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Stochastic Volatility Model with Burr Distribution Error: Evidence from Australian Stock Returns

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Abstract

The Stochastic Volatility (SV) models have been extensively used as alternative models to the well known ARCH and GARCH models in order to represent the volatility behavior in financial return series. In this paper, we study the SV models with error distribution following a class of thick-tailed distributions, called Mode-Centered Burr distribution, in the place of Normal distribution. Through empirical analysis on Australian stock returns data we illustrate that the SV model with error as Mode-Center Burr distribution is more appropriate than the basic SV model. Furthermore, an extension of the basic SV model is investigated, in the direction of allowing the volatility to follow a second-order autoregressive process. Properties of this model such as the kurtosis and autocorrelation function are derived.

Keywords: Autoregressive process, Burr distribution, time series forecasting.

1. Introduction

Volatility and forecasting of volatility have become key issues in financial markets as well as in risk management. Therefore, it is very important to have a good volatility model for forecasting future observations and thus forecasting volatility. In volatility modeling, one can formulate the conditional variance (volatility) as an observable function. The ARCH and GARCH models are the examples of this approach. In this paper, we consider the Stochastic Volatility (SV) model in which volatility is taken as an unobservable function. Generally, in SV models the distribution of returns, conditional on volatility, is assumed to be Normal. The main aim of this paper is to study the SV models by assuming that the conditional distribution of returns follows a class of thick-tailed distributions, called as Mode-Centered Burr distribution, whose properties are similar to the properties of a Normal distribution.

A description of the basic SV model is as follows. The random variable Y_t, for $t = 0, 1, \ldots, T$, represents the asset return at time t whose mean is assumed to be zero. The distribution of Y_t, conditional on its variance, is assumed to be Normal with mean zero and variance $\exp(V_t)$, where V_t follows an autoregressive order one (AR(1)) process.

In other words,

$$Y_t = \exp(V_t/2)\,\varepsilon_t, \tag{1}$$

$$V_t = \gamma + \phi\,V_{t-1} + \eta_t, \tag{2}$$

for $t = 0, 1, \ldots, T$, where the ε_t's are independent and identically distributed (i.i.d.) $N(0, 1)$ and η_t's are i.i.d. $N(0, \sigma_\eta^2)$. The arrays of η_t's and ε_t's are independent. Let $\theta = (\gamma, \phi, \sigma_\eta^2)$ be the parameter of the SV model; ϕ is the persistence parameter and σ_η denotes the volatility of volatility shock. Here, we restrict to the case that the SV model is covariance stationary, i.e. the persistence parameter $|\,\phi| < 1$. In this paper, we study the volatility model assuming that $\varepsilon_t, t = 0, 1, \ldots, T$, follow a class of thick-tailed distributions called Mode-Centered Burr distribution rather than a Normal distribution.

While the SV model is a good representation, from the theoretical viewpoint, of the behavior of the returns in the real financial markets, an important characteristic of the SV model is that the volatility is treated as a latent or an unobservable function. As a consequence, parameter estimation has been a major problem because of the difficulty in obtaining an exact expression for the likelihood function. Nonetheless, several non-likelihood-based and likelihood-based parameter estimation techniques have been developed. Furthermore, estimation by using Bayesian approach may be found, for example, in Araveeporn et al. (2010). We use the Maximum Likelihood method based on Efficient Importance Sampler procedure (ML-EIS) of Liesenfeld and Richard (2003 and 2006). They have shown that this approach is very accurate and efficient for the analysis of the basic SV model and its variants.

The paper is organized as follows. The proposed Mode-Centered Burr distribution and its properties are described in Section 2. Section 3 covers properties of the SV Burr model. In Section 4, we carry out an empirical analysis on Australian stock returns data in order to show the appropriateness of the proposed SV models. An extension of the basic SV model by allowing the AR(2) for volatility process is presented in Section 5.

2. Mode-Centered Burr Distribution

The Burr Type II distribution was originally defined by Burr (1942) in the form of the cumulative distribution function (cdf)

$$F(x) = \big(1 + \exp(-x)\big)^{-a},\ -\infty < x < \infty, \tag{3}$$

with parameter $a(> 0)$. The probability density function (pdf) is easily obtained by taking the first derivative of (3) and has the form

$$f(x) = a\,\exp(-x)\,\big(1 + \exp(-x)\big)^{-(a+1)},\ -\infty < x < \infty. \tag{4}$$

The mode of this distribution is at $x = \ln a$, which means that such a distribution has systematic varying mode as a varies. The distribution will have mode shifted to the negative values, for $0 < a < 1$, and the mode will be shifted to the positive values, for $a > 1$ (see Iriawan (1999)) for detailed discussion of (4) for various values of a).

Our aim is to have an alternative distribution for ε_t which has properties similar to the properties of the $N(0, 1)$ such as a fixed mode at $x = 0$, but with thicker tail than that of $N(0, 1)$. We achieve this by modifying (4) to ensure that the mode is fixed at $x = 0$. The resulting distribution is called

Mode-Centered Burr(a) distribution and its pdf is given by

$$f(x) = \exp(-x)\left(1 + \frac{\exp(-x)}{a}\right)^{-(a+1)}, \ -\infty < x < \infty. \tag{5}$$

At $x = 0$, for all a, the density value of the Mode-Centered Burr(a) distribution is always lower than the density value of $N(0,1)$. In fact, the density value of the Mode Centered Burr(a) distribution at $x = 0$ is $(1 + 1/a)^{-(a+1)} \leq 1/e \approx 0.3678$, attained when a goes to infinity. Whereas, the density value of $N(0,1)$ at $x = 0$ is $1/\sqrt{2\pi} \approx 0.3989$. By introducing a scale factor $c(> 0)$, the value of the densities of the Mode-Centered Burr(a) and $N(0,1)$ distributions at the mode can be made equal. The resulting pdf has the following form

$$f(x) = c\,\exp(-c\,x)\left(1 + \frac{\exp(-c\,x)}{a}\right)^{-(a+1)}, \ -\infty < x < \infty. \tag{6}$$

This distribution is called Mode-Centered Burr($c, a, 0, 1$) distribution, denoted as $MCB(c, a, 0, 1)$, with parameter c and a, and will be close to the $N(0,1)$ when we set $c = (1/\sqrt{2\pi})$ $(1 + 1/a)^{a+1}$. Note that the 0 and 1 indicate that $MCB(c, a, 0, 1)$ distribution has similarity to the $N(0,1)$.

In this paper, we use $MCB(c, a, 0, 1)$ distribution for the conditional distribution of returns (given the volatility) in the Stochastic Volatility (SV) model. Specifically, we use this distribution for the case $a = 1$. The pdf is given by

$$f(x) = c\,\exp(-c\,x)\,(1 + \exp(-c\,x))^{-2}, \ -\infty < x < \infty, \tag{7}$$

where $c = 4/\sqrt{2\pi}$.

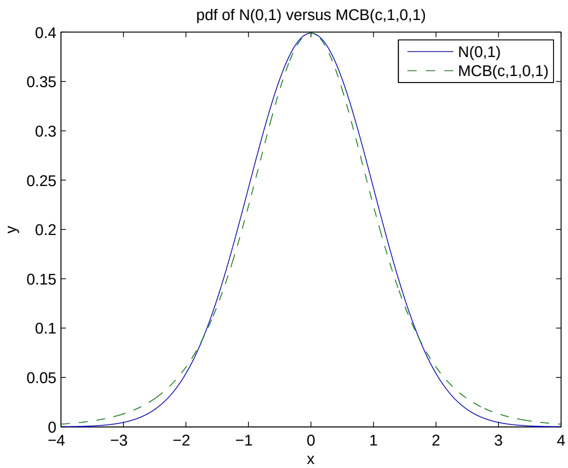

Figure 1 Densities of $N(0,1)$ and $MCB(c, 1, 0, 1)$ distribution

In Figure 1, we show the pdf of the $N(0,1)$ along with the $MCB(c, 1, 0, 1)$ distribution. It shows that $MCB(c, a, 0, 1)$ distribution is close to $N(0,1)$, but has a thicker tail. Some comparison of the two distributions will be presented in Table 1.

Table 1 Comparison of $N(0,1)$ and $MCB(c,1,0,1)$ distribution

	$N(0,1)$	$MCB(c,1,0,1)$
Second moment	1	1.2919
Fourth moment	3	17.8512
Kurtosis	3	10.6952

3. The Stochastic Volatility Burr Model

In this Section, we provide some of the interesting properties of the SV model, in particular, the properties of SV model with $MCB(c,a,0,1)$ distribution (SV Burr model, hereafter). The properties given here are (a) the predicted kurtosis, (b) the predicted autocorrelation function of squared returns and (c) the predicted autocorrelation function of absolute returns. The first two properties of the basic SV model have been reported in Liesenfeld and Jung (2000). We present these properties for general distribution of ε_t as follows. Details on the derivations of these properties can be found in Syuhada (2004).

Property 1 *The kurtosis of the SV model is* $\kappa = \exp\left(\frac{\sigma_\eta^2}{1-\phi^2}\right) E(\epsilon_t^4)/(E(\epsilon_t^2))^2$*, provided the* $E(\epsilon_t^4) < \infty$*. The term* $\exp\left(\frac{\sigma_\eta^2}{1-\phi^2}\right)$ *denotes the exponential value of the unconditional variance of log volatility, whilst* $E(\epsilon_t^4)/(E(\epsilon_t^2))^2$ *is the kurtosis of the model error.*

Property 2 *The autocorrelation function of the squared returns of the SV model is*

$$\rho(\tau) = \frac{\left(E(\epsilon_t^2)\right)^2 \Big(\exp\left(\sigma_\eta^2\, \phi^\tau/(1-\phi^2)\right) - 1\Big)}{E(\epsilon_t^4)\, \exp\left(\sigma_\eta^2/(1-\phi^2)\right) - \left(E(\epsilon_t^2)\right)^2}, \; \tau = 1, 2, \ldots.$$

From Property 2, it can be shown that the autocorrelation function of squared returns is positive and behaves exponentially with respect to the parameter ϕ. Also, the kurtosis of the error process, $E(\epsilon_t^4)/(E(\epsilon_t^2))^2$, plays an important role in the sense that different assumptions of error process may result in significant change in the autocorrelation function.

Now we consider the autocorrelation function of absolute returns of the SV model. Although Hsieh (1995) and Cont (2001) discussed the autocorrelation function of absolute returns, they did not provide an explicit expression of the function. We first define the absolute returns as $|y_t| = \sigma_t\,|\epsilon_t|$. The autocorrelation function of absolute returns is given by

$$\rho_{|y|}(\tau) = \mathrm{Cov}(|y_t|, |y_{t-\tau}|)/\mathrm{Var}(|y_t|),$$

where

$$\begin{aligned} \mathrm{Cov}(|y_t|, |y_{t-\tau}|) = {} & E\big(\exp(V_t/2 + V_{t-\tau}/2)\big)\, E(|\varepsilon_t|)\, E(|\varepsilon_{t-\tau}|) \\ & - E\big(\exp(V_t/2)\big)\, E\big(\exp(V_{t-\tau}/2)\big)\, E(|\varepsilon_t|)\, E(|\varepsilon_{t-\tau}|), \end{aligned}$$

and

$$\mathrm{Var}(|y_t|) = E\big(\exp(V_t)\big)\, E(\varepsilon_t^2) - \big(E(\exp(V_t/2))\big)^2 \big(E(|\varepsilon_t|)\big)^2.$$

Property 3 *The autocorrelation function of the absolute returns of the SV model is*

$$\rho_{|y|}(\tau) = \frac{\left(E|\varepsilon_t|\right)^2 \left(\exp\left(\sigma_\eta^2\, \phi^\tau/4(1-\phi^2)\right) - 1\right)}{\exp\left(\sigma_\eta^2/4(1-\phi^2)\right) E(\varepsilon_t^2) - (E|\varepsilon_t|)^2},\ \tau = 1, 2, \ldots,$$

where

$$\begin{aligned} Cov\big(|y_t|, |y_{t-\tau}|\big) &= \exp\left(\mu_V + \frac{1}{4}\,\sigma_V^2\right) \left[\exp\left(\frac{1}{4}\,\phi^\tau\,\sigma_V^2\right) - 1\right] \left(E(|\varepsilon_t|)\right)^2 \\ Var\big(|y_t|\big) &= \exp\left(\mu_V + \frac{1}{4}\,\sigma_V^2\right) \left[\exp\left(\frac{1}{4}\,\sigma_V^2\right) E(\varepsilon_t^2) - (E|\varepsilon_t|)^2\right]. \end{aligned}$$

From Property 2 and Property 3, we can observe a significant difference between the autocorrelation function of squared returns and the autocorrelation function of absolute returns in terms of the contribution of the error process. Here, in Property 3, the contributions of the error process are from the expected value of the absolute error process, $E|\varepsilon_t|$, and the second moment of error process, $E(\varepsilon_t^2)$, whereas in Property 2 the contribution of the error process comes from the second and fourth moments of the error distributions.

4. Empirical Analysis

4.1. Data

Our data is the daily stock returns of six companies listed on the Australian Stock Exchange (ASX). They are AMP (AMP Limited), NCP (News Corporation Limited), CBA (Commonwealth Bank of Australia), ERG (ERG Limited), LLC (Lend Lease Corporation Limited) and NAB (National Australia Bank) series, where the period of the series is about 10 year, except for AMP (4 year). Specifically, the periods are 15/06/1998 to 27/08/2002 (AMP), 07/09/1992 to 05/09/2002 (NCP, LLC, and NAB), 27/08/1992 to 27/08/2002 (CBA and ERG). For our analysis, we take the returns, y_t, centered about the sample mean, as

$$y_t = 100.\left[\ln\frac{p_t}{p_{t-1}} - \frac{1}{T}\sum_{t=1}^{T}\ln\frac{p_t}{p_{t-1}}\right],$$

where $p_t,\ t = 1, 2, \ldots, T$, denote the daily price series, and T the number of observations.

Table 2 The summary statistics

Statistic	AMP	NCP	CBA	ERG	LLC	NAB
T	1064	2530	2522	2497	2530	2523
Std Deviation	1.6508	2.2666	1.2173	3.6328	1.5393	1.3351
Skewness	-0.2090	0.5478	-0.1948	-0.2491	-1.1497	-0.7325
Kurtosis	6.7860	11.3165	5.2443	13.7589	15.0721	9.2226

Table 2 summarizes some statistics of the returns series. The number of observations are above 2000 for each series, except for the AMP. The empirical kurtosis is high, in the range of 5.2443 (CBA) to 15.0721 (LLC), which implies that the normality assumption for distribution of returns is doubtful. The values of skewness are far from zero (mostly negative), indicating an asymmetric property of the returns.

Further, from Table 3 we find that, in general, the first order autocorrelation coefficients of returns, $\rho(1)_y$, take the lowest values compared to the corresponding autocorrelation coefficient of squared returns, $\rho(1)_{y^2}$, and the autocorrelation function of absolute returns,$\rho(1)_{|y|}$. There is an exception for CBA series, where the autocorrelation coefficient of return reaches a higher value than the autocorrelation coefficient of squared and absolute returns. For NAB series, although the autocorrelation coefficient of returns is higher than that of squared returns, its value is still lower than the autocorrelation coefficient of absolute returns.

Table 3 The first order autocorrelation coefficient

Statistic	AMP	NCP	CBA	ERG	LLC	NAB
$\rho(1)_y$	0.054	0.028	0.104	-0.017	0.115	0.093
$\rho(1)_{y^2}$	0.148	0.172	0.070	0.163	0.408	0.079
$\rho(1)_{\|y\|}$	0.217	0.211	0.082	0.292	0.266	0.144
$BL_{y(20)}$	25.412	30.356	43.909	59.256	62.280	60.215
	(0.186)	(0.064)	(0.002)	(0.000)	(0.000)	(0.000)
$BL_{y^2(50)}$	198.049	349.046	150.319	374.455	529.430	295.316
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
$BL_{\|y\|(50)}$	368.478	1725.593	241.941	1035.488	981.242	457.795
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)

In addition, Box-Ljung (BL) statistic, given in Table 3 along with marginal significance levels in parentheses, is used to investigate whether there is a significant autocorrelation in certain series. We use 20 lags for the analysis of autocorrelation in returns and 50 lags for the analysis of autocorrelation in squared and absolute returns. Based on this BL statistic with 5% level of significance, we find that the AMP and NCP series have no significant autocorrelation in returns but have significant autocorrelation in squared and absolute returns. The rest of the series (CBA, ERG, LLC, and NAB) have significant autocorrelation in returns, squared and absolute returns. In summary, the data sets that we considered in this paper have many of the important features, specified in the current literature, that one would expect for financial returns. In particular, the return series have no or little significant autocorrelation in returns, have significant autocorrelation in squared and absolute returns. In the current literature such data sets are mostly studied using SV normal model. In the next section we illustrate that, for these data sets, the SV Burr model perform much better than the SV normal model.

4.2. Estimation Results

The estimates on ϕ for all SV models are given in Table 4. The estimations are based on a simulation sample size $N = 50$ and 3 EIS iterations. Generally, the estimates are greater than 0.90, except for CBA and ERG series under the SV Normal model. These high values indicate high persistence of volatility. We found that, except for the AMP, the estimates under the SV Burr model are higher than the corresponding estimates under the SV Normal model. The standard errors (in parentheses) are also lower under the SV Burr model compared to those of under the SV Normal model, which suggest that the SV Burr model perform better than the SV Normal model.

The predicted kurtosis of the SV models are given in Table 5. We can see that the SV Normal model does not predict the kurtosis close to the kurtosis observed in the data. Whereas the SV Mode-Centered Burr model gives the predicted kurtosis that is compatible with the empirical kurtosis, for all series.

Table 4 The estimates of ϕ

	AMP	NCP	CBA	ERG	LLC	NAB
SV Normal	0.9965	0.9496	0.8949	0.8529	0.9228	0.9166
	(0.0031)	(0.0127)	(0.0247)	(0.0275)	(0.0202)	(0.0190)
SV Burr	0.9958	0.9880	0.9526	0.9114	0.9909	0.9619
	(0.0042)	(0.0056)	(0.0140)	(0.0222)	(0.0062)	(0.0131)

Table 5 The predicted kurtosis

	AMP	NCP	CBA	ERG	LLC	NAB
Data	6.7860	11.3165	5.2443	13.7589	15.0721	9.2226
SV Normal	201.8911	6.2041	4.6413	8.0689	5.8836	4.5971
SV Burr	28.3355	7.1935	5.5828	10.7831	6.5547	5.5837

From Table 6 one can conclude that both SV Normal and SV Burr models predict the low values of first order autocorrelation coefficient of squared returns. However, the values are lower under the SV Burr model in comparison to that of under the SV Normal model. The predicted first order autocorrelation coefficients for the SV Normal model are close to those observed in the NCP and ERG series. Whereas under the SV Burr model, the predicted first order autocorrelation coefficients are close to those observed in the data for AMP, CBA, LLC and NAB series. As for the first order autocorrelation coefficient of absolute returns, it is shown that for at least three series (NCP, ERG, and NAB) the SV Normal model performs better than the SV Burr model.

Table 6 The predicted first order autocorrelation coefficient of squared returns/absolute returns

	Data	SV Normal	SV Burr
AMP	0.148/0.217	0.3252/0.5298	0.2083/0.5358
NCP	0.172/0.211	0.1909/0.2130	0.1133/0.2756
CBA	0.070/0.082	0.1312/0.1364	0.0680/0.1659
ERG	0.163/0.292	0.1875/0.2321	0.1392/0.3539
LLC	0.408/0.266	0.1765/0.1957	0.0998/0.2424
NAB	0.079/0.144	0.1331/0.1374	0.0687/0.1676

In conclusion, we have used the Mode-Centered Burr distribution as the error distribution in the SV model instead of Normal distribution. The main reason for using the Mode-Centered Burr distribution is that it has thicker tail compared to the Normal distribution. This characteristic enabled us to develop a better model in terms of capturing the stylized facts of returns. Our empirical analysis has shown that the SV model with the Mode-Center Burr distribution is more appropriate than the basic SV model. Preference of the SV Burr model over the SV Normal model for a given series can be assessed by observing high persistent volatility and capturing the stylized facts of returns such as high kurtosis and low first-order autocorrelation coefficients.

5. The SV Model with AR(2) Volatility Process

In this Section, we propose another extension for the basic SV model by allowing the volatility process to follow a second order autoregressive or AR(2) process. This extension is motivated by the work of Asai (2000) which developed the method to select the lag length of SV model. He stated that the unavailability of a method to select the lag length p of volatility process is one of the reasons for not using lag length $p > 1$ in empirical analysis of SV model. In his work, he extended the MCMC procedure of Kim et al. (1998) to approximate the exact likelihood of p^{th} order SV model. Then, the lag length of SV model is selected by using Bayes factors. From empirical results using daily returns, he found that there is strong support for taking lag length of two for the volatility process.

Our proposed SV model, called the SVAR(2) model, is defined as

$$y_t = \sigma_t \, \epsilon_t, \; \epsilon_t \sim \text{iid}(0,1) \tag{8}$$

$$(\sigma_t^2 | \sigma_{t-1}^2, \sigma_{t-2}^2) \sim \log N(\gamma + \phi_1 \, \ln \sigma_{t-1}^2 + \phi_2 \, \ln \sigma_{t-2}^2, \sigma_\eta^2), \tag{9}$$

where y_t, σ_t are the return and the volatility on day t, respectively. The notation i.i.d.$(0, 1)$ means $i.i.d.$ random variables with mean 0 and variance 1. The errors, ϵ_t and η_t, are unobservable, and hence σ_t is also unobservable. Moreover, ϵ_t and η_t are assumed to be stochastically independent.

5.1. Properties of The SVAR(2) Models

Let's consider the SV AR(2) model in Section 1 and express the volatility process as

$$\ln \sigma_t^2 = \gamma + \phi_1 \, \ln \sigma_{t-1}^2 + \phi_2 \, \ln \sigma_{t-2}^2 + \sigma_\eta \, \eta_t, \;\; \eta_t \sim \text{iid}N(0,1). \tag{10}$$

For $\phi_1 + \phi_2 < 1, \phi_2 - \phi_1 < 1, |\phi_2| < 1$ the process is stationary. Hereafter, we assume that these conditions are satisfied. Let $V_t = \ln \sigma_t^2$. The distributional properties of V_t are the following.

Property 4 *The conditional distribution of V_t is Normal with mean, $\gamma + \phi_1 \, V_{t-1} + \phi_2 \, V_{t-2}$, and variance, σ_η^2. The unconditional distribution of V_t is also Normal with mean*

$$E(V_t) = \frac{\gamma}{1 - \phi_1 - \phi_2}$$

and variance

$$Var(V_t) = \frac{1 - \phi_2}{1 + \phi_2} \, \frac{\sigma_\eta^2}{(1 - \phi_2)^2 - \phi_1^2}$$

We now derive the second moment and fourth moments of returns predicted by the SVAR(2) model. From (8), we obtain

$$E(y_t^2) = E\big(\exp(V_t)\big) \, E(\epsilon_t^2)$$

where

$$E\big(\exp(V_t)\big) = \exp\left(\frac{\gamma}{1 - \phi_1 - \phi_2} + \frac{1 - \phi_2}{1 + \phi_2} \, \frac{\sigma_\eta^2}{2\,((1 - \phi_2)^2 - \phi_1^2)}\right).$$

The fourth moment, $E(y_t^4)$, has the following form

$$E(y_t^4) = E\big(\exp(2\,V_t)\big) \, E(\epsilon_t^4)$$

where

$$E\big(\exp(2\,V_t)\big) = \exp\left(\frac{2\,\gamma}{1 - \phi_1 - \phi_2} + \frac{1 - \phi_2}{1 + \phi_2} \, \frac{2\sigma_\eta^2}{((1 - \phi_2)^2 - \phi_1^2)}\right)$$

Property 5 *The kurtosis predicted by the SVAR(2) model is*

$$\kappa = \exp\left(\frac{1-\phi_2}{1+\phi_2}\,\frac{\sigma_\eta^2}{(1-\phi_2)^2-\phi_1^2}\right)\,E(\varepsilon_t^4)/\big(E(\varepsilon_t^2)\big)^2,$$

where the $E(\varepsilon_t^2)$ and $E(\varepsilon_t^4)$ are the second and fourth moments of the error distribution. Here, we employ Normal and Mode-Centered Burr distributions as discussed in previous Section.

Table 7 The kurtosis for SVAR(2) models.

$\phi_1+\phi_2$	σ_η	SVAR(2) Normal	SVAR(2) Burr
0.85	0.17	3.2409	4.5373
0.90		3.3405	4.6767
0.95		3.6787	5.1502
0.99		7.7929	10.9100
0.85	0.05	3.0201	4.2281
0.90		3.0280	4.2392
0.95		3.0534	4.2747
0.99		3.2582	4.5615

Table 7 presents the kurtosis predicted by the SVAR(2) model under different assumptions of error process distribution. We can see that the kurtosis of SVAR(2) Burr are higher than those of SVAR(2) Normal. This feature occurs for all values of $\phi_1+\phi_2$ and σ_η given in the table. Unlike the SV model with AR(1) volatility process, the explicit expression of autocorrelation function of squared returns for SVAR(2) model is not easy to obtain. In order to calculate this autocorrelation function, we express V_t in terms of $V_{t-\tau}$ and $V_{t-\tau-1}$ with recursive coefficients. This result can be easily extended to SVAR(p) model ($p>2$).

Lemma 1 *Let $V_t=\ln\sigma_t^2$ so that (10) can be written as*

$$V_t=\gamma+\phi_1\,V_{t-1}+\phi_2\,V_{t-2}+\sigma_\eta\,\eta_t,\ \ \eta_t\sim iid\,N(0,1)$$

The above equation can be expressed as

$$V_t=(A_\tau)\,\gamma+(B_\tau)\,V_{t-\tau}+(C_\tau)\,V_{t-(\tau+1)}+(D_\tau)\,\sigma_\eta, \tag{11}$$

where

$$A_1=1, B_1=\phi_1, C_1=\phi_2\ \ and\ \ D_1=\eta_t,$$

and for $\tau\geq 2$,

$$A_\tau=A_{\tau-1}+B_{\tau-1},$$

$$B_\tau=\phi_1\,B_{\tau-1}+C_{\tau-1},$$

$$C_\tau=\phi_2\,B_{\tau-1},$$

$$D_\tau=D_{\tau-1}+B_{\tau-1}\,\eta_{t-(\tau-1)}.$$

Proof: By letting $V_t = \ln \sigma_t^2$, we obtain

$$V_t = \gamma + \phi_1 V_{t-1} + \phi_2 V_{t-2} + \sigma_\eta \eta_t. \tag{12}$$

Consequently,

$$V_{t-1} = \gamma + \phi_1 V_{t-2} + \phi_2 V_{t-3} + \sigma_\eta \eta_{t-1}.$$

We will express V_t as a function of $(V_{t-\tau}, V_{t-(\tau+1)})$, where $A_\tau, B_\tau, C_\tau, D_\tau$ are given above. We do this by induction method. For $\tau = 2$,

$$V_t = [1+\phi_1]\,\gamma + [\phi_1^2 + \phi_2]\,V_{t-2} + [\phi_2\,\phi_1]\,V_{t-3} + [\eta_t + \phi_1\,\eta_{t-1}]\,\sigma_\eta, \tag{13}$$

where

$$\begin{aligned} 1+\phi_1 &= A_2 = A_1 + B_1, \\ \phi_1^2 + \phi_2 &= \phi_1\,\phi_1 + \phi_2 = B_2 = \phi_1\,B_1 + C_1, \\ \phi_2\,\phi_1 &= C_2 = \phi_2\,B_1, \\ \eta_t + \phi_1\,\eta_{t-1} &= D_2 = D_1 + B_1\,\eta_{t-1}. \end{aligned}$$

Thus, we obtain $V_t = (A_2)\,\gamma + (B_2)\,V_{t-2} + (C_2)\,V_{t-3} + (D_2)\,\sigma_\eta$. It is true for $\tau = 2$.
We assume that the formula is true for $\tau = k$,

$$V_t = (A_k)\,\gamma + (B_k)\,V_{t-k} + (C_k)\,V_{t-(k+1)} + (D_k)\,\sigma_\eta,$$

where

$$\begin{aligned} A_k &= A_{k-1} + B_{k-1}, \\ B_k &= \phi_1\,B_{k-1} + C_{k-1}, \\ C_k &= \phi_2\,B_{k-1}, \\ D_k &= D_{k-1} + B_{k-1}\,\eta_{t-(k-1)}. \end{aligned}$$

Now, we prove this formula for $\tau = k+1$. We obtain

$$V_t = (A_{k+1})\,\gamma + (B_{k+1})\,V_{t-(k+1)} + (C_{k+1})\,V_{t-(k+1+1)} + (D_{k+1})\,\sigma_\eta,$$

where

$$\begin{aligned} A_{k+1} = 1 + \sum_{i=1}^{(k+1)-1} &\phi_1 \Big(\phi_1^{i-1} + (i-2)\,\phi_2\,\phi_1^{i-3} + (i-4)\,\phi_2^2\,\phi_1^{i-5} + \cdots\Big) \\ &+ \phi_2\Big(\phi_1^{i-2} + (i-3)\,\phi_2\,\phi_1^{i-4} + (i-5)\,\phi_2^2\,\phi_1^{i-6} + \cdots\Big) = A_k + B_k, \end{aligned}$$

$$\begin{aligned} B_{k+1} &= \phi_1 \Big(\phi_1^{(k+1)-1} + ((k+1)-2)\,\phi_2\,\phi_1^{(k+1)-3} + ((k+1)-4)\,\phi_2^2\,\phi_1^{(k+1)-5} + \cdots\Big) \\ &\quad + \phi_2\Big(\phi_1^{(k+1)-2} + ((k+1)-3)\,\phi_2\,\phi_1^{(k+1)-4} + ((k+1)-5)\,\phi_2^2\,\phi_1^{(k+1)-6} + \cdots\Big) \\ &= \phi_1\,B_k + C_k, \\ C_{k+1} &= \phi_2\Big(\phi_1^{(k+1)-1} + ((k+1)-2)\,\phi_2\,\phi_1^{(k+1)-3} + ((k+1)-4)\,\phi_2^2\,\phi_1^{(k+1)-5} + \cdots\Big) = \phi_2\,B_k, \\ D_{k+1} &= \eta_1 + \sum_{i=1}^{(k+1)-1} \Bigg(\phi_1 \Big(\phi_1^{i-1} + (i-2)\,\phi_2\,\phi_1^{i-3} + (i-4)\,\phi_2^2\,\phi_1^{i-5} + \cdots\Big) \\ &\quad + \phi_2\Big(\phi_1^{i-2} + (i-3)\,\phi_2\,\phi_1^{i-4} + (i-5)\,\phi_2^2\,\phi_1^{i-6} + \cdots\Big)\Bigg) \cdot \eta_{t-i} = D_k + B_k\,\eta_{t-k}. \end{aligned}$$

The autocorrelation function of squared returns y_t^2 is defined as

$$\rho(\tau) = \text{Cov}\big(y_t^2, y_{t-\tau}^2\big)/\text{Var}\big(y_t^2\big), \tag{14}$$

where

$$\text{Cov}\big(y_t^2, y_{t-\tau}^2\big) = \Big[E\Big(\exp\big(V_t + V_{t-\tau}\big)\Big) - \Big(E\big(\exp(V_t)\big)\Big)^2\Big]\,\big(E(\epsilon_t^2)\big)^2$$

and

$$\text{Var}\big(y_t^2\big) = \exp\big(2\,\mu_V + \sigma_V^2\big)\,\Big[\exp\big(\sigma_V^2\, E(\epsilon_t^4)\big) - \big(E(\epsilon_t^2)\big)^2\Big].$$

Note that μ_V and σ_V^2 are unconditional mean and variance of volatility. To evaluate (14), in particular, $\text{Cov}\big(y_t^2, y_{t-\tau}^2\big)$, we need to compute $E\big(\exp(V_t)\big)$ and $E(\exp(V_t + V_{t-\tau}))$. The derivations are given in the following proposition.

Proposition 1 *Let V_t and A_τ, B_τ, C_τ as in Lemma 1. Then,*

(i) $E\big(\exp(V_t)\big) = \exp\big(\mu_V + \frac{1}{2}\,\sigma_V^2\big)$,

(ii) $E\big(\exp(V_t + V_{t-\tau})\big) = \exp(A_\tau\,\gamma)\,\exp((1+B_\tau)\,\mu_V + \frac{1}{2}\,(1+B_\tau)^2\,\sigma_V^2) \times$
$\exp\big(C_\tau\,\mu_V + \frac{1}{2}\,C_\tau^2\,\sigma_V^2\big)\,\exp\big(\frac{1}{2}\,\sigma_{D_\tau}^2\,\sigma_\eta^2\big)$,
where $\sigma_{D_\tau}^2 = 1$ *for* $\tau = 1$, *and* $\sigma_{D_\tau}^2 = \sum_{j<\tau} 1 + B_{\tau-j}^2$ *for* $\tau = 2, 3, ...$

Proof: As $V_t \sim$ i.i.d.$N(\mu_V, \sigma_V^2)$, LHS of (i) is the moment generating function (mgf) of V_t. Hence, we can easily obtain

$$E\big(\exp(V_t)\big) = \exp\left(\mu_V + \frac{1}{2}\,\sigma_V^2\right). \tag{15}$$

To prove (ii) we note from Lemma 1 that

$$V_t + V_{t-\tau} = (A_\tau)\,\gamma + (1+B_\tau)\,V_{t-\tau} + (C_\tau)\,V_{t-\tau-1} + (D_\tau)\,\sigma_\eta^2,$$

The mgf of $V_t + V_{t-\tau}$, $E\big(\exp(V_t + V_{t-\tau})\big)$, is given by

$$E\big(\exp(V_t + V_{t-\tau})\big) = E\big(\exp(A_\tau\,\gamma)\big)\,E\Big(\exp[(1+B_\tau)\,V_{t-\tau}]\Big) \times$$
$$E\big(\exp(C_\tau\,V_{t-\tau-1})\big)\,E\big(\exp(D_\tau\,\sigma_\eta^2)\big).$$

By the assumption of stationarity of V_t and the Lemma 1, the above equation can be expressed as

$$E\big(\exp(V_t + V_{t-\tau})\big) = \exp(A_\tau\,\gamma)\,\exp((1+B_\tau)\,\mu_V + \frac{1}{2}\,(1+B_\tau)^2\,\sigma_V^2) \times$$
$$\exp\left(C_\tau\,\mu_V + \frac{1}{2}\,C_\tau^2\,\sigma_V^2\right)\,\exp\left(\frac{1}{2}\,\sigma_{D_\tau}^2\,\sigma_\eta^2\right)$$

where A_τ, B_τ, C_τ as in Lemma 1, $\sigma_{D_\tau}^2 = 1$ for $\tau = 1$, and $\sigma_{D_\tau}^2 = \sum_{j<\tau} 1 + B_{\tau-j}^2$ for $\tau = 2, 3, ...$

Figures 2 to 4 show the autocorrelation function of squared returns for SVAR(2) Normal and SVAR(2) Burr models, for several values of $\phi_1 + \phi_2$, i.e., $\phi_1 + \phi_2 = 0.90$, $\phi_1 + \phi_2 = 0.95$, $\phi_1 + \phi_2 = 0.99$. In the first few lags, the functions fluctuate significantly. Then, the functions

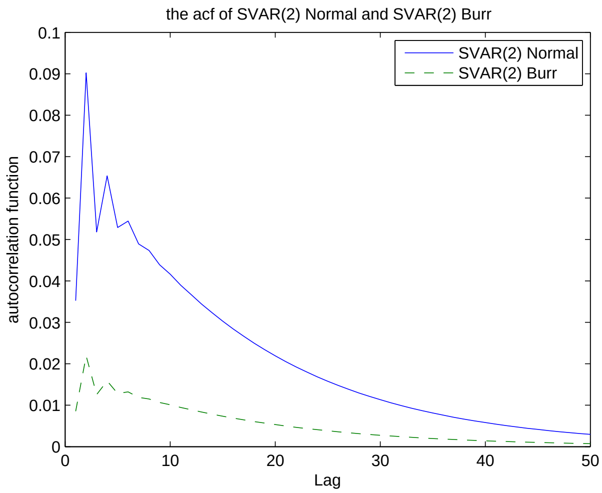

Figure 2 The autocorrelation function of squared returns of SVAR(2) Normal and SVAR(2) Burr models, $\phi_1 + \phi_2 = 0.90$

of all models decrease slowly as the lag increases. When $\phi_1 + \phi_2 = 0.99$, an indication of high persistence of volatility, the autocorrelation function decays very slowly and this is slower than those when $\phi_1 + \phi_2 = 0.90$ or $\phi_1 + \phi_2 = 0.95$.

This feature gives an indication that the ability of SVAR(2) model to capture the stylized fact of returns may be assessed through the high persistence parameter. In other words, if the SVAR(2) model gives high persistent volatility estimate, then it is likely to capture the low autocorrelation function of squared returns. Comparing all SVAR(2) models, one can conclude that the autocorrelation function of squared returns of SVAR(2) Burr model is lower and decay slower than that of SVAR(2) Normal model.

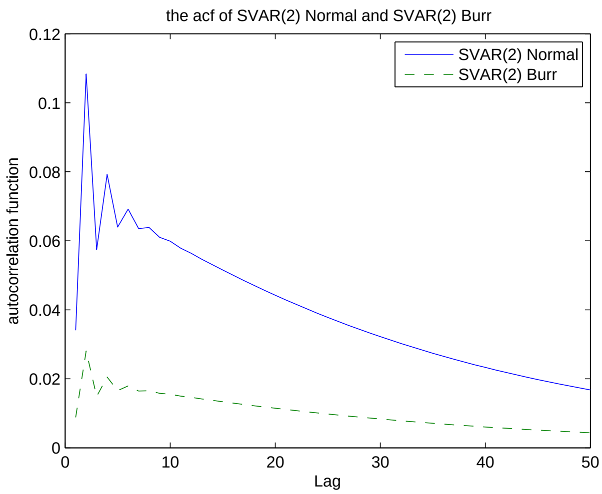

Figure 3 The autocorrelation function of squared returns of SVAR(2) Normal and SVAR(2) Burr models, $\phi_1 + \phi_2 = 0.95$

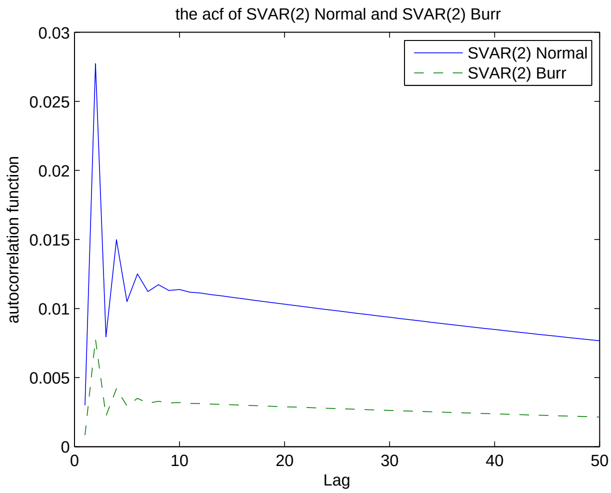

Figure 4 The autocorrelation function of squared returns of SVAR(2) Normal and SVAR(2) Burr models, $\phi_1 + \phi_2 = 0.99$

6. Conclusion

Volatility modeling through Stochastic Volatility (SV) model may be directed in two ways. Firstly, distributional assumption of the error or innovation changed to class of thick-tailed distribution. In the second direction, we may apply an AR(p) process for the volatility function. We have used, in this paper, a modified Burr distribution which is thick-tailed and comparable to the normal distribution and second-order AR for the volatility process. From the theoretical and empirical data of Australian stock returns, we find more appropriate SV models, in comparison to the basic SV model, for capturing empirical facts of returns and volatility. Furthermore, SV model with AR(2) volatility process has interesting properties for the autocorrelation function in which its shape is fluctuated in the first few lags before decay slowly.

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