size, the convergence of the estimated values toward the true parametric values tend to increase for each case. On the other hand, the amounts of posterior risks, based on each prior and loss function tend to decrease by increasing the sample size. It indicates that the estimators are consistent.

In comparison of non-informative priors, the Jeffreys prior provides the better estimates as the corresponding risks are least under said loss functions except QQLF. While the uniform and the exponential priors are equally efficient under SLELF, therefore they produce more efficient estimates as compared to the other informative priors. It can also be observed that performance of estimates under informative priors is better than those under non-informative priors and the Bayes estimates may turn out to be most efficient under gamma prior. This simply indicates the use of prior information that makes a different in terms of gain in precision.

The Credible intervals are in accordance with the point estimates, that is, the width of credible interval is inversely proportional to sample size. From the Tables 11-15, appended above, reveal that the effect of the prior information in the form of narrower width of interval. The Credible interval assuming gamma prior is much narrower than the credible intervals assuming informative and non-informative priors. The findings of the real life example are in accordance with the simulation study. Hence for Bayesian analysis of the parameter of the generalized logistic distribution, the use of gamma prior under type II left censored samples can be preferred. The study can further be extended by considering two parametric versions of the distribution under variety of circumstances.

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