Average Run Length of Cumulative Sum Control Chart by Markov Chain Approach for Zero-Inflated Poisson Processes

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Abstract

A Cumulative Sum (CUSUM) chart is an alternative effective to standard Shewhart control in order to monitor a small shift in process; however it is limited to normality assumption. An average run length (ARL) is common used to compare the performance of control charts which can be derived by Markov chain approach (MCA) and others mathematical tools. Consequently, this paper aims to propose closed form expression of the ARL based on MCA underlying zero-inflated Poisson (ZIP) processes of Cumulative Sum (CUSUM) chart which usually known as count data processes. Furthermore, the performance of the CUSUM chart is compared with an Exponentially Weighted Moving Average (EWMA) chart by studying the effect sizes of the probability of extra zeros in ZIP models and shift sizes in parameter of the ZIP processes for both of the control charts. The numerical results are obtained from the MCA which the performance of EWMA and CUSUM control charts are in the same manner for small shifts $(\delta \leq 0.01)$ and high probability of excess zero values $(\omega = 0.9)$. However, the CUSUM shows the poor performance for moderate to large shifts when compared with the EWMA chart.

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1. Introduction

A big role for Statistical Process Control (SPC) is to help a company analyze, control, measure, detect and improve in quality. The SPC have been implemented in many areas of applications-manufacturing and industry, computer intrusion and telecommunications network, finance and economics, health care and epidemiology, environmental statistics and etc. Quality characteristics can be described into two categories as physical and time orientation and sensory of product which terminate with continuous and discrete data, respectively. The popular latter types is number of defect and number of conformities that described by binomial and Poisson distributions, respectively. Due to digital economics, it is developing rapidly worldwide, then causes to improve process by decreasing product wastes or produce low defective proportion-namely zero-inflated Poisson (ZIP) model. Since, this model provides large numbers of zero count data then some alternatives models should be developed to overcome the underestimate problem of mean and variance.

In general, the classical attribute control chart as p and c charts have good performance for large shifts (Montgomery 2009) but the small and moderate shifts could be happened in realistic. Also, if the processes are drawn from ZIP model, then the classical p and c charts will signal too many false alarm rates when the process is in control. Consequently, these models are developed by Cohen (1963) by using maximum likelihood estimates to modify parameter for solving over-dispersion problem.

Recently, the control charts for ZIP are intensive studied by Areepong and Sukparungsee (2015, 2016) for detecting a change in parameter of ZIP models by proposing the explicit formulas for Moving Average (MA) and Double Moving Average (DMA) control charts. Chananet et al. (2015a) proposed the closed-form formula of ARL for Cumulative Sum (CUSUM) (Page 1954), Woodall and Adams 1993) and Exponentially Weighted Moving Average (EWMA) (Roberts 1959) charts by using Markov chain approach (MCA) when processes are modelled by zero-inflated negative binomial (ZINB) and derived the explicit formulas of ARL for ZINB moving average (MA) and double moving average (DMA) control charts (Chananet et al. 2015b).

A popularly performance characteristics of control charts is Average Run Length which is described by two states- when the process is in control state denoted by ARL_0 and when the process is out of control state denoted by ARL_1. Whereas the ARL_0 is the expectation of the observations before the control chart gives a false alarm that an in-control process has gone out of control. The ARL_1 means the expectation of the observations between a process going out of control and the control chart giving the alarm that the process has gone out-of-control. Ideally, the ARL_0 of an acceptable chart should be sufficient large and the ARL_1 should be minimum.

The evaluation methods of ARL have been proposed by many literatures which are intensively studied, for example the integral equation by using numerical quadrature method are presented to approximate the ARL of the CUSUM chart with normality process (Crowder 1978). Brook and Evans (1972) also implemented a finite-state Markov Chain Approach (MCA) for finding the ARL of EWMA chart. Lucas and Saccucci (1990) presented the closed form formulas for calculating the ARL by MCA and for designing the optimal EWMA control chart. The martingale approach is used to analytical derive the closed-form ARL when processes are drawn from light tailed distributions by Sukparungsee and Novikov (2006, 2008) and Areepong and Novikov (2008). The explicit form of ARL on Exponential CUSUM chart by deriving from integral equation are proved by Busaba et al. (2012) but restricted to continuous distributions. In addition, the explicit form of ARL can be calculated by numerical Gauss Legendre approximation for both EWMA and CUSUM when processes are serially correlated exponential white noise (Suriyakat et al. 2012, Petcharat et al. 2014, 2015, Somran et al. 2016). A simplest approach is often used when explicit formula or close-form formulas do not exist that is Monte Carlo (MC) simulation by Roberts (1959). MC is simple to program and test an accuracy of numerical results obtained from explicit formula and closed-form formulas. Due to the large number of trajectories, then it is very time consuming.

To overcome the limitation of each methods, the MCA can be evaluated the ARL when the process are underlying the zero-inflated Poisson (ZIP) on the CUSUM chart which proposed the closed-form of ARL in this paper. Additionally, the performance of the CUSUM chart are compared with the EWMA chart for small to large shifts and analyzed an effect of the proportion of exceed numbers of zero.

2.   Zero-Inflated Poisson Models

In many applications, the observations are frequently drawn from count data for describing the number of incidents per unit time or space. The discrete distributions as Poisson distribution is widely

modelled this kind of count data which the probability mass function (pmf.) of a random variable X is following

$$p(x;c) = P(X = x) = \frac{e^{-c}c^x}{x!}; \quad x = 0,1,2,..., \tag{1}$$

where X is number of count events occurring over unit time and space with parameter c. The real practice of Poisson dataset are represented as the number of nonconformities product per area in manufacturing, the number of network intrusion per hour in computer network, the number of infected patients per day in health science, etc. However, some dataset are not validated by modelling with Poisson distribution for example, the number of new Elephantiasis patients in Phetchaburi, Thailand, the number of major Thailand earthquakes per year (Unhapipat 2015), number of urinary tract infections for Human Immunodeficiency Virus (HIV) - infected patients (Broek 1995), unhealthy teeth (Böhning et al. 1999) and etc.

Since, the regular Poisson (c) model could be invalidated for too many zero-valued observations, then a simple way to model zero-inflated is to include an extra parameter considering as a probability of extra-zeros (ω) namely zero-inflated Poisson (ZIP) model (Cohen 1963, Lambert 1992). The probability mass function of this model can be written as following:

$$p(x;c,\omega) = \begin{cases} \omega + (1-\omega)e^{-c}; & x = 0 \\ (1-\omega)(\frac{e^{-c}c^x}{x!}); & x \neq 0 \end{cases} \tag{2}$$

where ω is the probability of the excess zero values by Poisson model. The mean and variance of the number of nonconforming products are calculated by

$$E(X) = c(1-\omega) \text{ and } V(X) = c(1-\omega)(1+\omega c).$$

3. Average Run Length of Zero-Inflated Poisson CUSUM Chart

Page (1954), who first introduced a cumulative sum (CUSUM) chart which are one of popular time-varying control charts correcting all observations in the sequence of sample values. An effective and efficient CUSUM control chart can provide the acceptable ARL_0 and control limit coefficient (h) in the continuous inspection chart. In this paper, the MCA by Brook and Evans (1972) and Lucas and Saccucci (1990) are used to derive the closed form formulas of ARL based on ZIP CUSUM control chart.

Assume X_t are a sequence of independent identically zero-inflated Poisson distributed (i.i.d.) random variables with parameters c and ω. The upper sided CUSUM control chart is designed to detect an upward shift in the mean and the statistic C_t is

$$C_t = \max\{0, X_t - k + C_{t-1}\}, \; t \in \{1, 2,...\}, \tag{3}$$

where k is a reference value and an initial value $C_0 = 0$. If the statistics C_t exceed the control limit (h), the process is issued a signal to out of control for the case of one-sided upper control chart.

The common criteria to measure the performance of the control charts is an Average Run Length (ARL) is used in practice to select the most appropriate control charts. Since, other approaches have a drawback, then one of the most effective approximation methods is the MCA. Therefore, the ARL can be evaluated by using Markov chain approach (MCA) of the ZIP CUSUM chart. This method is first proposed by Brook and Evans (1972) and intensively studied by Lucas and Saccucci (1990).

Consequently, the closed-form formulas of ARL for ZIP CUSUM control chart can be calculated which the procedures are shown as follows.

If the statistics $C_t = i$, then we will say that the control chart is in control state S_i, which each the control chart can be described as random walk over the states $S_0, S_1, ..., S_h$ where the S_h is an absorbing state to represents an out of control region above the control limit and S_0 is an initial state. The transition probability (P_{ij}) is the probability of moving from state i to state j where $j = 1, 2, ..., N$ in each step shown as following:

$$\begin{aligned} P_{ij} &= P\left(C_t \in S_j \mid C_{t-1} \in S_i\right) \\ &= P\left(X - k + C_{t-1} = j \mid C_{t-1} = i\right) \\ &= P\left(X = k + j - i\right), \quad i \neq h, 0 < j < h \end{aligned} \tag{4}$$

and $P_{i0} = P(X \leq k - i)$, $P_{ih} = P(X \geq k + h - i)$ and $P_{hh} = 1$. If the statistics C_t are move to an absorbing state, it is impossible to leave that state. Therefore the probability of leaving that state would be zero and it is shown as $P_{hh} = 1$, but it is not needed to consider calculating the ARL.

We can replace to the transition matrix $(\mathbf{P})$ and element of matrix (P_{ij}) as

$$\mathbf{P} = \begin{bmatrix} P_{11} & \cdots & P_{1n} & | & P_{1,n+1} \\ \vdots & \ddots & \vdots & | & \vdots \\ P_{n1} & \cdots & P_{nn} & | & P_{n,n+1} \\ --- & --- & --- & | & --- \\ 0 & \cdots & 0 & | & 1 \end{bmatrix} \text{ or } \mathbf{P} = \begin{bmatrix} \mathbf{R} & (\mathbf{I} - \mathbf{R})\mathbf{1} \\ \mathbf{0} & 1 \end{bmatrix} \tag{5}$$

where $\mathbf{R}$ contains the transition probabilities of going from one transient state to another state with $n \times n$ dimension, $\mathbf{I}$ is the $n \times n$ identity matrix, $\mathbf{1}$ is the $n \times 1$ column vector of ones and $\mathbf{0}$ is the $1 \times n$ row vector of zeros. The ARL based on t in control states can be obtained from (6) by Lucus and Saccucci (1990) where a positive integer (N) is given as following

$$ARL(t) = \sum_{i=1}^{\infty} iP(RL = i) \tag{6}$$

and then substitute $P(RL = i) = \mathbf{p}^T(\mathbf{R}^{i-1} - \mathbf{R}^i)\mathbf{1}$ into (6). The ARL can be rewritten as

$$\begin{aligned} ARL(t) &= \sum_{i=1}^{\infty} i\mathbf{P}^T(\mathbf{R}^{i-1} - \mathbf{R}^i)\mathbf{1} \\ &= \sum_{i=1}^{\infty} \mathbf{P}^T\mathbf{R}^{i-1}\mathbf{1} \\ &= \mathbf{P}^T(\mathbf{I} - \mathbf{R})^{-1}\mathbf{1} \end{aligned} \tag{7}$$

where $\mathbf{p}^T = (0, ..., 0, 1, 0, ..., 0)^T$ is the probability vector of an initial state with 1, corresponding to a specified state and zeros elsewhere (Chananet et al. 2015).

4. Numerical Results

The process is usually assumed that under in-control condition the parameter is known $c = c_0$. Due to the process variation, the in control parameter c_0 could be changed to be c_1 at unexpected times $v < \infty$. In this section, the numerical results obtained from the proposed closed-form

approximation of ZIP CUSUM are presented. The performance evaluation of the control chart based on assumption that observations are from ZIP model with in control parameter $c_0 = 1$ and 5 due to low nonconformities, $\omega = 0.1, 0.5$ and 0.9 defined as low, medium and high probability of zero in ZIP model, the desired $ARL_0 = 370$ and 500 for in control state. For an out of control state, the process mean shifts change from $(1-\omega)c_0$ to be $(1-\omega)c_1$ where $c_1 = (1+\delta)c_0$ and $\delta = 0.01, 0.05, 0.1, 0.25, 0.5$ and 1. In addition, the performance analysis of ZIP CUSUM is compared with the ZIP EWMA control charts.

Tables 1-4 show the numerically comparison results of ARL_1 of ZIPCUSUM obtained from Markov Chain Approach (MCA) in (7) with transient probability P_{ij} in (4). For the case of ZIP EWMA control chart, the ARL_1 values are approximated by MCA (Fatahi et al. 2012). An analysis of performance are compared the ARL_1 of ZIP CUSUM and EWMA when given $ARL_0 = 370$ and in control parameter of ZIP (c_0) is equal to 1 and 5 as shown on Tables 1 and 2, respectively. Whereas the magnitudes of shift are small ($\delta \le 0.01$), the CUSUM and EWMA control charts underlying ZIP model perform in the same manner, otherwise the EWMA chart is often superior to CUSUM chart with the minimum ARL_1.

In Tables 3 and 4, the performance results are guaranteed by given $ARL_0 = 500$ as presented in the same direction. The performance of CUSUM is as good as EWMA chart with small shifts and the performance of CUSUM is inferior to the EWMA control chart.

Table 1 Comparison ARL_1 of CUSUM versus EWMA when $c_0 = 1$ and $ARL_0 = 370$

ω	Control charts	Shifts (δ)					
		0.01	0.05	0.10	0.25	0.50	1.00
0.1	CUSUM ($k = 0.5, h = 333$)	365.81	351.96	336.06	295.96	246.89	185.46
	EWMA ($\lambda = 0.05, b = 1.07$)	349.63	280.14	217.90	116.10	54.91	22.68
0.5	CUSUM ($k = 0.5, h = 184$)	365.84	353.73	339.67	303.51	257.80	198.21
	EWMA ($\lambda = 0.05, b = 0.8425$)	351.23	282.65	218.12	117.41	58.75	23.77
0.9	CUSUM ($k = 0.5, h = 36$)	363.44	357.96	349.96	329.63	300.58	255.72
	EWMA ($\lambda = 0.05, b = 0.2965$)	364.98	342.82	317.83	257.38	189.60	117.14

Table 2 Comparison ARL_1 of CUSUM versus EWMA when $c_0 = 5$ and $ARL_0 = 370$

ω	Control charts	Shifts (δ)					
		0.01	0.05	0.10	0.25	0.50	1.00
0.1	CUSUM ($k=1, h=1300$)	369.53	359.98	348.73	318.83	278.98	223.22
	EWMA ($\lambda=0.05, b=5.535$)	360.28	295.05	205.97	75.57	28.97	12.05
0.5	CUSUM ($k=1, h=555$)	366.85	353.10	337.31	297.42	248.49	187.05
	EWMA ($\lambda=0.05, b=3.671$)	355.92	297.01	232.97	118.46	53.84	22.95
0.9	CUSUM ($k=0, h=182$)	367.78	363.06	357.32	341.13	317.22	278.34
	EWMA ($\lambda=0.05, b=1.197$)	363.25	338.49	310.92	254.86	175.97	105.36

Table 3 Comparison ARL_1 of CUSUM versus EWMA when $c_0 = 1$ and $ARL_0 = 500$

ω	Control charts	Shifts (δ)					
		0.01	0.05	0.10	0.25	0.50	1.00
0.1	CUSUM ($k=0.5, h=450$)	495.63	476.86	455.31	400.96	334.45	251.18
	EWMA ($\lambda=0.05, b=1.317$)	469.46	336.72	218.76	79.71	31.21	13.04
0.5	CUSUM ($k=0.5, h=250$)	496.71	480.25	461.15	412.01	349.92	268.95
	EWMA ($\lambda=0.05, b=0.864$)	466.87	363.60	272.19	133.96	61.01	25.87
0.9	CUSUM ($k=0.5, h=50$)	502.83	494.34	484.17	455.87	415.53	353.22
	EWMA ($\lambda=0.05, b=0.315$)	489.41	457.51	421.29	334.78	239.99	142.16

Table 4 Comparison ARL_1 of CUSUM versus EWMA when $c_0 = 5$ and $ARL_0 = 500$

ω	Control charts	Shifts (δ)					
		0.01	0.05	0.10	0.25	0.50	1.00
0.1	CUSUM ($k = 1, h = 1750$)	497.24	484.40	469.25	429.00	375.36	300.30
	EWMA ($\lambda = 0.05, b = 5.587$)	485.37	386.13	257.74	86.60	31.52	12.78
0.5	CUSUM ($k = 1, h = 750$)	495.58	476.99	455.65	401.71	335.56	252.51
	EWMA ($\lambda = 0.05, b = 3.73$)	477.68	386.76	293.28	139.14	59.93	24.65
0.9	CUSUM ($k = 1, h = 16$)	492.81	454.73	415.73	324.08	226.58	130.11
	EWMA ($\lambda = 0.05, b = 1.257$)	490.57	453.98	413.56	319.66	221.61	126.63

5. Conclusions

In this paper, the closed-form approximations of ARL for Cumulative Sum (CUSUM) are proposed when observations are drawn from zero-inflated Poisson (ZIP) model. Certainty, these approximations are much time saving in term of computational times. The CUSUM chart with ZIP model, namely ZIP CUSUM is compared the performance with Exponentially Weighted Moving Average (EWMA) chart, namely ZIP EWMA. The numerical results suggest no meaningful difference in the performance of CUSUM and EWMA control charts when the magnitudes of shift are very small ($\delta \leq 0.01$) and the probability of excess zeros are high ($\omega = 0.9$) as shown in Tables 1-4. Whereas the magnitudes of shift are moderate to large ($\delta > 0.01$) the performance of EWMA are superior to the CUSUM chart for both cases of $ARL_0 = 370$ and 500. In addition, the performance of control charts depend on the probability of excess zero values where the performance of the CUSUM and EWMA will decrease as the probability of excess zero values are increased. However, those control charts are flexible and simple to implement in real applications in order to vary the value of reference value (k) of CUSUM chart and weighted factor (λ) of EWMA chart in order to enhance the performance of the control charts.

References

Areepong Y, Novikov A. Martingale approach to EWMA control chart for changes in exponential distribution. J. Qual. Measure. Anal. 2008; 4(1): 197-203.

Areepong Y, Sukparungsee S. Explicit expression for the average run length of double moving average scheme for zero-inflated binomial process. Int. J. Appl. Math. Stat. 2015; 53(3): 33-43.

Areepong Y, Sukparungsee S. Approximation average run lengths of zero-inflated binomial GWMA chart with Markov chain approach. Far East J. Math. Sci. 2016; 99(3): 413-428.

Böhning D, Dierz E, Schlattmann P. The zero-inflated Poisson model and the decayed missing and filled teeth index in dental epidemiology, J. Roy. Stat. Soc. A Stat Soc. 1999; 162(2), 195-209.

Broek J. A score test for zero inflation in a Poisson distribution. Biometrics. 1995; 51(2): 738-743.

Brook D, Evans DA. An approach to the probability distribution of CUSUM run length. Biometrika. 1972; 9: 539-548.

Busaba J, Sukparungsee S, Areepong Y, Mititelu G. An analysis of average run length for CUSUM procedure with negative exponential data. Chiang Mai J. Sci. 2012; 39(2): 200-208.

Chananet C, Areepong Y, Sukparungsee S. A Markov chain approach for average run length of EWMA and CUSUM control chart based on ZINB Model. International J Int. J. Appl. Math. Stat. 2015; 53(1): 126-137.

Chananet C, Areepong Y, Sukparungsee S. An approximate formula for ARL in moving average chart with ZINB data. Thail Stat. 2015; 13(2): 209-222.

Cohen AC. Estimation in mixtures of discrete distributions. Proceedings of the International Symposium on Discrete Distributions, Montreal. 1963; 373-378.

Crowder SV. A simple method for studying run length distributions of exponentially weighted moving average charts. Technometrics. 1978; 29: 401-407.

Fatahi AA, Noorossana R, Dokouhaki P, Moghaddam BF. Zero inflated Poisson EWMA control chart for monitoring rare health-related events. Published on line, J. Mech. Med. Biol. 2012; 12(4): DOI: 10.1142/S0219519412500650.

Lambert D. Zero Inflated Poisson regression with and application to defects in manufacturing. Technometrics. 1992; 34(1): 1-14.

Lucas JM, Saccucci MS. Exponentially weighted moving average control schemes: properties and enhancements. Technometrics. 1990; 32(1): 1-29.

Montgomery DC. Introduction to statistical quality control. New York: Wiley; 2009.

Petcharat K, Areepong Y, Sukparungsee S, Mititelu G. Exact solution for average run length of CUSUM charts for MA(1) process. Chiang Mai J. Sci. 2014; 41(5.2): 1449-1456.

Petcharat K, Sukparungsee S, Areepong Y. Exact solution of the average run length for the cumulative sum charts for a moving average process of order q. Science Asia. 2015; 41: 141-147.

Roberts SW. Control chart tests based on geometric moving average. Technometrics. 1959; 1(3): 239-250.

Somran S, Sukparungsee S, Areepong Y. Analytic and numerical solutions of ARL of CUSUM procedure for exponentially distributed observations. Thail Stat. 2016; 14(1): 249-258.

Sukparungsee S, Novikov AA. Analytical approximations for detection of a change-point in case of light-tailed distributions. J. Qual. Measure. Anal. 2008; 4(2): 49-56.

Suriyakat W, Areepong Y. Sukparungsee S, Mititelu G. An analytical approach to EWMA control chart for AR(1) process observations with exponential white noise. Thail Stat. 2012; 10(1): 41-52.

Unhapipat S. Inference for zero-inflated Poisson distribution. PhD [dissertation]. Mahidol University; 2015.

Woodall W, Adams B. The statistical design of CUSUM charts. Qual. Eng. 1993; 5(4): 559-570.