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Measure of Rotatability for Second Order Response Surface Designs Using a Pair of Balanced Incomplete Block Designs

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Abstract

In this paper, measure of rotatability for second order response surface designs using a pair of balanced incomplete block designs without any additional set of designs points is suggested which enables us to assess the degree of rotatability for a given response surface design. Further, a comparative study on different methods of measure of rotatability using central composite designs, balanced incomplete block designs, pairwise balanced designs, symmetrical unequal block arrangements with two unequal block sizes and partially balanced incomplete block designs are also examined.

Keywords: Measure of rotatability, response surface designs, second order rotatable designs.

1. Introduction

Response surface methodology is a statistical technique which is very useful in design and analysis of scientific experiments. In many experimental situations the experimenter is concerned with explaining certain aspects of a functional relationship $Y = f(x_1, x_2, ..., x_v) + e$, where Y is the response, $x_1, x_2, ..., x_v$ are the levels of v-quantitative variables or factors and e is the random error. Response surface methods are useful where several independent variables influence a dependent variable. The independent variables are assumed to be continuous and controlled by the experimenter. The response is assumed to be a random variable. For example, if a chemical engineer wishes to find the temperature (x_1) and pressure (x_2) that maximizes the yield (response) of his process, the observed response Y may be written as a function of the levels of the temperature (x_1) and pressure (x_2) as $Y = f(x_1, x_2) + e$.

The concept of rotatability, which is very important in response surface designs, was proposed by Box and Hunter (1957). Das and Narasimham (1962) constructed rotatable designs through balanced incomplete block designs (BIBD). Narasimham et al. (1983) constructed second order rotatable designs through a pair of BIBDs. Khuri (1988) introduced a measure of rotatability for response surface designs. Draper and Pukelsheim (1990) studied another look at rotatability. Myers et al. (2016) studied on response surface methodology. Specifically, Park et al. (1993) introduced a new measure of rotatability for second order response surface designs and studied measure of rotatability for second order response surface designs using 3^k factorial and central composite designs to assess the degree of rotatability. Further, Victorbabu and Surekha (2013, 2014, 2015) studied measure of rotatability for second order response surface designs using incomplete block designs with unequal block sizes like pairwise balanced designs (PBD) and symmetrical unequal block arrangements (SUBA) with two unequal block sizes, partially balanced incomplete block designs (PBIBD) and BIBD respectively. In this paper, measure of rotatability for second order response surface designs using a pair of balanced incomplete block designs without any additional set of designs points is suggested which enables us to assess the degree of rotatability for a given response surface design.

2. Conditions for Second Order Rotatable Designs

Suppose we want to use the second order response surface design $D = ((x_{iu}))$ to fit the surface,

$$Y_u = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum_{i<j}\sum b_{ij} x_{iu} x_{ju} + e_u \tag{1}$$

where x_{iu} denotes the level of the i^{th} factor (i =1,2,…,v) in the u^{th} run (u =1,2,…,N) of the experiment and e_u's are uncorrelated random errors with mean zero and variance σ^2. Then D is said to be second order rotatable design (SORD) if the variance of the estimate of Y_u $(x_1, x_2, ..., x_v)$ with respect to each of independent variables (x_i) is only a function of the distance ($d^2 = \sum_{i=1}^{v} x_i^2$) of the point $(x_1, x_2, ..., x_v)$ from the origin (center) of the design. Such a spherical variance function for estimation of responses in the second order response surface is achieved if the design points satisfy the following conditions (Box and Hunter, 1957; Das and Narasimham, 1962).

$$\sum x_{iu} = 0,\ \sum x_{iu}x_{ju} = 0,\ \sum x_{iu}x_{ju}^2 = 0,\ \sum x_{iu}x_{ju}x_{ku} = 0,\ \sum x_{iu}^3 = 0,\ \sum x_{iu}x_{ju}^3 = 0,$$
$$\sum x_{iu}x_{ju}x_{ku}^2 = 0,\ \sum x_{iu}x_{ju}x_{ku}x_{lu} = 0;\ \text{for } i \neq j \neq k \neq l; \tag{2}$$

$$\text{(i) } \sum x_{iu}^2 = \text{constant} = N\lambda_2;\ \text{(ii) } \sum x_{iu}^4 = \text{constant} = cN\lambda_4;\ \text{for all } i \tag{3}$$

$$\sum x_{iu}^2 x_{ju}^2 = \text{constant} = N\lambda_4;\ \text{for } i \neq j \tag{4}$$

$$\frac{\lambda_4}{\lambda_2^2} > \frac{v}{(c+v-1)} \tag{5}$$

$$\sum x_{iu}^4 = c\sum x_{iu}^2 x_{ju}^2 \tag{6}$$

where c, λ_2 and λ_4 are constants.

The variances and covariances of the estimated parameters are

$$V(\hat{b}_0) = \frac{\lambda_4(c+v-1)\sigma^2}{N\left[\lambda_4(c+v-1) - v\lambda_2^2\right]},$$

$$V(\hat{b}_i) = \frac{\sigma^2}{N\lambda_2},$$

$$V(\hat{b}_{ij}) = \frac{\sigma^2}{N\lambda_4},$$

$$V(\hat{b}_{ii}) = \frac{\sigma^2}{(c-1)N\lambda_4}\left[\frac{\lambda_4(c+v-2) - (v-1)\lambda_2^2}{\lambda_4(c+v-1) - v\lambda_2^2}\right],$$

$$Cov(\hat{b}_0,\hat{b}_{ii}) = \frac{-\lambda_2\sigma^2}{N[\lambda_4(c+v-1)-v\lambda_2^2]},$$

$$Cov(\hat{b}_{ii},\hat{b}_{jj}) = \frac{(\lambda_2^2-\lambda_4)\sigma^2}{(c-1)N\lambda_4[\lambda_4(c+v-1)-v\lambda_2^2]} \text{ and other covariances vanish.} \quad (7)$$

The variance of the estimated response at the point $(x_{10}, x_{20}, \ldots, x_{v0})$ is

$$V(\hat{y}_0)=V(\hat{b}_0)+\left[V(\hat{b}_i)+2\operatorname{cov}(\hat{b}_o,\hat{b}_{ii})\right]d^2+V(\hat{b}_{ii})d^4+\sum x_{i0}^2x_{j0}^2\left[V(\hat{b}_{ij})+2\operatorname{cov}(\hat{b}_{ii},\hat{b}_{jj})-2V(\hat{b}_{ii})\right]. \quad (8)$$

The coefficient of $\sum x_{i0}^2 x_{j0}^2$ in the above equation (8) is simplified to $(c-3)\sigma^2/(c-1)N\lambda_4$. A second order response surface design D is said to be a SORD, if in this design c=3 and all the other conditions (2) to (7) hold.

3. Conditions of Measure of Rotatability for Second Order Response Surface Designs

Following Box and Hunter (1957), Das and Narasimham (1962), Park et al (1993), conditions (2) to (6) and (7) give the necessary and sufficient conditions for a measure of rotatability for any general second order response surface designs. Further we have,

$V(b_i)$ are equal for i,

$V(b_{ii})$ are equal for i,

$V(b_{ij})$ are equal for i, j, where $i \neq j$,

$$Cov(b_i,b_{ii}) = Cov(b_i,b_{ij}) = Cov(b_{ii},b_{ij}) = Cov(b_{ij},b_{il}) = 0 \text{ for all } i \neq j, j \neq l, l \neq i. \quad (9)$$

Park et al. (1993) suggested that if the conditions in (2) to (6) together with (7) and (9) are met, then the following measure $(P_v(D))$ given below can be used to assess the degree of rotatability for any general second order response surface design (cf. Park et al., 1993, page 661).

$$P_v(D) = \frac{1}{1+R_v(D)}. \quad (10)$$

where

$$R_V(D)=\left[\frac{N}{\sigma^2}\right]^2 \frac{6v\left[V(\hat{b}_{ij})+2\operatorname{cov}(\hat{b}_{ii},\hat{b}_{jj})-2V(\hat{b}_{ii})\right]^2(v-1)}{\lambda_4^2(v+2)^2(v+4)(v+6)(v+8)g^8}, \quad (11)$$

and g is the scaling factor.

On simplification of $V(\hat{b}_{ij})+2\operatorname{cov}(\hat{b}_{ii},\hat{b}_{jj})-2V(\hat{b}_{ii})$ becomes $\dfrac{(c-3)\sigma^2}{(c-1)N\lambda_4}$.

Thus, $$R_V(D) = \left[\frac{c-3}{c-1}\right]^2 \frac{6v(v-1)}{\lambda_4^2(v+2)^2(v+4)(v+6)(v+8)g^8}. \quad (12)$$

4. Second Order Rotatable Designs Using a Pair of BIBDs

The method of construction of SORD using two suitably chosen BIBDs, without any additional set of points can be obtained as follows (c.f. Narasimham et al. 1983).

Result: If $D_1 = (v, b_1, r_1, k_1, \lambda_1)$ and $D_2 = (v, b_2, r_2, k_2, \lambda_2)$ are two BIBDs in 'v' treatments with

$r_1 <$ or $> 3\lambda_1$ and $r_2 >$ or $< 3\lambda_2$ respectively and using "multiplication" in Das and Narasimham (1962) sense, then the design points, $[1-(v,b_1,r_1,k_1,\lambda_1)]\times 2^{t(k_1)} \cup [a-(v,b_2,r_2,k_2,\lambda_2)]\times 2^{t(k_2)}$ give a v-dimensional SORD in $N = b_1 \times 2^{t(k_1)} + b_2 \times 2^{t(k_2)}$ points with $a^4 = -\dfrac{(r_1-3\lambda_1)}{(r_2-3\lambda_2)} 2^{t(k_1)-t(k_2)}$.

5. Measure of Rotatability for Second Order Response Surface Designs Using a Pair of Balanced Incomplete Block Designs

In this section new measure of rotatability for second order response surface designs using a pair of BIBDs without any additional set of points is suggested.

Theorem: *The design points,* $[1-(v,b_1,r_1,k_1,\lambda_1)]\times 2^{t(k_1)} \cup [a-(v,b_2,r_2,k_2,\lambda_2)]\times 2^{t(k_2)}$ *will give a v-dimensional measure of second order response surface design using a pair of BIBDs in* $N = b_1 \times 2^{t(k_1)} + b_2 \times 2^{t(k_2)}$ *design points without any additional set of points with 'a' pre-fixed and* $c = \dfrac{r_1 \times 2^{t(k_1)} + r_2 \times 2^{t(k_2)} a^4}{\lambda_1 \times 2^{t(k_1)} + \lambda_2 \times 2^{t(k_2)} a^4}$.

Proof: For the design points generated from a pair of BIBDs, the simple symmetric conditions (2) are true. Further conditions (3) and (4) are true as follows.

$$\sum x_{iu}^2 = r_1 \times 2^{t(k_1)} + r_2 \times 2^{t(k_2)} a^2 = N\lambda_2 \tag{13}$$

$$\sum x_{iu}^4 = r_1 \times 2^{t(k_1)} + r_2 \times 2^{t(k_2)} a^4 = cN\lambda_4 \tag{14}$$

$$\sum x_{iu}^2 x_{ju}^2 = \lambda_1 \times 2^{t(k_1)} + \lambda_2 \times 2^{t(k_2)} a^4 = N\lambda_4 \tag{15}$$

From (14) and (15) we get $c = \dfrac{r_1 \times 2^{t(k_1)} + r_2 \times 2^{t(k_2)} a^4}{\lambda_1 \times 2^{t(k_1)} + \lambda_2 \times 2^{t(k_2)} a^4}$.

The measure of rotatability values for second order response surface design using a pair of BIBDs can be obtained as follows: $P_v(D) = \dfrac{1}{1+R_v(D)}$.

where $R_V(D) = \left[\dfrac{c-3}{c-1}\right]^2 \dfrac{6v(v-1)}{\lambda_4^2 (v+2)^2 (v+4)(v+6)(v+8) g^8}$

and the scaling factor

$$g = \begin{cases} \dfrac{1}{a}, \text{if } a \le \sqrt{2^{t(k_1)-t(k_2)}\left(\dfrac{b_1-r_1}{r_2}\right) + \dfrac{b_2}{r_2}} \\[2ex] \dfrac{1}{\sqrt{2^{t(k_1)-t(k_2)}\left(\dfrac{b_1-r_1}{r_2}\right) + \dfrac{b_2}{r_2}}}, \text{otherwise} \end{cases}$$

Corollary (i):

Taking $D_1 = (v = v, b_1 = 1, r_1 = 1, k_1 = 1, \lambda_1 = 1)$ and $D_2 = (v = v, b_2 = v, r_2 = 1, k_2 = 1, \lambda_2 = 1)$, we get Park et al.(1993) measure of rotatability for second order response surface designs using central composite designs as a particular case.

Corollary (ii):

Taking $D_1 = (v = v, b_1 = b, r_1 = r, k_1 = k, \lambda_1 = \lambda)$ and $D_2 = (v = v, b_2 = v, r_2 = 1, k_2 = 1, \lambda_2 = 0)$, we get Victorbabu and Surekha (2015) measure of rotatability for second order response surface designs using BIBD as a particular case.

Example: We illustrate Theorem with measure of rotatability for second order response surface design using a pair of BIBDs with parameters $D_1 = (v = 5, b_1 = 5, r_1 = 4, k_1 = 4, \lambda_1 = 3)$ and $D_2 = (v = 5, b_2 = 10, r_2 = 4, k_2 = 2, \lambda_2 = 1)$. For the above pair of BIBDs the design points $[1-(5,5,4,4,3)]2^{t(4)} \cup [a-(5,10,4,2,1)]2^{t(2)}$ will give a measure of rotatability for second order response surface design in N=120 design points for five factors.

From (13), (14) and (15) we have,

$$\sum x_{iu}^2 = 64 + 16a^2 = N\lambda_2 \tag{16}$$

$$\sum x_{iu}^4 = 64 + 16a^4 = cN\lambda_4 \tag{17}$$

$$\sum x_{iu}^2 x_{ju}^2 = 48 + 4a^4 = N\lambda_4 \tag{18}$$

From (17) and (18), we can obtain the rotatability value by taking c = 3. Hence, we get a SORD with a = 2.1147 and c = 3. Instead of taking a = 2.1147, suppose we take a = 1.6, we get c = 2.275267334. The scaling factor g=0.625, $R_v(D) = 0.069009013$ and $P_v(D) = 0.9354$.

Table 1 gives the values of measure of rotatability for second order response surface designs using a pair of BIBDs for $5 \le v \le 15$ (v: number of factors).

6. Conclusions

The concept of rotatability is important criteria in response surface designs. If the rotatability in response surface designs is unachievable, it is good to achieve nearly rotatability in response surface designs. In such cases we study the sensitivity of the spherical variance function for the disturbance that occur due to the modifications in these values of “a” and “c”. It is further noted that the range of a, $c \ge 1$, range of g is $0 \le g \le 1$ and range of $R_v(D)$ is $R_v(D) \ge 0$. It can be verify that $P_v(D)$ is one if and only if the design is rotatable and it is smaller than one for a non-rotatable designs.

In this paper, measure of rotatability for second order response surface designs using a pair of BIBDs without taking any additional set of design points has been proposed which enables us to assess the degree of rotatability for a given second order response surface designs. The comparison with the previous studies like measure of rotatability for second order response surface designs using central composite designs, BIBD, PBD, SUBA with two unequal block sizes, pair of PBIBD are also presented in this paper in Table 2 for ready reference.

Table 1 Values of measure of rotatability for second order response surface designs using a pair of BIBDs

a	c	g	$R_v(D)$	$P_v(D)$
D_1=(5,5,4,4,3); D_2=(5,10,4,2,1), N=120, a^* = 2.1147				
1.0	1.538461538	1.0000	0.074657518	0.93050
1.1	1.623303930	0.909090909	0.098787343	0.91010
1.2	1.726239200	0.833333333	0.114368144	0.89740
1.3	1.846002652	0.769230769	0.117774831	0.89460
1.6	2.275267334	0.6250	0.069009013	0.93540
1.9	2.721641412	0.534522483	0.010721377	0.98940
2.1147	2.999949725	0.534522483	$1.586101656\times10^{-10}$	0.99999
2.2	3.096698433	0.534522483	4.355658361×10^{-4}	0.99960
2.5	3.373317013	0.534522483	2.438715945×10^{-3}	0.99760
2.8	3.564421988	0.534522483	2.306652167×10^{-3}	0.99770
3.1	3.693345893	0.534522483	1.563992085×10^{-3}	0.99840
3.4	9.313345272	0.534522483	0.042416149	0.95930
3.7	9.498535976	0.534522483	0.023126002	0.97740
4.0	9.626865672	0.534522483	0.012993756	0.98720
D_1=(6,6,5,5,4); D_2=(6,15,5,2,1), N=156, a^* = 1.9343				
1.0	1.470588235	1.0000	0.063283466	0.9405
1.1	1.564380644	0.909090909	0.115796628	0.8962
1.2	1.680240793	0.833333333	0.126164496	0.8880
1.3	1.818005844	0.769230769	0.121975717	0.8913
1.6	2.339670829	0.6250	0.052235451	0.9504
1.9	2.933322081	0.526315789	3.381954099×10^{-4}	0.9997
1.9343	2.999929707	0.516982887	$6.849925222\times10^{-10}$	0.99999
2.2	3.478146179	0.512989176	0.012716212	0.9874
2.5	3.910329171	0.512989176	0.017133563	0.9832
2.8	4.022546266	0.512989176	0.012771593	0.9874
3.1	4.446249773	0.512989176	7.964624232×10^{-3}	0.9921
3.4	4.599020541	0.512989176	4.680885688×10^{-3}	0.9953
3.7	4.705038097	0.512989176	2.717434104×10^{-3}	0.9973
4.0	4.779411765	0.512989176	1.590792504×10^{-3}	0.9984

Table 1 (Continued)

a	c	g	$R_v(D)$	$P_v(D)$
D_1=(7,7,4,4,2); D_2=(7,21,6,2,1), N=196, a^* = 1.2779				
1.0	2.444444444	1.0000	6.359876410×10^{-3}	0.9937
1.1	2.618801576	0.909090909	4.621449256×10^{-3}	0.9954
1.2	2.823379924	0.833333333	1.384480221×10^{-3}	0.9986
1.2779	3.000032378	0.769230769	$1.692641151\times10^{-10}$	0.99999
1.3	3.052348449	0.6250	1.568140547×10^{-4}	0.9998
1.6	3.801231310	0.526315789	0.057771982	0.9454
1.9	4.478516173	0.454545454	0.241550546	0.8054
2.2	4.981721908	0.426401432	0.479332143	0.6760
2.5	5.320053121	0.426401432	0.414952934	0.7067
2.8	5.539340335	0.426401432	0.206655940	0.8287
3.1	5.681122767	0.426401432	0.103503393	0.9060
3.4	8.270278193	0.426401432	0.053635848	0.9491
3.7	5.83624686	0.426401432	0.028699980	0.9721
4.0	5.878787879	0.426401432	0.015919108	0.9843
D_1=(8,14,7,4,3); D_2=(8,28,7,2,1), N=336, a^* = 1.1892				
1.0	2.692307692	1.0000	1.725267755×10^{-3}	0.9983
1.1	2.840791438	0.909090909	7.801517587×10^{-4}	0.9992
1.1892	2.999986325	0.840901446	$7.012886969\times10^{-13}$	1.0000
1.2	3.020918599	0.833333333	2.051519789×10^{-5}	0.99997
1.3	3.230504641	0.769230769	3.425679372×10^{-3}	0.9966
1.6	3.981717834	0.6250	0.119292284	0.8934
1.9	4.762872472	0.526315789	0.524693419	0.6559
2.2	5.419222257	0.454545454	1.155780742	0.4639
2.5	5.903304774	0.4000	1.809639167	0.3559
2.8	6.237738479	0.357142857	2.359177888	0.2977
3.1	6.463355313	0.353553390	1.333218565	0.4286
3.4	6.615473352	0.353553390	0.706094699	0.5861
3.7	6.719180146	0.353553390	0.374704302	0.7274
4.0	6.791044776	0.353553390	0.215557395	0.8227

Table 1 (Continued)

a	c	g	$R_v(D)$	$P_v(D)$
D_1=(10,18,9,45,4); D_2=(10,45,9,2,1), N=468, a^* = 1.1892				
1.0	2.647058824	1.0000	2.022884443×10^{-3}	0.9980
1.1	2.815885159	0.909090909	9.198825456×10^{-4}	0.9991
1.1892	2.999984045	0.840901446	$1.000275266\times10^{-11}$	1.0000
1.2	3.024433428	0.833333333	2.441208768×10^{-5}	0.99998
1.3	3.272410520	0.769230769	4.197587101×10^{-3}	0.9958
1.6	4.211407491	0.6250	0.152967887	0.8673
1.9	5.279979747	0.526315789	0.728002599	0.5787
2.2	6.260663122	0.454545454	1.726784626	0.3667
2.5	7.038592509	0.4000	2.866020614	0.2587
2.8	7.605832783	0.357142857	3.896638544	0.2042
3.1	8.003249591	0.333333333	3.631474242	0.2159
3.4	8.278236974	0.333333333	1.970487213	0.3366
3.7	8.469068574	0.333333333	1.082369891	0.4802
4.0	8.602941176	0.333333333	0.613173761	0.6199
D_1=(12,22,11,6,5); D_2=(12,44,11,3,2), N=1056, a^* = 1.3375				
1.0	2.5000	1.0000	2.820294864×10^{-3}	0.9972
1.1	2.621448696	0.909090909	2.716684011×10^{-3}	0.9973
1.2	2.766763848	0.833333333	1.570889486×10^{-3}	0.9984
1.3	2.933125131	0.769230769	1.804965010×10^{-4}	0.9998
1.3375	3.000035146	0.747663551	5.54627392×10^{-11}	0.99999
1.6	3.506475933	0.6250	0.019556899	0.9808
1.9	4.067217058	0.526315789	0.118443908	0.8941
2.2	4.512732756	0.454545454	0.278350319	0.7823
2.5	4.827388535	0.4000	0.441582554	0.6937
2.8	5.038239377	0.357142857	0.575878568	0.6346
3.1	5.177583557	0.322580645	0.324638705	0.7549
3.4	5.270248744	0.747663551	0.171483719	0.8536
3.7	5.332840381	0.747663551	0.093100921	0.9148
4.0	5.375939850	0.747663551	0.052150607	0.9504

Table 1 (Continued)

a	c	g	$R_v(D)$	$P_v(D)$
D_1=(13,26,12,6,5); D_2=(13,13,4,4,1), N=1040, a^* = 1.5651				
1.0	2.545454545	1.0000	1.852486396×10^{-3}	0.9982
1.1	2.714218182	0.909090909	1.275825672×10^{-3}	0.9987
1.2	2.674794593	0.833333333	2.881821297×10^{-3}	0.9971
1.3	2.755454609	0.769230769	2.481797393×10^{-3}	0.9975
1.5651	3.000014779	0.638936809	$1.989783653\times10^{-11}$	1.0000
1.6	3.033442876	0.6250	1.076968593×10^{-4}	0.9999
1.9	3.305317361	0.526315789	0.014551290	0.9857
2.2	3.521324972	0.454545454	0.054410242	0.9484
2.5	3.673885350	0.4000	0.104329088	0.9055
2.8	3.776116061	0.357142857	0.149811962	0.8697
3.1	3.843676876	0.322580645	0.185690660	0.8434
3.4	3.888605452	0.312347523	0.131196754	0.8840
3.7	3.918952912	0.312347523	0.072737922	0.9322
4.0	3.939849624	0.312347523	0.041313969	0.9603
D_1=(15,15,7,7,3); D_2=(15,35,7,3,1), N=1240, a^* = 1.4142				
1.0	2.5200	1.0000	1.821170686×10^{-3}	0.9982
1.1	2.601650952	0.909090909	2.33405994×10^{-3}	0.9977
1.2	2.704467354	0.833333333	2.170285605×10^{-3}	0.9978
1.3	2.829625299	0.769230769	1.119391993×10^{-3}	0.9989
1.4142	2.999978080	0.707113562	$2.797959125\times10^{-11}$	1.0000
1.6	3.334310850	0.6250	0.010770950	0.9893
1.9	3.975596847	0.526315789	0.151949787	0.8681
2.2	4.638406262	0.454545454	0.564692431	0.6391
2.5	5.223984143	0.4000	1.214036896	0.4517
2.8	5.689531227	0.357142857	1.941830169	0.3399
3.1	6.037404568	0.322580645	2.614400307	0.2767
3.4	6.28949158	0.294117647	3.172479417	0.2397
3.7	6.470239022	0.294117647	3.609767388	0.2169
4.0	6.6000	0.294117647	2.407115170	0.2935

Table 2 Comparison of different methods of measure of rotatability for second order response surface designs at different factors of "v" and different levels of "a"

v =2 factors; a	CCD	BIBD	PBD	SUBA	Pair of PBIBD	Pair of BIBD
	N=9, a^* = 1.4142	-	-	-	-	-
1.0	0.4675	-	-	-	-	-
1.3	0.9801	-	-	-	-	-
1.6	0.9509	-	-	-	-	-
1.9	0.6078	-	-	-	-	-
2.2	0.2509	-	-	-	-	-
2.5	0.0932	-	-	-	-	-
2.8	0.0369	-	-	-	-	-
3.1	0.0159	-	-	-	-	-
3.4	7.4666×10^{-3}	-	-	-	-	-
3.7	3.7438×10^{-3}	-	-	-	-	-
4.0	1.9868×10^{-3}	-	-	-	-	-
v =3 factors; a	CCD	BIBD (3,3,2,2,1)	PBD	SUBA	Pair of PBIBD	Pair of BIBD
	N=15, a^* = 1.6818	N=19, a^* = 1.1892	-	-	-	-
1.0	0.0333	0.9948	-	-	-	-
1.3	0.3425	0.9883	-	-	-	-
1.6	0.9720	0.6367	-	-	-	-
1.9	0.8439	0.1891	-	-	-	-
2.2	0.3652	0.0519	-	-	-	-
2.5	0.1242	0.0402	-	-	-	-
2.8	0.0457	0.0374	-	-	-	-
3.1	0.0189	0.0359	-	-	-	-
3.4	8.5981×10^{-3}	0.0350	-	-	-	-
3.7	4.2345×10^{-3}	0.0344	-	-	-	-
4.0	2.2207×10^{-3}	0.0340	-	-	-	-
v =4 factors; a	CCD	BIBD (4,4,3,3,2)	PBD	SUBA	Pair of PBIBD	Pair of BIBD
	N=25, a^* = 2.0000	N=41, a^* = 1.8612	-	-	-	-
1.0	3.4017×10^{-3}	0.9379	-	-	-	-
1.3	0.0350	0.8344	-	-	-	-
1.6	0.2699	0.8647	-	-	-	-
1.9	0.9367	0.9915	-	-	-	-
2.2	0.7810	0.4342	-	-	-	-
2.5	0.2699	0.1082	-	-	-	-
2.8	0.0869	0.0328	-	-	-	-
3.1	0.0326	0.0251	-	-	-	-
3.4	0.0140	0.0223	-	-	-	-
3.7	6.6466×10^{-3}	0.0208	-	-	-	-
4.0	3.4017×10^{-3}	0.0199	-	-	-	-

Table 2 (Continued)

v =5 factors; a	CCD N=27, a^* = 2.0000	BIBD (5,5,4,4,3) N=91, a^* = 2.5149	PBD -	SUBA -	Pair of PBIBD -	Pair of BIBD (5,5,4,4,3) (5,10,4,2,1) N=120, a^* = 2.1147
1.0	1.3106×10^{-3}	0.8862	-	-	-	0.9305
1.3	0.0138	0.6049	-	-	-	0.8946
1.6	0.1244	0.3919	-	-	-	0.9354
1.9	0.8506	0.3437	-	-	-	0.9894
2.2	0.7461	0.4892	-	-	-	0.9996
2.5	0.2576	0.9959	-	-	-	0.9976
2.8	0.0820	0.2884	-	-	-	0.9977
3.1	0.0307	0.0593	-	-	-	0.9984
3.4	0.0132	0.0184	-	-	-	0.9593
3.7	6.2414×10^{-3}	8.9158×10^{-3}	-	-	-	0.9774
4.0	3.1937×10^{-3}	7.5895×10^{-3}	-	-	-	0.9872

v =6 factors; a	CCD N=45, a^* = 2.3784	BIBD (6,10,5,3,2) N=93, a^* = 1.4142	PBD -	SUBA (6,11,7,3,4,2,9,4) N=189, a^* = 2.5149	Pair of PBIBD (6,4,2,3,1,0) (6,3,1,2,0,1) N=44, a^* = 1.1892	Pair of BIBD (6,6,5,5,4) (6,15,5,2,1) N=156, a^* = 1.9343
1.0	2.4247×10^{-4}	0.9970	-	0.9657	0.9944	0.9405
1.3	2.2334×10^{-3}	0.9973	-	0.8145	0.9873	0.8913
1.6	0.0152	0.9560	-	0.5710	0.6186	0.9504
1.9	0.0991	0.4443	-	0.4320	0.1775	0.9997
2.2	0.6350	0.0968	-	0.5054	0.0482	0.9874
2.5	0.8583	0.0240	-	0.9951	0.0156	0.9832
2.8	0.2581	7.5905×10^{-3}	-	0.2232	9.2810×10^{-3}	0.9874
3.1	0.0766	2.8834×10^{-3}	-	0.0382	8.8889×10^{-3}	0.9921
3.4	0.0284	1.2480×10^{-3}	-	0.0108	8.6561×10^{-3}	0.9953
3.7	0.0124	5.9465×10^{-4}	-	3.9947×10^{-3}	8.5099×10^{-3}	0.9973
4.0	5.9840×10^{-3}	3.0503×10^{-4}	-	1.7532×10^{-3}	8.4140×10^{-3}	0.9984

v =7 factors; a	CCD N=79, a^* = 2.8284	BIBD (7,7,4,4,2) N=127, a^* = 2.0000	PBD -	SUBA -	Pair of PBIBD -	Pair of BIBD (7,7,4,4,2) (7,21,6,2,1) N=196, a^* = 1.2779
1.0	4.7481×10^{-5}	0.9825	-	-	-	0.9937
1.3	4.1104×10^{-4}	0.9170	-	-	-	0.9998
1.6	2.4468×10^{-3}	0.8531	-	-	-	0.9454
1.9	0.0122	0.9610	-	-	-	0.8054
2.2	0.0591	0.6922	-	-	-	0.6760
2.5	0.3162	0.1405	-	-	-	0.7067
2.8	0.9860	0.0325	-	-	-	0.8287
3.1	0.3602	0.0102	-	-	-	0.9060s
3.4	0.0854	3.9505×10^{-3}	-	-	-	0.9491
3.7	0.0289	1.7518×10^{-3}	-	-	-	0.9721
4.0	0.0121	8.5717×10^{-4}	-	-	-	0.9843

Table 2 (Continued)

v =8 factors; a	CCD	BIBD (8,14,7,4,3)	PBD	SUBA	Pair of PBIBD	Pair of BIBD (8,14,7,4,3) (8,28,7,2,1)
	N=81, a^* = 2.8284	N=241, a^* = 2.0000	-	-	-	N =336, a^* = 1.1892
1.0	3.0720×10^{-5}	0.9935	-	-	-	0.9983
1.3	2.6598×10^{-4}	0.9647	-	-	-	0.9966
1.6	1.5844×10^{-3}	0.9249	-	-	-	0.8934
1.9	7.9087×10^{-3}	0.9773	-	-	-	0.6559
2.2	0.0391	0.7631	-	-	-	0.4639
2.5	0.2303	0.1649	-	-	-	0.3559
2.8	0.9862	0.0343	-	-	-	0.2977
3.1	0.3832	9.7734×10^{-3}	-	-	-	0.4286
3.4	0.0934	3.5109×10^{-3}	-	-	-	0.5861
3.7	0.0317	1.4781×10^{-3}	-	-	-	0.7274
4.0	0.0134	6.9682×10^{-4}	-	-	-	0.8227

v =9 factors; a	CCD	BIBD (9,18,8,4,3)	PBD (9,11,5,5,4,3,2)	SUBA (9,15,7,3,5,6,9,3)	Pair of PBIBD	Pair of BIBD (9,18,8,4,3) (9,36,8,2,1)
	N =147, a^* = 3.3636	N =307, a^* = 1.6818	N =195, a^* = 1.6818	N =259, a^* = 2.0000	-	N =432, a^* = 0.9457
1.0	6.6526×10^{-6}	0.9987	0.9969	0.9936	-	0.99999
1.3	5.5886×10^{-5}	0.9948	0.9882	0.9649	-	0.9702
1.6	3.1236×10^{-4}	0.9982	0.9962	0.9252	-	0.7699
1.9	1.3768×10^{-3}	0.9369	0.8886	0.9774	-	0.4927
2.2	5.3555×10^{-3}	0.4115	0.3010	0.7640	-	0.3250
2.5	0.0203	0.0879	0.0633	0.0742	-	0.2424
2.8	0.0839	0.0212	0.0166	0.0344	-	0.2004
3.1	0.3565	6.5093×10^{-3}	5.5471×10^{-3}	9.8210×10^{-3}	-	0.1841
3.4	0.9569	2.4219×10^{-3}	2.1995×10^{-3}	3.5281×10^{-3}	-	0.3002
3.7	0.1663	1.0372×10^{-3}	9.8748×10^{-4}	1.4854×10^{-3}	-	0.4419
4.0	0.0412	4.9315×10^{-4}	4.8611×10^{-4}	7.0024×10^{-4}	-	0.5860

v =10 factors; a	CCD	BIBD (10,18,9,5,4)	PBD (10,11,5,5,4,2)	SUBA (10,11,5,4,5,5,6,2)	Pair of PBIBD (10,8,4,5,2,0) (10,5,1,2,0,1)	Pair of BIBD (10,18,9,45,4) (10,45,9,2,1)
	N =149, a^* = 3.3636	N =309, a^* = 2.2134	N =197, a^* = 1.6818	N =197, a^* = 1.6818	N =148, a^* = 1.6818	N =468, a^* = 1.1892
1.0	4.9196×10^{-6}	0.9932	0.9972	0.9972	0.9881	0.9980
1.3	4.1328×10^{-5}	0.9587	0.9896	0.9896	0.9648	0.9958
1.6	2.3101×10^{-4}	0.8843	0.9966	0.9966	0.9916	0.8673
1.9	1.0185×10^{-3}	0.8639	0.9005	0.9005	0.8379	0.5787
2.2	3.9659×10^{-3}	0.9990	0.3283	0.3283	0.2754	0.3667
2.5	0.0151	0.4544	0.0712	0.0712	0.0703	0.2587
2.8	0.0634	0.0822	0.0188	0.0188	0.0220	0.2042
3.1	0.3475	0.0199	6.2890×10^{-3}	6.2890×10^{-3}	8.2723×10^{-3}	0.2159
3.4	0.9615	6.4369×10^{-3}	2.4947×10^{-3}	2.4947×10^{-3}	3.5645×10^{-3}	0.3366
3.7	0.1835	2.5367×10^{-3}	1.1202×10^{-3}	1.1202×10^{-3}	1.6948×10^{-3}	0.4802
4.0	0.0462	1.1444×10^{-3}	5.5151×10^{-4}	5.5151×10^{-4}	8.6839×10^{-4}	0.6199

Table 2 (Continued)

$v=12$ factors; a	CCD	BIBD (12,22,11,6,5)	PBD	SUBA (12,15,7,4,6,3,12,3)	Pair of PBIBD (12,8,4,6,2,0) (12,6,1,2,0,1)	Pair of BIBD (12,22,11,6,5) (12,44,11,3,2)
	$N=281$, $a^*=4.000$	$N=729$, $a^*=2.8284$	-	$N=505$, $a^*=2.3784$	$N=280$, $a^*=2.0000$	$N=1056$, $a^*=1.3375$
1.0	8.7743×10^{-7}	0.9939	-	0.9956	0.9897	0.9972
1.3	7.2628×10^{-6}	0.9565	-	0.9708	0.9495	0.9998
1.6	3.9381×10^{-5}	0.8359	-	0.9022	0.9081	0.9808
1.9	1.6412×10^{-4}	0.6492	-	0.8334	0.9767	0.8941
2.2	5.7850×10^{-4}	0.5202	-	0.9071	0.7928	0.7823
2.5	1.8465×10^{-3}	0.5690	-	0.8779	0.2178	0.6937
2.8	5.6637×10^{-3}	0.9859	-	0.1982	0.0541	0.6346
3.1	0.0178	0.2622	-	0.0390	0.0173	0.7549
3.4	0.0637	0.0401	-	0.0108	6.7026×10^{-3}	0.8536
3.7	0.2010	0.0102	-	3.8232×10^{-3}	2.9768×10^{-3}	0.9148
4.0	1.0000	3.5093×10^{-3}	-	1.6016×10^{-3}	1.4575×10^{-3}	0.9504
$v=13$ factors; a	CCD	BIBD (13,26,12,6,5)	PBD (13,15,7,7,6,5,3)	SUBA	Pair of PBIBD	Pair of BIBD (13,26,12,6,5) (13,13,4,4,1)
	$N=283$, $a^*=4.000$	$N=859$, $a^*=2.6321$	$N=987$, $a^*=2.8284$	-	-	$N=1040$, $a^*=1.5651$
1.0	7.1843×10^{-7}	0.9970	0.9961	-	-	0.9982
1.3	5.9466×10^{-6}	0.9782	0.9719	-	-	0.9975
1.6	3.2244×10^{-5}	0.9151	0.8874	-	-	0.9999
1.9	1.3438×10^{-4}	0.8098	0.7356	-	-	0.9857
2.2	4.7371×10^{-4}	0.7579	0.6103	-	-	0.9484
2.5	1.5124×10^{-3}	0.9137	0.6447	-	-	0.9055
2.8	4.6421×10^{-3}	0.7130	0.9892	-	-	0.8697
3.1	0.0147	0.1234	0.3041	-	-	0.8434
3.4	0.0528	0.0254	0.0465	-	-	0.8840
3.7	0.2210	7.3736×10^{-3}	0.0113	-	-	0.9322
4.0	1.0000	2.6946×10^{-3}	3.7518×10^{-3}	-	-	0.9603
$v=14$ factors; a	CCD	BIBD	PBD (14,15,7,7,6,3)	SUBA	Pair of PBIBD	Pair of BIBD
	$N=285$, $a^*=4.0000$	-	$N=989$, $a^*=2.8284$	-	-	-
1.0	5.9971×10^{-7}	-	0.9966	-	-	-
1.3	4.9640×10^{-6}	-	0.9752	-	-	-
1.6	2.6916×10^{-5}	-	0.8994	-	-	-
1.9	1.1218×10^{-4}	-	0.7593	-	-	-
2.2	3.9546×10^{-4}	-	0.6398	-	-	-
2.5	1.2627×10^{-3}	-	0.6730	-	-	-
2.8	3.8780×10^{-3}	-	0.9904	-	-	-
3.1	0.0123	-	0.3314	-	-	-
3.4	0.0444	-	0.0524	-	-	-
3.7	0.2255	-	0.0128	-	-	-
4.0	1.0000	-	4.2529×10^{-3}	-	-	-

Table 2 (Continued)

v =15 factors; a	CCD	BIBD (15,15,7,7,3)	PBD	SUBA (15,16,8,5,9,6,10,4)	Pair of PBIBD	Pair of BIBD (15,15,7,7,3) (15,35,7,3,1)
	N =287, a^* = 4.0000	N =991, a^* = 2.8284	-	N =2079, a^* = 4.0000	-	N =1240, a^* = 1.4142
1.0	5.0874×10^{-7}	0.9970	-	0.9923	-	0.9982
1.3	4.2110×10^{-6}	0.9780	-	0.9424	-	0.9989
1.6	2.2834×10^{-5}	0.9098	-	0.7671	-	0.9893
1.9	9.5163×10^{-5}	0.7808	-	0.4796	-	0.8681
2.2	3.3550×10^{-4}	0.6673	-	0.2514	-	0.6391
2.5	1.0714×10^{-3}	0.6991	-	0.1340	-	0.4517
2.8	3.2917×10^{-3}	0.9915	-	0.0826	-	0.3399
3.1	0.0104	0.3588	-	0.0635	-	0.2767
3.4	0.0380	0.0588	-	0.0676	-	0.2397
3.7	0.1981	0.0145	-	0.1319	-	0.2169
4.0	1.0000	4.7993×10^{-3}	-	1.0000	-	0.2935
v =16 factors; a	CCD	BIBD	PBD	SUBA	Pair of PBIBD	Pair of BIBD
	N =289, a^* = 4.0000	-	-	-	-	-
1.0	4.3749×10^{-7}	-	-	-	-	-
1.3	3.6213×10^{-6}	-	-	-	-	-
1.6	1.9636×10^{-5}	-	-	-	-	-
1.9	8.1836×10^{-5}	-	-	-	-	-
2.2	2.8852×10^{-4}	-	-	-	-	-
2.5	9.2151×10^{-4}	-	-	-	-	-
2.8	2.8320×10^{-3}	-	-	-	-	-
3.1	8.9786×10^{-3}	-	-	-	-	-
3.4	0.0328	-	-	-	-	-
3.7	0.1752	-	-	-	-	-
4.0	1.0000	-	-	-	-	-
v =17 factors; a	CCD	BIBD	PBD	SUBA	Pair of PBIBD	Pair of BIBD
	N =291, a^* = 4.0000	-	-	-	-	-
1.0	3.8062×10^{-7}	-	-	-	-	-
1.3	3.1505×10^{-6}	-	-	-	-	-
1.6	1.7083×10^{-5}	-	-	-	-	-
1.9	7.1199×10^{-5}	-	-	-	-	-
2.2	2.5103×10^{-4}	-	-	-	-	-
2.5	8.0182×10^{-4}	-	-	-	-	-
2.8	2.4647×10^{-3}	-	-	-	-	-
3.1	7.8205×10^{-3}	-	-	-	-	-
3.4	0.0287	-	-	-	-	-
3.7	0.1560	-	-	-	-	-
4.0	1.0000	-	-	-	-	-

a^* indicates exact rotatability level/value.

CCD (Park et al. 1993); BIBD (Victorbabu and Surekha, 2015); PBD (Victorbabu and Surekha, 2013); SUBA with two unequal block sizes (Victorbabu and Surekha, 2013); Pair of PBIBD (Victorbabu and Surekha, 2014); Pair of BIBD (Present method).

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