Thailand Statistician
January 2017; 15(1): 1-10
http://statassoc.or.th
Contributed paper

Bulk Service Queuing System with Impatient Customers: A Computational Approach

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Received: 19 January 2016
Accepted: 15 June 2016

Abstract

The paper investigates a $M/M^{(b,b)}/1$ queuing model with bulk service. The server serves the customers in batches of fixed size b, and the service time is assumed to be exponentially distribution. Customers arrive to the system as a Poisson process and may renege after waiting in the queue for an exponentially distributed time. The reneging of a customer depends on the state of the system. The model is analyzed to find the different measures of effectiveness of the model. The approach adopted is based on embedded Markov chains.

Keywords: Queuing, Poisson arrival, batch service, reneging customers, embedded Markov chain.

1. Introduction

Bulk service is a common phenomenon in real life. Some examples of a bulk service system are telecommunication, transportation process, production process, airline scheduling, to name a few. Over the last two decades many studies have been carried out to analyze bulk service queues with various arrival processes and service time distributions. Powell and Humblet (1986) used general control strategy for bulk service queue. Jayaraman et al. (1994) investigated a general bulk service queue with arrival rate dependent on server-breakdown. Willmot and Drekic (2001) proposed transient analysis for the M^x /M /∞ queue. Dshalalow (2001) described briefly D-policy for bulk queuing system. See also Jaiswal (1964), Downton (1986), Jain and Singh (2005). However, very few authors considered bulk service queuing system with impatient customer. Shawky, and El–Paoumy (2008) studied a truncated hyper-Poisson queue with general bulk service rule, finite holding capacity and impatient customers and attempted to obtain an analytical solution to the problem, which is applicable only when the system has finite holding capacity. They could find only the explicit expression for the expected queue length. Shinde and Patankar (2012) investigated a state dependent bulk service system with server vacation when customers may be impatient. Their analysis is based on shifting operator and Rouche's Theorem involving. However, the methods used by various authors become very complicated when state dependent balking or state dependent reneging is introduced.

In this paper, we suggest a different approach to compute the performance measures of a system. We use embedded Markov chain to analyze a batch service queuing model with state dependent reneging. The batch size "*b*" is assumed to be fixed, so that whenever there is less than "*b*" customers in the system, the server waits till at least "*b*" customers are available. Customers or units are assumed to arrive in a Poisson manner, and the service time is exponentially distributed. An arriving customer who has to wait reneges (abandon later) after an exponentially distributed time, if his service has not yet begun. As different customers may have different delay thresholds, just as different customers have different utility functions, the reneging times of the successive customers may be taken to be independently and identically distributed.

Our analysis is based on steady state distribution of the semi Markov process. There is no bound for system capacity, so the state space of the Markov chain is infinite (countable). We check the stability condition to ensure the existence and uniqueness of the steady state distribution, and we use finite approximation of infinite transition probability matrix (TPM) to obtain the steady state distribution. Thus, we can avoid using transformations and carrying out tedious calculations of roots to get the values of the steady state probabilities. By our approach we can get a good approximation of the steady state distribution for any given set of system parameters, and it can be improved by our choice of the finite order of the TPM, that is, increase in the order will make the approximation better.

The paper is organized as follows. Section 2 gives the assumptions and notations. Section 3 analyzes the model to obtain the steady state distribution of the number of customers in the system. In Section 4 expressions for the measures of effectiveness of the system are obtained, while a comparison of the derived formulae of the performance measures and those obtained from a simulation study is given in Section 5. Some concluding remarks on the approach are made in Section 6.

2. Assumptions and Notations

Assumptions:

The queuing system is governed by the following assumptions:

(i) Customers arrive to the system one by one as a Poisson process with mean arrival rate λ.

(ii) The service times are independently and identically distributed as exponential with mean $1/\mu_1$.

(iii) The waiting time of the server since last service until the number of waiting customers is at least equal to its serving capacity *b*, does not affect the next service time.

(iv) The waiting time in the queue of a reneging customer has an exponential distribution with mean $1/\mu_2$.

(v) The service station has an infinite waiting capacity.

(vi) $\lambda < b\mu_1$

Notations:

$X(\nu)$ number of customer in the system at time "ν"

t_m *m*-th time epoch at which the system size changes.

X_m number of customers in the system after the *m*-th transition.

3. Analysis of the model

Let us define

$$I = \{0, 1, 2, \ldots\},\ I^{+} = \{1, 2, \ldots\},\ \boldsymbol{X} = \{X_m: m = 0, 1, 2,\ldots\},\ \boldsymbol{t} = \{t_m: m = 1, 2,\ldots\}.$$

We have that $X_m = X(t_m+) = \lim_{h\to 0^+} X(t_m + h)$, for $m \geq 1$. Now, for all $m \geq 1$, $t_m \geq 0$ and $j \in$ I, we have

$$\begin{aligned} &\mathrm{P}[X_{m+1} = j,\ t_{m+1} - t_m \leq \nu \mid X_0, X_1,\ldots, X_m;\ t_1, t_2,\ldots, t_m] \\ &\quad = \mathrm{P}[X_{m+1} = j,\ t_{m+1} - t_m \leq \nu \mid X_m]. \end{aligned} \tag{1}$$

Hence, $(\boldsymbol{X}, \boldsymbol{t})$ is a Markov-renewal process.

Our interest lies in finding the distribution of X_m as $m\to\infty$ and of $X(\nu)$ as $\nu\to\infty$, that is, we want to find the steady state distribution of the number of customers in the system at any point of time.

Let $Q(i, j, \nu)$ denote the probability that if the system is in state i after a transition, then the next transition will occur after at most ν units of time and the system will move to state j, i.e.

$$Q(i, j, \nu) = \mathrm{P}[X_{m+1} = j,\ t_{m+1} - t_m \leq \nu \mid X_m = i]. \tag{2}$$

Let $Q(\nu)$ denote a matrix whose (i, j)-th element is $Q(i, j, \nu)$. Let, further, P be the transition probability matrix (TPM) of the discrete time Markov chain $\{X_m\}$ underlying the Markov renewal process $(\boldsymbol{X}, \boldsymbol{t})$, and $P(i, j)$ denote the one-step transition probability of going from state i to state j. Then,

$$P(i, j) = \lim_{\nu\to\infty} Q(i, j, \nu) \tag{3}$$

Let T_i denote the unconditional time for which the system remains in state i. The distribution of T_i can be obtained in the following way:

Suppose at some time point "t^*" the system reaches the state 'i'. If, $i < b$, the state of the system can change only due to an arrival or a reneging. Let X denote the time for the next arrival and Y denote the time for the next departure due to reneging. Then, $T_i = \min(X, Y)$.

Now, according to our assumptions, $X \sim \exp(\lambda)$ and $Y \sim \exp(i\mu_2)$, since there are "i" customers waiting in the queue. Hence,

$$\Pr(T_i > \nu) = \Pr[X > \nu, Y > \nu] = \Pr[X > \nu]\Pr[Y > \nu] = e^{-(\lambda + i\mu_2)\nu},$$

which shows that $T_i \sim$ exponential $(\lambda + i\mu_2)$.

The situation $i \geq b$ occurs when there are b customers in service, none of whom will renege, and $(i\text{-}b)$ customers are in the waiting line, who may renege., In this situation, the state of the system can change due to a new arrival, a reneging or a service completion. Let X, Y and Z denote respectively the times for the next arrival, the next reneging and the next service completion, Then, $T_i = \min(X, Y, Z)$. By our assumptions, $X \sim$ exponential(λ), $Y \sim$ exponential$((i-b)\mu_2)$ and $Z \sim$ exponential(μ_1), and, therefore, arguing as before, we have that $T_i \sim$ exponential$(\lambda + (i-b)\mu_2 + \mu_1)$.

Thus we have,

$$T_i \sim \exp\ (\lambda + i\mu_2), \qquad \text{when } i < b$$

$$T_i \sim \exp\ (\lambda + (i-b)\mu_2 + \mu_1), \quad \text{when } i \geq b.$$

Hence, the process $\{X(t)\}$ is a Markov process.

Noting that both i, the system size before transition, and j, the system size after transition, can vary from 0 to ∞, we obtain the one-step transition probabilities $P(i, j)$ as follows:

$$P(i,j) = \begin{cases} 1 & \text{if } i = 0 \text{ and } j = 1 \\ \lambda/(\lambda + i\mu_2) & \text{if } 0 \le i < b \text{ and } j = i+1 \\ i\mu_2/(\lambda + i\mu_2) & \text{if } 0 \le i < b \text{ and } j = i-1 \\ \lambda/(\lambda + (i-b)\mu_2 + \mu_1) & \text{if } i \ge b \text{ and } j = i+1 \\ (i-b)\mu_2/(\lambda + (i-b)\mu_2 + \mu_1) & \text{if } i \ge b \text{ and } j = i-1 \\ \mu_1/(\lambda + (i-b)\mu_2 + \mu_1) & \text{if } i \ge b \text{ and } j = i-b. \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

To study the Markov chain $\{X_m\}$, we make use of the following theorem by Pakes (1969):

Theorem 1 (Pakes, 1969): *If* $|\gamma_i|<\infty$ *for all i, where* $\gamma_i = E[X_{m+1} - X_m \mid X_m = i]$, *and if* $\lim\sup_{i\to\infty} \gamma_i < 0$ *in an irreducible and aperiodic Markov chain* $\{X_m: m = 1, 2, \ldots\}$, *then it is ergodic.*

We then have the following observation on the Markov chain $\{X_m\}$:

Theorem 2 *The Markov chain* $\{X_m\}$ *is positive recurrent.*

Proof: The state of the chain can get increased by one unit due to an arrival, and decreased by one unit due to reneging or decreased by "*b*" units due to service completion. If "*b*" is an even number then starting from a state, the system can return to that state either in an odd number of steps or in an even number of steps. The above claim is clear from the following example:

Example 1: Let us assume that b is an even number. Suppose the system is at the state "0" and the server is idle. Then, it can go to the state "b" in b steps if there be "b" consecutive arrivals, and then can get back to the state "0" in a single step due to a service completion, provided there are no arrivals during that time. This means, the number of steps for the system to return to the state "0" is $b+1$, which is an odd number. Let us consider another way in which the system starting at state "0" returns to the same state. Suppose there are "$b-1$" consecutive arrivals which takes the system to state "$b-1$" in $b-1$ steps, and then it gets back to state "0" in $b-1$ steps due to "$b-1$" consecutive reneging of the customers. Hence, in this case, the number of steps to return to the state "0" is $2b-2$, which is an even number. Thus, the periodicity of the state "0" is one.

Since the chain is irreducible, clearly the periodicity of the chain must be 1.

Thus, for b even, the chain is aperiodic.

If "*b*" is an odd number, then starting from a state, the system can return to that state only in an odd number of steps, which is clear from the following example:

Example 2: Consider the system to be at the state "0" and the server is idle. Due to arrivals, service completions and reneging, the system goes through different states. The number of steps required for the system to again return to state "0" can be calculated in the following way:

Suppose there are "$n_1 b + n_2$"(where "n_2" be any positive integer less than "*b*") arrivals in that interval, and "$n_1 - n_3$" service completions, where $0 \le n_3 \le n_1$. Then for the system to go back to state "0", there must be $n_1 b + n_2 - (n_1 - n_3 b) = n_3 b + n_2$ reneging. Thus, the number of steps to return to state "0" is $n_1 b + n_2 + (n_1 - n_3) + n_3 b + n_2 = n_1(b+1) + n_3(b-1) + 2n_2 = n_0$, say, since every arrival

increases the state by unity and every service completion decreases the state by "b", while every reneging decreases the state by unity. Clearly, n_0 is an even number, whatever be $n_1, n_2, n_3 \in \{0,1,2,...\}$, $n_1 \geq n_3$. Hence, the number of steps to return to "0" is always even.
Since the chain is irreducible, it follows that the periodicity of the chain is 2, which is the greatest common divisor of all even numbers.
Hence, for odd b, the chain is of periodicity 2.

Case 1: "b" is an even integer, the Markov chain $\{X_m\}$ is aperiodic.
For b even,

$$\begin{aligned}\gamma_i &= \mathrm{E}[X_{m+1} - X_m \mid X_m = i].\\ &= 1, && \text{when } i = 0\\ &= (\lambda - i\mu_2)/(\lambda + i\mu_2), && \text{when } 0 < i < b\\ &= [(2\lambda - (b-1)\mu_1)/(\lambda + (i-b)\mu_2 + \mu_1)] - 1, && \text{when } i \geq b.\end{aligned}$$

Hence, $\gamma_i \to -1$ as $i \to \infty$. This indicates that $|\gamma_i| < \infty$ and $\limsup_{i\to\infty} \gamma_i < 0$**.**

Then, by virtue of Theorem 1, it follows that the Markov chain $\{X_m\}$ is ergodic.

Case 2: "b" is an odd integer, the chain $\{X_m\}$ is periodic with periodicity 2.
In this case,

$$\Pr[X_{m+2} = i \mid X_m = i] > \Pr[X_{m+2} = i \mid X_{m+1} = i+1]\ \Pr[X_{m+1} = i+1 \mid X_m = i] > 0.$$

Hence, $\Pr[X_{2m+2} = i \mid X_{2m} = i] > 0$. But this gives the one step transition probability "$p(i, i)$" for the chain $\{X_{2m}\}$. Thus the chain $\{X_{2m}\}$ is aperiodic (Hoel et al., 1972).
Consider

$$\delta_i = \mathrm{E}[X_{2m+2} - X_{2m} \mid X_{2m} = i],\ i \geq 0.$$

Since we have only a finite number of non-zero elements in each row and column of the transition matrix P, for given i, δ_i will be the sum of a finite number of non-zero values. Hence, δ_i is finite for all i.
When $i \geq 2b$,

$$\delta_i = 2\cdot p(i,i+1)\cdot p(i+1,i+2) - 2\cdot p(i,i-1)\cdot p(i-1,i-2) - (b-1)\cdot\{p(i,i+1)\cdot p(i+1,i+1-b) + p(i,i-b)\cdot p(i-b,i-b+1)\} - (b+1)\cdot\{p(i,i-1)\cdot p(i-1,i-1-b) + p(i,i-b)\cdot p(i-b,i-b-1)\} - 2b\cdot p(i,i-b)\, p(i-b,i-2b)$$

Hence, $\delta_i \to -2$ as $i \to \infty$.
This shows that $|\delta_i| < \infty$, and $\limsup_{i\to\infty} \delta_i < 0$.

Then, by virtue of Theorem 1, the Markov chain $\{X_{2m}\}$ is ergodic.
Now, consider $\tau_y^{(2)} = \min\{2n: X_{2n} = y \mid X_0 = y\}$ and $\tau_y^{(1)} = \min\{n: X_n = y \mid X_0 = y\}$.

As the chain $\{X_{2m}\}$ is ergodic, $\mathrm{E}[\tau_y^{(2)}] < \infty$.

But, $\tau_y^{(2)} = \tau_y^{(1)}$, since the Markov chain $\{X_m\}$is of periodicity 2.

Hence, $\mathrm{E}[\tau_y^{(1)}] < \infty$.

Thus, the chain $\{X_m\}$ is positive recurrent.
Hence, whatever be "b", the chain $\{X_m\}$ is positive recurrent.
Thus the theorem is proved.

As $\{X_m\}$ is irreducible and positive recurrent, there exists a unique stationary distribution π, which is characterized by the equations

$$\pi_j = \sum_{i\geq 0} \pi_i P(i,j)\text{, for all } j \geq 1;\ \sum_{j\geq 0} \pi_j = 1. \tag{5}$$

(Hoel et al., 1972)

Equations (5) give an infinite system of equations, which is rather difficult to solve. In order to compute the steady-state probabilities, we therefore make use of a modification of a well-known procedure due to Seneta (1968), which is the north-west corner truncation of an infinite-dimensional TPM to a finite one. The modified method is due to Wolf (1980). The method works well when the elements in the TPM become negligible as one proceeds to the eastern and southern sides of the infinite-dimensional transition probability matrix.

Let us consider a special sequence of TPM $\{P_m, m > 0\}$, constructed using P, such that

$$P_m(i,j) = \begin{cases} 0, & \text{if } i \geq m+1 \text{ or, } j \geq m+1 \\ P(i,0) + \sum_{j'>m} P(i,j'), & \text{if } i < m+1 \text{ and } j = 0 \\ P(i,j), & \text{if } i < m+1 \text{ and } 0 < j < m+1. \end{cases} \tag{6}$$

Then, P_m has exactly one stationary distribution, π^m, and it converges to π (Wolf, 1979).
If $\lim_{t\to\infty} P[X(t)=j] = \nu_j$, then $\nu_j^m = M_j \prod_j^m \big/ \sum_i M_j \prod_j^m$ will converge to ν_j, where $M_i = E[T_i]$.

Using the statistical software R, we can compute π^m as the eigen vector of the transpose of P_m corresponding to the eigen value 1 and it exists because we are considering an irreducible and positive recurrent transition probability matrix of order "m". Then, using the formula

$$\nu_j^m = M_j \prod_j^m \big/ \sum_i M_j \prod_j^m \text{ for large value of "}m\text{"},$$

we can approximate ν_j.

4. Performance Measures

We now evaluate the performance measure as follows:

(i) Expected number of customer in the system = $E = \lim_{m\to\infty} \sum_j j\nu_j^m$

(ii) Standard deviation of number of customer in the system = $\sigma = \sqrt{\lim_{m\to\infty}\left[\sum_j j^2 \nu_j^m - E^2\right]}$

(iii) Expected queue length = $Q = \lim_{m\to\infty} \sum_{j=0}^{b-1} j\nu_j^m + \sum_{j\geq b} (j-b)\nu_j^m$

(iv) Busy period probability = $P_B = \lim_{m\to\infty} \sum_{j\geq b} \nu_j^m$

(v) Average reneging rate = $R = \lim_{m\to\infty}\left[\sum_{i<b} i\mu_2 \nu_i^m + \sum_{i\geq b} (i-b)\mu_2 \nu_i^m\right]$

(vi) Proportion of customers served, that is the proportion of customers who left after service completion.

To obtain (vi), we consider a large value of "t" as well as a large number of transitions, say M transitions in the system size, so that the value of the M^{th} transition, viz. t_M, is high. If we have n_1 arrivals and n_2 service completions during that time, then we can say that in the long run the proportion of customers served under steady state situation is

$$P^{*} \approx \left(\frac{Total\ number\ of\ customer\ served}{Total\ number\ of\ customer\ arrived}\right) = bn_2 / n_1$$

Now,

$$(n_1 / M) \approx \lim_{n\to\infty} P[X_{n+1} - X_n = 1] = \lim_{m\to\infty} \sum_{i\geq 0} p(i, i+1)\Pi_i^m = P_1 \text{, say}$$

and $(n_2 / M) \approx \lim_{n\to\infty} P[X_{n+1} - X_n = -b] = \lim_{m\to\infty} \sum_{i\geq b} p(i, i-b)\Pi_i^m = P_2$, say.

Hence, $P^{*} = \left(\frac{bn_2}{n_1}\right) = \left(\frac{bn_2 / M}{n_1 / M}\right) \approx \left(\frac{bP_2}{P_1}\right)$. (7)

Though we do not get the above measures of performance in closed forms we can approximate them for large "*m*". In fact, these measures are convergent, which is very evident from Figure 1. With increase in the value of '*m*' one can make the approximation better up to any desired level of accuracy.

5. Computations of Performance Measures

Using R software, we have computed all the performance measures mentioned above. Simulation studies are carried out for each set of values of the system parameters. In each case, we have simulated 500 transitions and used 1000 replications.

Values obtained using the derived formula are very close to the simulated values of the performance measures. As this method is highly based on the consideration of large value of '*m*', so we have computed all the measures for *m* = 100(1)199 and checked their convergence.

Table 1 compares the measures of performance based on the derived formulae with those obtained by simulation, and these are found to be considerably close. It is also observed that other parameters remaining constant, an increase in the service capacity (*b*) of the server results in a decrease in the expected queue length, the chance of the server remaining idle and the rate of reneging. The variation in the number of customers in the system also decreases with increase in *b*. In Figure 1, we graphically show the convergence of the various measures of performance to their true values as the order of the transition probability matrix increases.

6. Conclusions

The paper analyzes a $M/M^{(b,b)}/1$ queuing model with reneging using embedded Markov chain. It derives the performance measures of the system and carries out a simulation study to show the closeness of simulated values to those derived by the algebraic expressions. The procedure adopted is less tedious than the usual Laplace transform approach used to analyze a queuing system, which requires the generation of complex zeroes and therefore poses several numerical difficulties. The method may be used to analyze more general type of bulk service queues.

Acknowledgements

The authors thank the anonymous referees for reading the paper with care and making fruitful suggestions, which helped to improve the presentation.

Table 1 Values of performance measures based on derived formulae and their simulated values for given sets of model parameters

λ	μ_2	b	μ_1	E	E (*simul.*)	Q	Q (*simul.*)	σ^2	σ^2 (*simul.*)	P_B	P_B (*simul.*)	$P_{*(\%)}$	$P_{*(\%)}$ (*simul.*)	R	R (*simul.*)
2	0.050	9	0.25	14.41	14.48	8.01	8.07	72.54	72.39	0.7128	0.71293	79.9674	79.7349	0.4006	0.3825
		10	0.25	14.07	13.63	7.59	7.36	70.30	66.68	0.6482	0.62724	81.0274	81.9310	0.3794	0.3521
		11	0.25	13.94	13.74	7.43	7.30	68.79	65.52	0.5922	0.58585	81.4277	81.6801	0.3714	0.3117
		11	0.20	16.91	16.00	9.22	8.79	91.59	90.59	0.6996	0.65553	76.9507	77.4029	0.4610	0.4726
		12	0.20	16.70	16.54	8.92	8.79	90.71	87.72	0.6475	0.64546	77.6968	78.1187	0.4461	0.4422
		13	0.20	16.61	16.82	8.82	16.82	90.13	94.08	0.5997	0.60253	77.9601	77.4286	0.4408	0.4239
4	0.025	14	0.30	32.93	32.97	21.38	21.41	406.75	398.27	0.8251	0.82595	86.6367	86.6886	0.5345	0.4971
		14	0.50	16.63	16.21	9.08	8.84	113.73	106.24	0.5390	0.52649	94.3226	94.4824	0.2271	0.2325
		15	0.30	30.87	30.23	19.13	18.65	370.97	362.14	0.7826	0.77121	88.0459	88.4408	0.4782	0.4972
	0.08	10	0.50	14.87	14.63	8.18	7.98	81.88	80.15	0.6692	0.66479	83.6498	83.9291	0.6540	0.6728
		12	0.35	20.10	20.58	11.30	11.59	127.00	130.50	0.7400	0.75100	77.4554	76.5877	0.9018	0.8854
		13	0.50	14.60	14.73	7.86	7.93	75.52	77.23	0.5186	0.52296	84.2743	84.3844	0.6290	0.5988

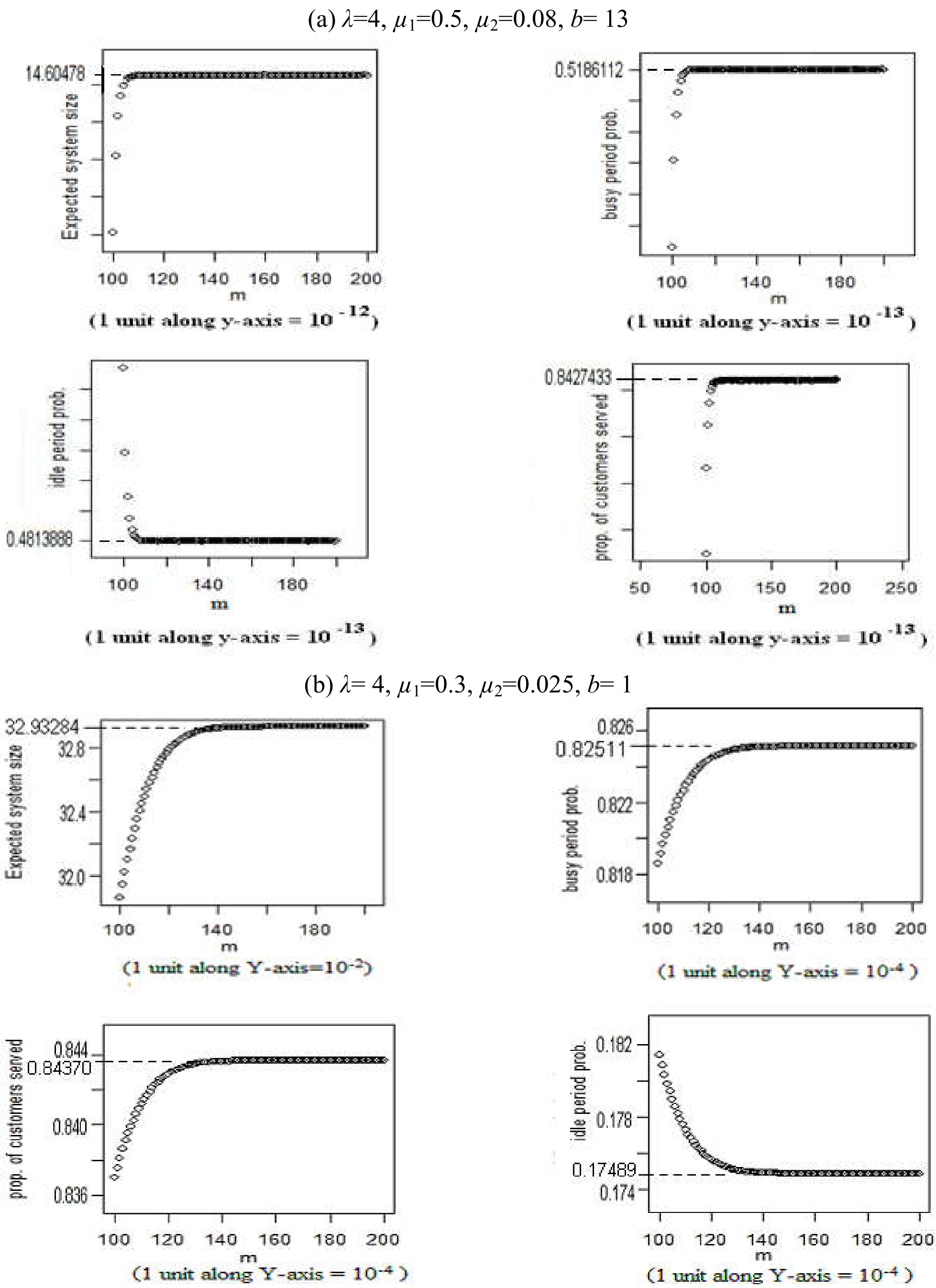

Figure 1 Graphs showing the convergence of the performance measures as the order (m) of TPM increases

For each graph, the value to which the curve converges is indicated by dotted line.

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