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# Recurrence Relations of Moments of Generalized Order Statistics in a Compound Rayleigh Distribution

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**Abstract**

In this paper, recurrence relations for single and product moments of generalized order statistics (gos) from the Compound Rayleigh distribution (CRD) have been established. These relations are deduced for moments of order statistics and upper record values. Further, this distribution has been characterized through the recurrence relation for a single moment of generalized order statistics from CRD.

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**Keywords**: Single and product moments, order statistics, upper record values and characterization.

## 1. Introduction

Kamps (1995) introduced the concept of generalized order statistics (gos) as follows:
Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed (iid) random variable (rv) with the *df*, $F(x)$ and the *pdf* $f(x)$. Let $n \in N$, $n \geq 2$, $k > 0$, where $k$ is a positive integer, $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, 2, \ldots, n-1\}$. Then $X(r, n, \tilde{m}, k)$, $r = 1, 2, \ldots, n$ are called gos if their joint *pdf* is given by

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[\bar{F}(x_i)]^{m_i} f(x_i)\right)[\bar{F}(x_n)]^{k-1} f(x_n) \tag{1}$$

on the cone $F^{-1}(0+) < x_1 \leq x_2 \leq \cdots \leq x_n < F^{-1}(1)$ of $\mathbb{R}^n$. Here $\bar{F}(x) = 1 - F(x)$.

The *pdf* of $r^{th}$ $m-gos$ is given by (Kamps, 1995):

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r - 1} g_m^{r-1}[F(x)] f(x) \tag{2}$$

and the joint *pdf* of $X(r, n, m, k)$ and $X(s, n, m, k)$, $1 \leq r < s \leq n$, is given by

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m g_m^{r-1}[F(x)]$$
$$\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1} f(x)f(y)\,,\ x<y \qquad (3)$$

where

$$C_{r-1}=\prod_{i=1}^{r}\gamma_i\ ,\quad \gamma_i=k+(n-i)(m+1)\,,$$

and

$$g_m(x)=\int_0^x(1-t)^m dt = h_m(x)-h_m(0)=\begin{cases}-\dfrac{1}{m+1}(1-(1-x)^{m+1}), & m\neq -1\\ -\ln(1-x) & ,\quad m=-1\end{cases},\qquad x\in[0,1)$$

$$h_m(x)=\begin{cases}-\dfrac{1}{m+1}(1-x)^{m+1}, & m\neq -1\\ -\log(1-x) & ,\quad m=-1\end{cases}\qquad x\in[0,1).$$

The result given in the paper can be used to compute the moments of ordered random variables, if the parent distribution follows the CRD, since recurrence relations reduce the amount of direct computation and hence reduce the time and labour.

The recurrence relations based on generalized order statistics have received considerable attention in recent years. Many authors derived the recurrence relations for generalized order statistics for different distributions. See, Ahmad and Fawzy (2002), AL-Hussaini et al. (2005), Khan et al. (2007), Kumar and Khan (2013), Khan et al. (2015a, 2015b) among others.

We say that an absolutely continuous random variable (rv) $X$ has a compound Rayleigh distribution CRD $(\alpha,\beta)$ if its cumulative distribution function (cdf) $F(x)$ is as follows,

$$F(x;\alpha,\beta)=1-\left(1+\frac{x^2}{\beta}\right)^{-\alpha},\quad x>0\,,\alpha>0,\beta>0 \qquad (4)$$

where $\alpha$ , $\beta$ are shape and scale parameters respectively.

The corresponding probability density function ($pdf$) $f(x)$ is given by

$$f(x:\alpha,\beta)=2\alpha\beta^{\alpha}x(\beta+x^2)^{-(\alpha+1)}\,, x>0\,,\alpha>0,\beta>0. \qquad (5)$$

The two parameter compound Rayleigh distribution CRD $(\alpha,\beta)$ provides a population model which is useful in several areas of statistics, including life testing and reliability. The compound Rayleigh distribution CRD is a special case of the three- parameter Burr type- XII distribution. For more details properties on CRD $(\alpha,\beta)$ (see Sagheer and Ahsanullah, 2015).

Therefore, it is easy to see that from (4) and (5), we have

$$\bar{F}(x)=\frac{1}{2\alpha}[\beta x^{-1}+x]f(x). \qquad (6)$$

The relation in (6) will be exploited in this paper to derive some recurrence relations for the moments of ordered random variables from the compound Rayleigh distribution CRD $(\alpha,\beta)$.

It appears from literature that no attention has been paid on the characterization of the compound Rayleigh distribution CRD $(\alpha,\beta)$ based on generalized order statistics.

## 2. Recurrence Relations for Single Moments

**Theorem 1** *For the compound Rayleigh distribution given (5) and* $n \in \mathbb{N},\ m \in \mathbb{R}\ ,\ 2 \le r \le n$

$$E[X^{j}(r,n,m,k)] - E[X^{j}(r-1,n,m,k)] = \frac{j\beta}{2\alpha\gamma_r} E[X^{j-2}(r,n,m,k)] + \frac{j}{2\alpha\gamma_r} E[X^{j}(r,n,m,k)]. \tag{7}$$

**Proof:** From (2), we have

$$E[X^{j}(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j [\bar{F}(x)]^{\gamma_r - 1} g_m^{r-1}(F(x)) f(x) dx. \tag{8}$$

Integrating by parts taking $[\bar{F}(x)]^{\gamma_r-1} f(x)$ as the part to be integrated, we get

$$E[X^{j}(r,n,m,k)] = E[X^{j}(r-1,n,m,k)] + \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j [\bar{F}(x)]^{\gamma_r} g_m^{r-1}(F(x)) f(x) dx.$$

The constant of integration vanishes since the integral considered in (8) is definite integral. On using relation (6), we obtain

$$E[X^{j}(r,n,m,k)] - E[X^{j}(r-1,n,m,k)] = \frac{jC_{r-1}}{\gamma_r (r-1)!}\int_0^\infty x^{j-1} [\bar{F}(x)]^{\gamma_r} g_m^{r-1}[F(x)] \left\{ \frac{1}{2\alpha}[\beta x^{-1} + x] f(x) \right\} dx$$

$$E[X^{j}(r,n,m,k)] - E[X^{j}(r-1,n,m,k)] = \frac{j\beta}{2\alpha\gamma_r} E[X^{j-2}(r,n,m,k)] + \frac{j}{2\alpha\gamma_r} E[X^{j}(r,n,m,k)].$$

**Remark 1** *Setting* $m = 0, k = 1$ *in the Theorem 1, we obtain the recurrence relations for the single moments of order statistics of the compound Rayleigh distribution in the form*

$$E[X^{j}_{r:n}] - E[X^{j}_{r-1:n}] = \frac{j}{2\alpha(n-r+1)}\left\{ \beta E[X^{j-2}_{r:n}] + E[X^{j}_{r:n}] \right\}.$$

**Remark 2** *Setting* $m = -1, k = 1$ *in the Theorem 1, we get the recurrence relations for the single moments of upper* $k-$*record of the compound Rayleigh distribution in the form*

$$E[X^{j}_{U(r)}]^k - E[X^{j}_{U(r-1)}]^k = \frac{j}{2\alpha k}\left\{ \beta E[X^{j-2}_{U(r)}]^k + E[X^{j}_{U(r)}]^k \right\}$$

as obtained by Khan and Khan (2016).

## 3. Recurrence Relations for Product Moments

**Theorem 2** *For the compound Rayleigh distribution given (5) and* $n \in \mathbb{N},\ m \in \mathbb{R}\ ,\ 1 \le r \le s \le n-1$

$$E[X^{i}(r,n,m,k)\ X^{j}(s,n,m,k)] - E[X^{i}(r,n,m,k)\ X^{j}(s-1,n,m,k)] = \frac{j}{2\alpha\gamma_s}\left\{ \beta E[X^{i}(r,n,m,k)\ X^{j-2}(s,n,m,k)] + E[X^{i}(r,n,m,k)\ X^{j}(s,n,m,k)] \right\}. \tag{9}$$

**Proof:** From (3), we have

$$E[X^{i}(r,n,m,k)\ X^{j}(s,n,m,k)] = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty x^i [\bar{F}(x)]^m g_m^{r-1}[F(x)] f(x) I(x) dx \tag{10}$$

where

$$I(x)=\int_x^\infty y^j[\bar{F}(y)]^{\gamma_s-1}[h_m(F(y)-h_m(F(x)]^{s-r-1}f(y)\,dy.$$

Solving the integral in $I(x)$ by parts and substituting the resulting expression in (10), we get

$$E[X^i(r,n,m,k)\ X^j(s,n,m,k)]-E[X^i(r,n,m,k)\ X^j(s-1,n,m,k)]=$$

$$\frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\beta\int_x^\beta x^i y^{j-1}[\bar{F}(x)]^m f(x)\,g_m^{r-1}[F(x)]\ \times[h_m(F(y)-h_m(F(x)]^{s-r-1}[F(y)]^{\gamma_s}\,dy\,dx.$$

The constant of integration vanishes since the integral in $I(x)$ is definite integral. On using relation (6), we obtain

$$E[X^i(r,n,m,k)\ X^j(s,n,m,k)]-E[X^i(r,n,m,k)\ X^j(s-1,n,m,k)]$$

$$=\frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty x^i y^{j-1}[\bar{F}(x)]^m\,g_m^{r-1}[F(x)]\,f(x)$$

$$\times[h_m(F(y)-h_m(F(x)]^{s-r-1}[F(y)]^{\gamma_s-1}\left\{\frac{1}{2\alpha}[\beta y^{-1}+y]\,f(y)\right\}dy\,dx$$

and hence the Theorem.

**Remark 3** *Setting* $m=0,k=1$ *in the Theorem 2, we obtain the recurrence relations for the product moments of order statistics of the compound Rayleigh distribution in the form*

$$E[X_{r,s:n}^{i\ j}]-E[X_{r,s-1:n}^{i\ j}]=\frac{j}{2\alpha\,(n-s+1)}\left\{\beta\,E[X_{r,s:n}^{i\ j-2}]+E[X_{r,s:n}^{i\ j}]\right\}.$$

**Remark 4** *Setting* $m=-1,k=1$ *in the Theorem 2, we get the recurrence relations for the product moments of upper* $k-$*record of the compound Rayleigh distribution in the form*

$$E[X^i{}_{U(r)}\,X^j{}_{U(s)}]^k-E[X^i{}_{U(r)}\,X^j{}_{U(s-1)}]^k=\frac{j}{2\alpha\,k}\left\{\beta\,E[X^i{}_{U(r)}X^{j-2}{}_{U(s)}]^k+E[X^i{}_{U(r)}\,X^j{}_{U(s)}]^k\right\}$$

as obtained by Khan and Khan (2016).

## 4. Characterization

This section discusses the characterization of CRD. Characterization of a probability distribution plays an important role in probability and statistics. Before a particular probability distribution model is applied to fit the real world data, it is necessary to confirm whether the given continuous probability distribution satisfies the underlying requirements. A probability distribution can be characterized through the various methods mainly conditional expectations, L-moments and recurrence relations. In this paper, we establish the characterization result based on recurrence relation of single moments of generalized order statistics. In recent years, there has been a great interest in the characterizations of probability distributions through recurrence relations based on $gos$.

Theorem 3 is characterized based on the generalization of Müntz-Szász Theorem (Hwang and Lin, 1984), which states that on a space $L(a,b)$ of summable functions defined on the interval $(a,b)$, a sequence of functions $f_n(x)$ is complete on $(a,b)$ if for any $g\in L(a,b)$ the equalities

$$\int_a^b f_n(x)g(x)dx=0\qquad n=1,2,...,$$

imply that $g(x)=0$ almost everywhere on $(a,b)$.

**Theorem 3** *The necessary and sufficient condition for a random variable* $X$ *to be distributed with pdf given by (5) is that,*

$$E[X^j(r,n,m,k)] = E[X^j(r-1,n,m,k)] + \frac{j\beta}{2\alpha\gamma_r} E[X^{j-2}(r,n,m,k)] + \frac{j}{2\alpha\gamma_r} E[X^j(r,n,m,k)] \tag{11}$$

*if and only if*

$$\bar{F}(x) = \frac{1}{2\alpha}[\beta x^{-1} + x] f(x).$$

**Proof:** The necessary part follows immediately from (7). On the other hand if the recurrence relation in (11) is satisfied, then

$$j\beta E[X^{j-2}(r,n,m,k)] + j E[X^j(r,n,m,k)] = 2\alpha\gamma_r E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)]$$

$$j\beta E[X^{j-2}(r,n,m,k)] + j E[X^j(r,n,m,k)] = 2\alpha\gamma_r \left[ \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j [\bar{F}(x)]^{\gamma_r - 1} g_m^{r-1}[F(x)] f(x) dx \right) - \left( \frac{(r-1)C_{r-1}}{\gamma_r (r-1)!}\int_0^\infty x^j [\bar{F}(x)]^{\gamma_r + m} g_m^{r-2}[F(x)] f(x) dx \right]$$

$$= 2\alpha\gamma_r \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j [\bar{F}(x)]^{\gamma_r} g_m^{r-2}[F(x)] f(x) \left[ \frac{g_m[F(x)]}{[\bar{F}(x)]} - \frac{(r-1)[\bar{F}(x)]^m}{\gamma_r} \right] dx. \tag{12}$$

Let

$$h(x) = -\frac{[\bar{F}(x)]^{\gamma_r} g_m^{r-1}[F(x)]}{\gamma_r}. \tag{13}$$

Differentiating both sides of (13), we get

$$h'(x) = [\bar{F}(x)]^{\gamma_r} g_m^{r-2}[F(x)] f(x) \left\{ \frac{g_m[F(x)]}{[\bar{F}(x)]} - \frac{(r-1)[\bar{F}(x)]^m}{\gamma_r} \right\}.$$

Thus

$$j\beta E[X^{j-2}(r,n,m,k)] + j E[X^j(r,n,m,k)] = \frac{2\alpha\gamma_r C_{r-1}}{(r-1)!}\int_0^\infty x^j h'(x) dx. \tag{14}$$

Integrating RHS in (14) by parts and using the value of $h(x)$ from (13),

$$\beta E[X^{j-2}(r,n,m,k)] + E[X^j(r,n,m,k)] = \frac{2\alpha\gamma_r C_{r-1}}{(r-1)!}\int_0^\infty x^{j-1} [\bar{F}(x)]^{\gamma_r} g_m^{r-1}[F(x)] dx$$

$$\frac{\beta C_{r-1}}{(r-1)!}\int_0^\infty x^{j-2} [\bar{F}(x)]^{\gamma_r - 1} g_m^{r-1}[F(x)] f(x) dx + \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j [\bar{F}(x)]^{\gamma_r - 1} g_m^{r-1}[F(x)] f(x) dx$$

$$= \frac{2\alpha\gamma_r C_{r-1}}{(r-1)!}\int_0^\infty x^{j-1} [\bar{F}(x)]^{\gamma_r} g_m^{r-1}[F(x)] dx.$$

Which reduces to

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j-1} [\bar{F}(x)]^{\gamma_r - 1} g_m^{r-1}[F(x)] f(x) \left[ \beta x^{-1} + x - 2\alpha \frac{[\bar{F}(x)]}{f(x)} \right] dx = 0. \tag{15}$$

Now applying a generalization of Müntz-Szász Theorem to (15)

$$\left[\beta x^{-1} + x - 2\alpha \frac{[\bar{F}(x)]}{f(x)}\right] = 0.$$

Which gives

$$\bar{F}(x) = \frac{1}{2\alpha}[\beta x^{-1} + x] f(x).$$

This proves that $f(x)$ has the form as in (5).

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