Thailand Statistician
July 2017; 15(2): 128-148
http://statassoc.or.th
Contributed paper

Modeling of Claim Severity through the Mixture of Exponential Distribution and Computation of its Probability of Ultimate Ruin

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Received: 28 July 2016
Accepted: 10 May 2017

Abstract

In this paper we have discussed the infinite time ruin probabilities in continuous time in a compound Poisson process with a constant premium rate for the mixture of exponential claims. Firstly, we have fitted the mixture of two exponential and the mixture of three exponential to a set of claim data and thereafter, have computed the probability of ultimate ruin through a method giving its exact expression and then through a numerical method, namely the method of product integration. The derivation of the exact expression for ultimate ruin probability for the mixture of three and mixture of two exponential is done through the moment generating function of the maximal aggregate loss random variable. Consistencies are observed in the values of ultimate ruin probabilities obtained by both the methods.

Keywords: Classical risk model, maximal aggregate loss, product integration.

1. Introduction

There has always been an emphasis on understanding Statistical modeling of the Insurance scenario. One of the main challenges of an actuary is to depict uncertainty involved in the claim arrival pattern and in the claim severity pattern through some probability models with the objective of risk assessment. Risk assessment leads to a systematic control of the reserves of the company thereby constituting one of the main ingredients of risk management.

The estimates of the individual loss amounts (claim severities) is done under the assumption that for a particular contract at any point in time, there exists a probability distribution governing the loss amount for any loss event occurring at that time. Loss modeling in the domain of insurance deals with probability model building and Statistical techniques for estimating and testing the model parameters which describe two vital components involving uncertainty in the domain of insurance namely the claim arrival pattern and the amount of each claim (claim severities). A good introduction to the subject of fitting distribution to losses is given in Hogg and Klugman (1984).

A mixed exponential model is a widely used model for modeling the claim severity. The fact that a mixed exponential model as a claim severity model leads to mathematical tractability in the computation of some of the actuarial quantities of interest like the probability of ruin, moments of the time to ruin, probability function of the number of claims until ruin etc, has made it one of the most widely used model for modeling the claim severity in Actuarial science. The reference Keatinge (1999) provides justification why a mixture of exponential distribution is an appropriate choice for actuarial modeling. Perhaps, from the point of distribution fitting, it is the mixture of exponential distribution which among all other potential models for modeling the claim severity, provides the optimum balance between the goodness of fit and the smoothening of the data. As stated in Keatinge (1999), a mixture of exponential distribution has a survival function whose derivative changes at a slower rate as the claim size gets larger and larger and approaches zero asymptotically, as claim size approaches infinity. This is a desirable property for it ensures some amount of smoothness while retaining the goodness of fit.

Considering the importance of distribution fitting to claim severities in the domain of insurance and the suitability of the mixture of exponential distribution as a loss model justified both on the basis of its relevance to reality and scope for mathematical tractability in computing related actuarial quantities like probability of ultimate ruin, moments of the time to ruin etc, we present the following objectives of the work stated in this paper

(1) To fit a mixture of two exponential and mixture of three exponential to a set of insurance claim data using the multi parameter Newton-Raphson method and to test for the goodness of fit for these models.

(2) To derive an exact expression for the probability of ultimate ruin for the mixture of two and the mixture of three exponential using the moment generating function of the maximal aggregate loss random variable.

(3) To compute the probability of ultimate ruin for these models using a numerical approach namely the method of product integration.

The first part of the paper deals with fitting of a mixture of two exponential and a mixture of three exponential to a set of claim data using the multi parameter Newton-Raphson method as stated in Keatinge (1999). The next part deals with the computation of the probability of ultimate ruin for these two distributions using two approaches, firstly, using the moment generating function (m.g.f.) of the maximal aggregate loss random variable as outlined in Bowers et al (1998) and then using a numerical algorithm namely the method of product integration. The concluding section deals with results and discussions.

2. Methodology

2.1. Fitting of the mixture of exponential distribution

The survival function of the mixture of exponential distribution is a completely monotone function and hence it satisfies the requirement of a model that would ensure smoothness in the fitting (Keatinge, 1999). A mixture of exponential distribution has a decreasing failure rate which is a desirable characteristic of a loss model.

The probability density function of a mixture of n exponential distribution is given by

$$f(x) = \sum_{i=1}^{n} w_i \lambda_i e^{-\lambda_i x}, \tag{1}$$

where $x > 0$, $\lambda_i > 0$ for $i = 1, 2..., n$ and $w_1, w_2, ..., w_n$ denote a series of non-negative weights satisfying $\sum_{i=1}^{n} w_i = 1.$ In particular, when $n = 2,$ the probability density function of the mixture of two exponential is given by

$$f(x) = w_1 \lambda_1 e^{-\lambda_1 x} + w_2 \lambda_2 e^{-\lambda_2 x}, \tag{2}$$

where $x>0$, $\lambda_i, w_i > 0$ for $i = 1,2$ and $\sum_{i=1}^{2} w_i = 1$ and when $n = 3,$ the probability density function of the mixture of three exponential is given by

$$f(x) = w_1\lambda_1 e^{-\lambda_1 x} + w_2\lambda_2 e^{-\lambda_2 x} + w_3 e^{-\lambda_3 x}, \tag{3}$$

where $x>0,$ $\lambda_i, w_i > 0$ for $i = 1,2,3$ and $\sum_{i=1}^{3} w_i = 1.$

In fitting the mixture of two exponential and mixture of three exponential to our claim data, we have used the method of maximum likelihood estimation which in these cases were implemented through the multi parameter Newton-Raphson method, a brief introduction to which is given in the appendix. Maximum likelihood estimation technique has been our choice because of many desirable Statistical characteristics, it possesses. The parameters for both of our models were estimated through an algorithm extracted in some sense from Keatinge (1999). It needs mentioning that this algorithm is almost the maximum likelihood estimation technique implemented via the multi parameter Newton-Raphson method with slight modifications. Another important point is that this algorithm even specializes in identifying the number of exponentials to be fitted through a set of conditions called the Karush Kuhn Tucker (KKT) conditions. Although, this was a desirable feature of the algorithm, yet we have not gone into the complexity of verifying the KKT conditions at each step of the algorithm and have just concentrated on implementing the algorithm for $K = 2$ and $K = 3,$ where "K" is the number of exponentials to be combined to get the desired mixture of exponential. For further details on the mixture of exponential distribution, refer to Jewel (1982) and Lindsay (1981).

We first concentrate on fitting the mixture of two exponential and then the mixture of three exponential. We give the general algorithm which will be valid for both mixture of two and mixture of three exponential. Also, it needs mentioning that we have categorized the data into a number of intervals and hence, have used the features of the algorithm to deal with grouped data.

The following is a brief sketch of the algorithm as extracted from Keatinge (1999):

(1) Begin with an initial set of values of w_i's and λ_i's. The closer these values are to the final estimate, faster will be the convergence. Although, there is no hard and fast rule for selecting the initial values, yet one can try with a number of set of initial values and the one maximizing the likelihood (in a rough way through trial and error) can be taken as the set of initial values to start the algorithm.

(2) Implement the Newton's method to find the indicated change in the parameters and call this the Newton step. Each λ_i is a parameter and all but one of the w_i's are parameters. We must set w_1 equal to one minus the sum of the other w_i's.

(3) Adjust the parameters by the amount of the Newton step. If all the λ_i's remain positive and if all the w_i's remain between zero and one and if the log-likelihood function increases, then go to the next iteration. If the values at any particular iteration don't satisfy these conditions, then try a backward Newton step, then half a forward, then half a backward step, then a quarter of a forward step and so on, until the values satisfy all of these conditions.

(4) Carry out the iterations so long the estimates get stabilized i.e. no change observed in their values in the subsequent iterations.

After obtaining the parameter estimates, we have gone for assessing the goodness of fit, firstly through some graphical displays and then through the chi-square goodness of fit test.

In implementing the multi parameter Newton-Raphson, We require the gradient and the hessian matrices for the mixture of exponential distribution for grouped data and we cite them as given below:

For grouped data, the log-likelihood function is given by

$$L(w_1, w_2, \ldots, \lambda_1, \lambda_2, \lambda_3, \ldots) = \{p(x < b_1\}^{a_1} \prod_{k=2}^{g-1} \{p(b_{k-1} < x < b_k)\}^{a_k} \{p(x > b_{g-1})\}^{a_g}$$

$$= a_1 \log(1 - S(b_1)) + \sum_{k=2}^{g-1} a_k \log(S(b_{k-1}) - S(b_k)) + a_g \log(S(b_{g-1}))$$

$$= a_1 \log\left(\sum_{i=1}^{n} w_i \left(1 - e^{-\lambda_i b_i}\right)\right) + \sum_{k=2}^{g-1} a_k \log\left(\sum_{i=1}^{n} w_i \left(e^{-\lambda_i b_{k-1}} - e^{-\lambda_i b_k}\right)\right)$$

$$+ a_g \log\left(\sum_{i=1}^{n} w_i \left(e^{-\lambda_i b_{g-1}}\right)\right),$$

where $S(.)$ is the survival function of the mixture of exponential distribution, g is the number of groups, $a_1, a_2, \ldots, a_g$ are the number of observations in each group, $b_1, b_2, \ldots, b_{g-1}$ are the group boundaries and n is the number of exponentials to be mixed.

The survival function of the mixture of exponential is given by

$$S(x) = \sum_{i=1}^{n} w_i e^{-\lambda_i x}.$$

The derivatives required for constructing the gradient matrix are

$$\frac{\partial \log L}{\partial \lambda_i} = \sum_{k=1}^{g} a_k \left(\frac{\partial \log L}{\partial \lambda_i}\right)_k = \sum_{k=1}^{g} a_k \frac{w_i(-b_{k-1}e^{-\lambda_i b_{k-1}} + b_k e^{-\lambda_i b_k})}{\sum_{j=1}^{n} w_j (e^{-\lambda_j b_{k-1}} - e^{-\lambda_j b_k})}, i = 1, 2, \ldots, n$$

$$\frac{\partial \log L}{\partial w_i} = \sum_{k=1}^{g} a_k \left(\frac{\partial \log L}{\partial w_i}\right)_k = \sum_{k=1}^{g} a_k \frac{\left(e^{-\lambda_i b_{k-1}} - e^{-\lambda_i b_k}\right) - \left(e^{-\lambda_1 b_{k-1}} - e^{-\lambda_1 b_k}\right)}{\sum_{j=1}^{n} w_j \left(e^{-\lambda_j b_{k-1}} - e^{-\lambda_1 b_k}\right)}, i = 1, 2, \ldots, n.$$

The following derivatives are required for constructing the hessian matrix.

$$\frac{\partial^2 \log L}{\partial \lambda_i^2} = \sum_{k=1}^{g} a_k \left[\frac{w_i \left(b_{k-1}^2 e^{-\lambda_i b_{k-1}} - b_k^2 e^{-\lambda_i b_k}\right)}{\sum_{j=1}^{3} w_j \left(e^{-\lambda_j b_{k-1}} - e^{-\lambda_j b_k}\right)} - \left(\left(\frac{\partial \log L}{\partial \lambda_i}\right)_k\right)^2\right], \; i = 1, 2, \ldots, n$$

$$\frac{\partial^2 \log L}{\partial \lambda_i \partial \lambda_l} = \sum_{k=1}^{g} a_k \left[-\left(\frac{\partial \log L}{\partial \lambda_i}\right)_k \left(\frac{\partial \log L}{\partial \lambda_l}\right)_k\right], \; i = 2, 3, \ldots, n; \; l = 2, 3, \ldots, n$$

$$\frac{\partial^2 \log L}{\partial w_i \partial w_l} = \sum_{k=1}^{g} a_k \left[-\left(\frac{\partial \log L}{\partial w_i}\right)_k \left(\frac{\partial \log L}{\partial w_l}\right)_k\right], \; i = 2, 3, \ldots, n; \; l = 2, 3, \ldots, n$$

$$\frac{\partial^2 \log L}{\partial\lambda_1 \partial w_i} = \sum_{k=1}^{g} a_k \left[\frac{\left(-b_{k-1}e^{-\lambda_i b_{k-1}} + b_k e^{-\lambda_i b_k}\right)}{\sum_{j=1}^{n} w_j \left(e^{-\lambda_j b_{k-1}} - e^{-\lambda_j b_k}\right)} - \left(\frac{\partial \log L}{\partial \lambda_1}\right)_k \left(\frac{\partial \log L}{\partial w_i}\right)_k \right], \; i = 2,3,\ldots,n$$

$$\frac{\partial^2 \log L}{\partial\lambda_i \partial w_i} = \sum_{k=1}^{g} a_k \left[\frac{\left(-b_{k-1}e^{-\lambda_1 b_{k-1}} + b_k e^{-\lambda_1 b_k}\right)}{\sum_{j=1}^{n} w_j \left(e^{-\lambda_j b_{k-1}} - e^{-\lambda_j b_k}\right)} - \left(\frac{\partial \log L}{\partial \lambda_i}\right)_k \left(\frac{\partial \log L}{\partial w_i}\right)_k \right], \; i = 2,3,\ldots,n$$

$$\frac{\partial^2 \log L}{\partial\lambda_i \partial w_l} = \sum_{k=1}^{g} a_k \left[-\left(\frac{\partial \log L}{\partial \lambda_i}\right)_k \left(\frac{\partial \log L}{\partial w_l}\right)_k \right], \; i = 2,3,\ldots.,n; \; l = 2,3,\ldots,n; \; i \neq l.$$

2.2. Classical risk model

Let $\{U(t)\}_{t\geq 0}$ denote the surplus process of an insurer as

$$U(t) = u + ct - W(t),$$

where $u \geq 0$ is the initial surplus, c is the rate of premium income per unit time and $\{W(t)\}_{t\geq 0}$ is the aggregate claim process and we have $W(t) = \sum_{i=1}^{M(t)} X_i$ where $\{M(t)\}_{t\geq 0}$ is a homogeneous Poisson process with parameter λ, X_i denotes the amount of the ith claim and $\{X_i\}_{i=1}^{\infty}$ is a sequence of iid random variables with distribution function F such that $F(0) = 0$ and probability density function f. We denote $E(X_1^k)$ by p_k. Also we have $c = (1+\theta)\lambda p_1$, ($p_1$ is the mean of the claim severity distribution) where θ is the security loading factor.

Let T_u denote the time to ruin from initial surplus u so that $T_u = \inf\{t : U(t) < 0\}$ and define $\psi(u) = P\{T_u < \infty\} = 1 - \chi(u)$ and $\psi(u,t) = P\{T_u \leq t\}$. $\psi(u)$ is known as the ultimate ruin probability whereas $\psi(u,t)$ is the finite time ruin probability. For a detailed discussion on the classical risk model and the probability of ruin, see Grandell (1991), Panjer and Willmot (1992), Klugman et al (1998) and Asmussen (2000).

The classical risk model is inbuilt with some assumptions which make it deviate from real life situations. The assumptions like the independence between claim severity distribution and claim number distribution, no effect of interest, taxation and inflation on surplus, independence of the intensity parameter λ with time etc. are not so realistic but still the classical risk model constitutes the basis of many models in insurance mathematics. Probability of ruin is a very important component of the operational risk theory. Apart from issuing warning signals to the company about probable insolvency, it renders other benefits in terms of long range planning for the use of insurer's funds.

2.3. Computing the exact probability of ruin for mixture of exponential distribution

The maximal aggregate loss random variable is defined as

$$L = \max_{t\geq 0} \{w(t) - ct\}, \tag{4}$$

i.e., it is the maximum of the excess of aggregate claims over premium received. Again, since $w(t) - ct = 0$ for $t = 0$, it follows that $L \geq 0$..

To obtain the distribution function (d.f.) of L, we proceed as follows

$$1-\psi(u) = P\{u(t) \geq 0 \text{ for all } t\} \text{ (by definition)}$$
$$= P\{u + ct - w(t) \geq 0 \text{ for all } t\}$$
$$= P\{w(t) - ct \leq u \text{ for all } t\}. \tag{5}$$

But $w(t) - ct \leq u$ for all t is equivalent to $\max_{t\geq 0}\{w(t) - ct\} \leq u$ and therefore, (5) is equivalent to

$$1-\psi(u) = P\{L \leq u\}, u \geq 0, \tag{6}$$

i.e. the complement of the probability of ultimate ruin can be interpreted as the distribution function of L.

Putting $u = 0$ in (6), we have

$$1-\psi(0) = P(L \leq 0) = P(L = 0), \text{ since } L \geq 0.$$

This leads us to conclude that the distribution of L is of mixed type with a point mass of $1-\psi(0)$ at the origin with the remaining probability distributed continuously over positive values of L.
Now,

$$M_L(r) = E\left(e^{rL}\right) = e^{r.0}P(L=0) + \int_0^\infty e^{ur} f_L(u)du, \tag{7}$$

where $f_L(u)$ is the pdf of the random variable $L.$ But from (6), we have the pdf of L as

$$f_L(u) = \frac{d}{du}\left(1-\psi(u)\right) = -\frac{d\psi(u)}{du} = -\psi(u). \tag{8}$$

Using (8) in (7), we have

$$M_L(r) = 1-\psi(0) + \int_0^\infty e^{ur}\left(-\psi'(u)\right)du. \tag{9}$$

Again, we know

$$\psi(0) = \frac{1}{1+\theta}. \tag{10}$$

(Formula (13.5.2) of Bowers et al., 1998)
Hence,

$$M_L(r) = \frac{\theta}{1+\theta} + \int_0^\infty e^{ur}\left(-\psi'(u)\right)du. \tag{11}$$

Following Formula (13.6.8) of Bowers et al. (1998), the moment generating function (m.g.f.) of the maximal aggregate loss random variable L is

$$M_L(r) = \frac{\theta p_1 r}{1+(1+\theta)p_1 r - M_X(r)}, \tag{12}$$

where $M_X(r)$ is the moment generating function of the underlying claim severity distribution.
This can alternatively be put in the form

$$M_L(r) = \frac{\theta}{1+\theta} + \frac{1}{1+\theta}\left\{\frac{\theta\left[M_X(r)-1\right]}{1+\left(1+\theta\right)p_1 r - M_X(r)}\right\}. \tag{13}$$

Comparing (11) and (13), we have

$$\int_0^\infty e^{ur}\left(-\psi'(u)\right)du = \frac{1}{1+\theta}\left\{\frac{\theta\left[M_X(r)-1\right]}{1+\left(1+\theta\right)p_1 r - M_X(r)}\right\}. \tag{14}$$

This formula is used for finding the exact expression for $\psi(u)$ on a family of claim amount distributions. One of such claim severity distributions is the mixture of exponential distribution. For the mixture of exponential distribution whose probability density function is given by (1), its m.g.f. is given by

$$M_X(r) = \sum_{i=1}^{n} w_i \frac{\lambda_i}{\lambda - r_i}.$$

Substituting the expression for $M_X(r)$ in the R.H.S of (14) and recognizing that the R.H.S of the result is a rational function of r, which by applying the method of partial fraction can be written in the following form

$$\int_0^\infty e^{ur} \left(-\psi'(u)\right) du = \sum_{i=1}^{n} \frac{C_i R_i}{R_i - r},$$

where C_i 's $(i = 1, 2, ..., n)$ are some constants and R_i 's $(i = 1, 2, ...n)$ are the roots of the Lundberg's equation $1 + (1+\theta) p_1 r - M_X(r) = 0.$ The only function which satisfies this and for which $\psi(\infty) = 0$ is $\psi(u) = \sum_{i=1}^{n} C_i e^{-R_i u}$, which is the expression for the probability of ruin for mixture of n exponential claim severity distribution.

We shall make use of the above procedure to find the exact expression for the probability of ultimate ruin for the mixture of exponential distribution. In particular, we shall obtain exact expressions for the probability of ultimate ruin when the claim severity distribution is taken as the mixture of two exponential and the mixture of three exponential which have been fitted to our data. We give in details the procedure for mixture of three exponential whereas the case for the mixture of two exponential can be derived in a similar fashion.

a) Exact expression for the probability of ruin in case of mixture of three exponential

As given in (3), the probability density function of the mixture of three exponential distribution is given by

$$f(x) = w_1 \lambda_1 e^{-\lambda_1 x} + w_2 \lambda_2 e^{-\lambda_2 x} + w_3 e^{-\lambda_3 x}, \tag{15}$$

where $x > 0,\ \lambda_i, w_i > 0\ (i = 1, 2, 3)$ and $\sum_{i=1}^{3} w_i = 1.$

And its mean is given by

$$p_1 = \frac{w_1}{\lambda_1} + \frac{w_2}{\lambda_2} + \frac{w_3}{\lambda_3},$$

and its moment generating function is given by

$$M_X(r) = \frac{a_1 r^2 + b_1 r + k_1}{(\lambda_1 - r)(\lambda_2 - r)(\lambda_3 - r)},$$

where $a_1 = w_1\lambda_1 + w_2\lambda_2 + w_3\lambda_3$, $b_1 = -\left\{w_1\lambda_1(\lambda_2 + \lambda_3) + w_2\lambda_2(\lambda_1 + \lambda_3) + w_3\lambda_3(\lambda_1 + \lambda_2)\right\}$ and $k_1 = \lambda_1\lambda_2\lambda_3$.

Also,

$$(\lambda_1 - r)(\lambda_2 - r)(\lambda_3 - r) = a_2 r^3 + b_2 r^2 + c_2 r + d_2,$$

where $a_2 = -1, b_2 = \lambda_1 + \lambda_2 + \lambda_3, c_2 = -(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)$ and $d_2 = \lambda_1\lambda_2\lambda_3$.

Therefore

$$M_X(r) - 1 = \frac{a_3 r^3 + b_3 r^2 + c_3 r + d_2}{(\lambda_1 - r)(\lambda_2 - r)(\lambda_3 - r)}, \tag{16}$$

where $a_3 = -a_2, b_3 = a_1 - b_2, c_3 = b_1 - c_2, d_3 = 0.$

Also, we have

$$1+(1+\theta)p_1 r - M_X(r) = \frac{a_4 r^4 + b_4 r^3 + c_4 r^2 + d_4 r + e_4}{(\lambda_1 - r)(\lambda_2 - r)(\lambda_3 - r)}, \tag{17}$$

where $a_4 = a_2(1+\theta)p_1, b_4 = a_2 + b_2(1+\theta)p_1, c_4 = b_2 + c_2(1+\theta)p_1 - a_1$, $d_4 = c_2 + d_2(1+\theta)p_1 - b_1$ and $e_4 = 0$.

Hence,

$$\frac{1}{1+\theta}\left\{\frac{\theta[M_X(r)-1]}{1+(1+\theta)p_1 r - M_X(r)}\right\} = \frac{\theta}{1+\theta}\left\{\frac{a_3 r^2 + b_3 r + c_3}{a_4 r^3 + b_4 r^2 + c_4 r + d_4}\right\}, \tag{18}$$

implying

$$\frac{1}{1+\theta}\left\{\frac{\theta[M_X(r)-1]}{1+(1+\theta)p_1 r - M_X(r)}\right\} = \frac{C_1 R_1}{R_1 - r} + \frac{C_2 R_2}{R_2 - r} + \frac{C_3 R_3}{R_3 - r}, \tag{19}$$

where R_1, R_2 and R_3 are the roots of the Lundberg equation

$$a_4 r^3 + b_4 r^2 + c_4 r + d_4 = 0.$$

Hence, if R_1, R_2 and R_3 are the roots of the cubic equation $a_4 r^3 + b_4 r^2 + c_4 r + d_4 = 0.$ Then

$$\frac{\theta}{1+\theta}\left\{\frac{a_3 r^2 + b_3 r + c_3}{a_4 r^3 + b_4 r^2 + c_4 r + d_4}\right\} = \frac{\theta}{1+\theta}\left\{\frac{a_3 r^2 + b_3 r + c_3}{k(R_1 - r)(R_2 - r)(R_3 - r)}\right\}, \tag{20}$$

where k is to be determined such that $(R_1 - r)(R_2 - r)(R_3 - r) = a_4 r^3 + b_4 r^2 + c_4 r + d_4$, thereby giving $k = -a_4$. Now,

$$\frac{a_3 r^2 + b_3 r + c_3}{k(R_1 - r)(R_2 - r)(R_3 - r)} = \frac{M_1}{R_1 - r} + \frac{M_2}{R_2 - r} + \frac{M_3}{R_3 - r}, \tag{21}$$

M_i's $(i = 1,2,3)$ are to be determined by the method of partial fractions thereby giving

$M_1 = \dfrac{a_3 R_1^2 + b_3 R_1 + c_3}{k(R_2 - R_1)(R_3 - R_1)}, M_2 = \dfrac{a_3 R_2^2 + b_3 R_2 + c_3}{k(R_1 - R_2)(R_3 - R_2)}$ and $M_3 = \dfrac{a_3 R_3^2 + b_3 R_3 + c_3}{k(R_1 - R_3)(R_2 - R_3)}.$

Hence, restating (19), we have

$$\frac{1}{1+\theta}\left\{\frac{\theta[M_X(r)-1]}{1+(1+\theta)p_1 r - M_X(r)}\right\} = \frac{C_1 R_1}{R_1 - r} + \frac{C_2 R_2}{R_2 - r} + \frac{C_3 R_3}{R_3 - r}, \tag{22}$$

where $C_i = \dfrac{\theta}{1+\theta}\dfrac{M_i}{R_i}$ $, i = 1,2,3.$

Therefore, the probability of ultimate ruin in case of mixture of three exponential is given by

$$\psi(u) = C_1 e^{-R_1 u} + C_2 e^{-R_2 u} + C_3 e^{-R_3 u}. \tag{23}$$

Similarly, the expression for the exact probability of ruin for the mixture of two exponential distribution can be derived as follows.

Here,

$$M_X(r) = \frac{a_1 r + b_1}{a_2 r^2 + b_2 r + c_2},$$

where $a_1 = -(w_1\lambda_1 + w_2\lambda_2), b_1 = \lambda_1\lambda_2,$

$$(\lambda_1 - r)(\lambda_2 - r) = a_2 r^2 + b_2 r + c_2,$$

where $a_2 = 1, b_2 = -(\lambda_1 + \lambda_2), c_2 = \lambda_1 \lambda_2.$

Therefore,

$$M_X(r) - 1 = \frac{a_3 r^2 + b_3 r}{a_2 r^2 + b_2 r + c_2},$$

where $a_3 = -a_2$ and $b_3 = a_1 - b_2$. Again,

$$1 + (1+\theta) p_1 r - M_X(r) = \frac{a_4 r^3 + b_4 r^2 + c_4 r + d_4}{a_2 r^2 + b_2 r + c_2},$$

where $a_4 = a_2(1+\theta) p_1, b_4 = a_2 + (1+\theta) p_1 b_2, c_4 = b_2 + (1+\theta) p_1 c_2 - a_1$ and $d_4 = 0.$

Therefore,

$$\frac{1}{1+\theta}\left\{\frac{\theta[M_X(r) - 1]}{1 + (1+\theta) p_1 r - M_X(r)}\right\} = \frac{C_1 R_1}{R_1 - r} + \frac{C_2 R_2}{R_2 - r},$$

where R_1 and R_2 are the roots of the quadratic equation $a_4 r^2 + b_4 r + c_4 = 0$ and $C_1 = \frac{\theta}{1+\theta}\frac{M_1}{R_1}$, $C_2 = \frac{\theta}{1+\theta}\frac{M_2}{R_2}$, $k = a_4$, $M_1 = \frac{a_3 R_1 + b_3}{k(R_2 - R_1)}$ and $M_2 = \frac{a_3 R_2 + b_3}{k(R_1 - R_2)}$.

Hence, the probability of ultimate ruin in case of mixture of two exponential is given by

$$\psi(u) = C_1 e^{-R_1 u} + C_2 e^{-R_2 u}. \tag{24}$$

2.4. Product integration

Product integration which was traditionally used to numerically solve Volterra integral equation of the second type (Delves and Mohamed 1989) can also be used to compute the ultimate ruin probabilities, especially while dealing with heavy tailed claim severity distributions (Ramsay and Usabel 1997).

For calculating the infinite time ruin probability numerically, one has to solve the following integral equation (Gerber, 1979)

$$\psi(x) = \frac{\lambda}{c}\int_0^x \psi(x-y)\{1 - F(y)\}dy + \frac{\lambda}{c}\int_x^\infty \{1 - F(y)\}dy \tag{25}$$

where $\psi(0) = \frac{\lambda p_1}{c}$, $c = \lambda p_1 (1+\theta)$ and $p = \int_0^\infty \{1 - F(y)\}dy$, which can also be put in the form (of a Volterra integral equation of the second kind)

$$\psi(u) = \frac{1}{1+\theta}\left\{A(u) + \int_0^u K(u,t)\psi(t)dt\right\}, u \ge 0, \tag{26}$$

where $A(u) = \frac{1}{p_1}\int_0^\infty \{1 - F(y)\}dy$ and $K(u,t) = \frac{1 - F(u-t)}{p_1}$, $0 \le t \le u.$

The earlier approach to the numerical computation of $\psi(u)$ was based on the discretization of the underlying claim severity distribution and then computing $\psi(u)$ recursively based on some initial conditions. However, one of the main drawbacks of the recursive schemes is that they are slow and less accurate because the quadrature rule employed in the recursive schemes are usually of the low order. The method of Product integration as justified in Ramsay and Usabel (1997) is relatively faster and more accurate. In this paper, we have adopted the method of product integration to compute numerically the probability of ultimate ruin in case of our fitted mixture of three exponential and the mixture of two exponential. Following is the highlight of the execution procedure of the method of product integration as extracted from Ramsay and Usabel (1997). It needs to be noted that equation (26) gives rise to equation (25) as a special case. Hence the execution process of this equation will lead to the numerical solution of (25) at which we are interested and this has been being executed by the method of product integration.

The Volterra integral equation of the second kind is given by

$$X(s) = Y(s) + \int_a^s K(s,t)X(t)dt, a \le s \le b, \tag{27}$$

where $K(.,.)$ is the kernel (and is known) and $X(.)$ is the unknown function to be determined. If $K(.,.)$ or one of it's low order derivative is badly behaved in one of its arguments, Newton Cotes integration formulae may produce inaccurate results or converge slowly. To deal with such situations, when $K(.,.)$ is badly behaved, Delves and Mohamed (1989) and Linz (1985) recommends the use of product integration.

We first factorize $K(s,t)$ as

$$K(s,t) = P(s,t)\overline{K}(s,t),$$

where $\overline{K}(s,t)$ is smooth and well behaved and can be accurately approximated by a suitable Langrange's Interpolation Polynomial and $P(s,t)$ is badly behaved.

The interval $\left[a,b\right]$ is divided into "n" subintervals $\left\{h_i\right\}$ where $h_i = s_{i+1} - s_i, i = 0,1,2,...,n-1$ and $a = s_0 < s_1 ..... < s_n = b$. A quadrature rule of the form

$$\int_a^{s_i} P(s_i,t)\overline{K}(s,t)X(t)dt = \sum_{j=0}^{i} w_{ij}\overline{K}(s_i,t_j)X(t_j), \tag{28}$$

where $t_i = s_i$ for $i = 0,1,2,...n$ is used to approximate the integral appearing in (27). The weights are determined by ensuring that the rule of equation (28) is exact when $\overline{K}(s,t)X(t)$ is a polynomial in t of degree $\le d$.

Roughly speaking, $\overline{K}(s,t)X(t)$ is approximated by a quadrature rule (a polynomial of certain degree by Taylor's expansion) and then it is termed with $P\left(s_i,t\right)$ to form the integrand of $\int_a^{s_i} P(s_i,t)\overline{K}(s,t)X(t)dt$ and on being evaluated, it would result in a sum of the form $\sum_{j=0}^{i} w_{ij}\overline{K}(s_i,t)X(t_j)$, where w_{ij} are the weights of the terms of this series determined by suitably identifying the coefficients in the quadrature rule.

It is assumed that $\left(d+1\right)$ moments μ_{ij} exist for this is necessary for the application of the method of Product Integration and for each i, the moments μ_{ij} can be calculated as

$$\mu_{ij} = \int_a^{s_i} t^j P\left(s_i,t\right)dt, j = 0,1,...,d. \tag{29}$$

Also, if $y = f(x)$ is a function of x and $y_0 = f(x_0)$ and $y_1 = f\left(x_1\right)$, then by Langrange's Interpolation formula, we have

$$f(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0} \left(i.e., d = 1 \right), \tag{30}$$

and substituting $y_0 = \overline{K}\left(s_i, t_j\right) X(t_j)$ and $y_1 = \overline{K}\left(s_i, t_{j+1}\right) X(t_{j+1}),$ we have

$$\overline{K}(s,t)X(t) \approx \frac{\left(t_{j+1} - t\right)}{h_j} \overline{K}\left(s_i, t_j\right) X(t_j) + \frac{\left(t - t_j\right)}{h_j} \overline{K}\left(s_i, t_{j+1}\right) X(t_{j+1}). \tag{31}$$

Hence,

$$\int_a^{s_i} P(s_i,t)\overline{K}(s,t)X(t) \approx \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} P(s_i,t) \left\{ \frac{\left(t_{j+1} - t\right)}{h_j} \overline{K}\left(s_i, t_j\right) X(t_j) + \frac{\left(t - t_j\right)}{h_j} \overline{K}\left(s_i, t_{j+1}\right) X(t_{j+1}) \right\} dt$$

$$= \sum_{j=0}^{i} w_{ij} \overline{K}\left(s_i, t\right) X\left(t_j\right), \tag{32}$$

where $w_{i0} = \int_{t_0}^{t_1} P(s_i,t) \frac{\left(t_1 - t\right)}{h_0} dt, j = 0$, $w_{ii} = \int_{t_{i-1}}^{t_i} P(s_i,t) \frac{\left(t - t_{i-1}\right)}{h_{i-1}} dt, j = i$ and

$$w_{ij} = \int_{t_j}^{t_{j+1}} P(s_i,t) \frac{\left(t_{j+1} - t\right)}{h_j} dt + \int_{t_{j-1}}^{t_j} P(s_i,t) \frac{\left(t - t_{j-1}\right)}{h_{j-1}} dt, j = 1,2,3,...,i-1.$$

Introducing two new variables

$$v_{ij} = \int_{t_j}^{t_{j+1}} \left(t_{j+1} - t\right) P(s_i,t) dt \text{ and } c_{ij} = \int_{t_j}^{t_{j+1}} P\left(s_i, t\right) dt.$$

Weights can be put in the form

$$w_{i0} = \frac{v_{i0}}{h_0}, \; w_{ij} = \frac{v_{ij}}{h_j} + c_{ij} - \frac{v_{i,j-1}}{h_{j-1}}, \; j = 1,2,....,i-1 \text{ and } w_{ii} = c_{i,i-1} - \frac{v_{i,i-1}}{h_{i-1}}.$$

And finally, the estimate of $X(s)$ is given by $\widehat{X_n}(s_n)$, where $\widehat{X_n}(s_i)$ are obtained recursively by

$$\widehat{X_n}(s_i) = y(s_i) + \sum_{j=0}^{i} w_{ij} \overline{K}(s_i, t_j) \widehat{X_n}(t_j), i = 1,2,...,n,$$

where $\widehat{X_n}(s_0) = y(a).$

For accelerating the convergence, we have used the Richardson's extrapolation technique. (Ramsay 1992b; Ramsay and Usabel 1997).

a) Product integration for the computation of the ultimate probability of ruin for mixture of three exponential distributed claims

We have used product integration to compute the ultimate probability of ruin for the mixture of three exponentially and mixture of two exponentially distributed claims taking an illustrative value of θ as $\theta = 0.3$.

Here,

$$F(t) = 1 - \left(w_1 e^{-\lambda_1 t} + w_2 e^{-\lambda_2 t} + w_3 e^{-\lambda_3 t} \right),$$

and

$$K(u,t) = \frac{1 - F(u-t)}{p_1(1+\theta)} = \frac{w_1 e^{-\lambda_1 (u-t)} + w_2 e^{-\lambda_2 (u-t)} + w_3 e^{-\lambda_3 (u-t)}}{p_1(1+\theta)}.$$

As for this distribution, all the moments μ_{ij} exists for any finite s, product integration can be used. Let

$$P(s,t) = K(s,t)$$

$$\overline{K}(s,t) = \begin{cases} 1, \text{ if } 0 \le t \le s \\ 0, \text{ otherwise.} \end{cases}$$

b) Computation of the weights when the claim severity is mixture of three exponential

The weights are given by

$$v_{ij} = \frac{1}{p_1(1+\theta)}\left\{t_{j+1}M_1 + M_2\right\},$$

where

$$M_1 = \frac{w_1 e^{-\lambda_1 u}}{\lambda_1}\left\{e^{\lambda_1 t_{j+1}} - e^{\lambda_1 t_j}\right\} + \frac{w_2 e^{-\lambda_2 u}}{\lambda_2}\left\{e^{\lambda_2 t_{j+1}} - e^{\lambda_2 t_j}\right\} + \frac{w_3 e^{-\lambda_3 u}}{\lambda_3}\left\{e^{\lambda_3 t_{j+1}} - e^{\lambda_3 t_j}\right\},$$

$$M_2 = \frac{w_1 e^{-\lambda_1 u}}{\lambda_1}\left\{\left(t_{j+1}e^{\lambda_1 t_{j+1}} - t_j e^{\lambda_1 t_j}\right) - \frac{1}{\lambda_1}\left(e^{\lambda_1 t_{j+1}} - e^{\lambda_1 t_j}\right)\right\} + \frac{w_2 e^{-\lambda_2 u}}{\lambda_2}\left\{\left(t_{j+1}e^{\lambda_2 t_{j+1}} - t_j e^{\lambda_2 t_j}\right) - \frac{1}{\lambda_2}\left(e^{\lambda_2 t_{j+1}} - e^{\lambda_2 t_j}\right)\right\}$$
$$+ \frac{w_3 e^{-\lambda_3 u}}{\lambda_3}\left\{\left(t_{j+1}e^{\lambda_3 t_{j+1}} - t_j e^{\lambda_3 t_j}\right) - \frac{1}{\lambda_3}\left(e^{\lambda_3 t_{j+1}} - e^{\lambda_3 t_j}\right)\right\},$$

and

$$c_{ij} = \frac{1}{p_1(1+\theta)}\left\{\frac{w_1 e^{-\lambda_1 u}}{\lambda_1}\left\{e^{\lambda_1 t_{j+1}} - e^{\lambda_1 t_j}\right\} + \frac{w_2 e^{-\lambda_2 u}}{\lambda_2}\left\{e^{\lambda_2 t_{j+1}} - e^{\lambda_2 t_j}\right\} + \frac{w_3 e^{-\lambda_3 u}}{\lambda_3}\left\{e^{\lambda_3 t_{j+1}} - e^{\lambda_3 t_j}\right\}\right\}.$$

Similar expressions can be derived for the weights for the mixture of two exponential. All the computations were done using R software (2013).

3. Results and Discussions

Our data is a set of 160,000 claim amounts spread over a period of 6 months i.e. from April, 2013 to September, 2013 obtained from Bajaj Allianz General Insurance Company, India from its motor insurance portfolio covering all its branches in India. No adjustment was made for inflation for the time horizon is narrow. It needs to be mentioned that the data is utilized more for the illustration of the various methodologies rather than for the extraction of any concrete meaningful conclusion on the operational aspect of the Insurance Company under concern. Since the inter arrival time of claim was difficult to track, an illustrative value of the intensity parameter was taken as $\lambda = 32.427$.

In fitting the mixture of two exponential, the parameter estimates got stabilized at the 1048^{th} iteration whereas in case of the mixture of three exponential, the values got stabilized at the 32^{th} iteration. Table 1 and Table 2 display the parameter estimates for the mixture of two exponential and the mixture of three exponential.

Table 1 Parameter estimates for mixture of two exponential

Parameters	Estimates
λ_1	2.148864e-05
λ_2	2.148712e-05
w_2	9.999962e-01
$w_1 = 1 - w_2$	3.800000e-06

Value of the log-likelihood is -316945.6

Table 2 Parameter estimates for mixture of three exponential

Parameters	Estimates
λ_1	1.066956e-05
λ_2	7.979466e-05
λ_3	1.005759e-05
w_2	6.769671e-01
w_3	2.244830e-01
$w_1 = 1 - w_2 - w_3$	9.854990e-02

Value of the log-likelihood is -256633.6

a) Testing the goodness of fit

We have initially assessed the goodness of fit by some graphical displays. Figure 1 indicates the histogram of the claim data. The histogram indicates that a long and a light tailed distribution might be a suitable model for our claim data, thereby justifying the use of the mixture of two exponential and the mixture of three exponential for our data. Figure 2 indicates the histogram for a set of data simulated from a mixture of three exponential with the values of the parameters as those of the estimated values. It needs mentioning that we have used the inverse transform algorithm to generate random observations from the mixture of three exponential. The transcendental equation for the inverse transform method equating the distribution function of the mixture of three exponential with that of an uniform random variable lying between 0 and 1 was solved using the uniroot function of R. This complexity encountered in generating random observations from the mixture of exponential distribution limited us to drawing just 100 observations from the mixture of three exponential. Hence the histogram from the simulated data and the QQ plot were constructed only for the case of mixture of three exponential skipping the figures in case of mixture of two exponential. Histogram of the simulated data from the mixture of three exponential bears resemblance with the histogram of the observed data indicated in Figure 1. Additionally, the QQ plot for the mixture of three exponential (Figure 3) indicate the potentiality of the mixture of three exponential for being adequate for modeling our claim data.

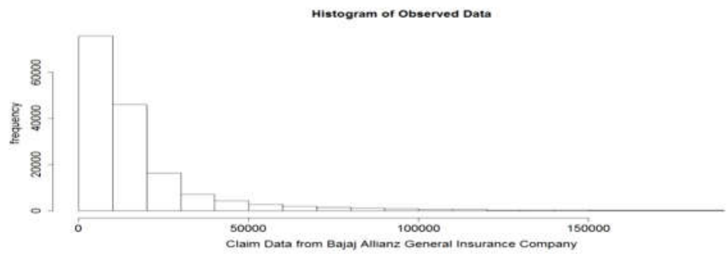

Figure 1 Histogram of the observed claim data on motor insurance

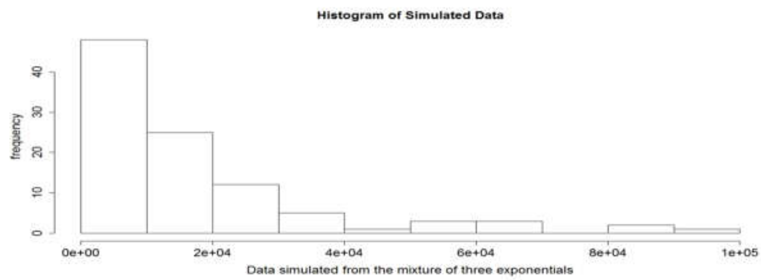

Figure 2 Histogram for a data set simulated from a mixture of three exponential with $\hat{\lambda}_1 = 1.066956e-05$, $\hat{\lambda}_2 = 7.979466e-05$, $\hat{\lambda}_3 = 1.005759e-05$, $\hat{w}_1 = 0.0985499$, $\hat{w}_2 = 6.769671e-01$ and $\hat{w}_3 = 2.24483e-01$

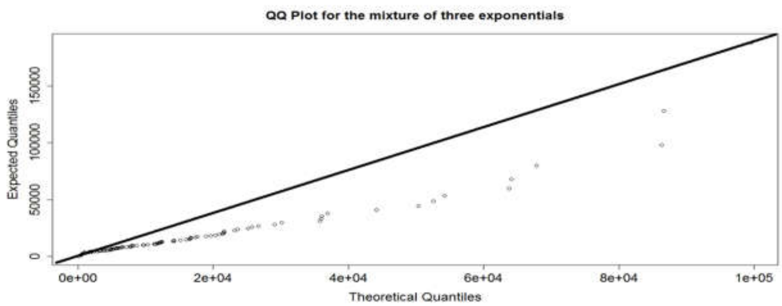

Figure 3 QQ plot between the empirical quantiles estimated from the motor insurance data and the theoretical quantiles for the mixture of three exponential with $\hat{\lambda}_1 = 1.066956e-05$, $\hat{\lambda}_2 = 7.979466e-05$, $\hat{\lambda}_3 = 1.005759e-05$, $\hat{w}_1 = 0.0985499$, $\hat{w}_2 = 6.769671e-01$ and $\hat{w}_3 = 2.24483e-01$

Since the data have been categorized into a number of class intervals, we used the chi square goodness of fit test for assessing the fit of the mixture of two exponential and the mixture of three exponential to the observed claim data.

For obtaining the expected frequency (E_i) in the i^{th} class interval, we have used the following formula

$$E_i = \{F(b_i, \lambda_1, \lambda_2, ..., w_1, w_2, ...) - F(b_{i-1}, \lambda_1, \lambda_2, ..., w_1, w_2, ...)\} \times N,$$

where b_i is the upper class boundary of the i^{th} class interval, N is the total observed frequency and $F(., \lambda_1, \lambda_2, ..., w_1, w_2, ...)$ is the cumulative distribution function of the mixture of exponential with parameters $\lambda_1, \lambda_2, ..., w_1, w_2, ...$ .Table 3 shows the class intervals, observed frequencies and the expected frequencies under mixture of two exponential model and mixture of three exponential for our data.

For the mixture of two exponential, the calculated value of the chi-square statistics is 12.384870. The tabulated value of the chi-square statistics with 20-1-3=16 df at 5% level of significance is 26.296230. Since the calculated value of chi-square is less than the tabulated value of chi-square with 16 df at 5% level of significance, we have reasons to believe that the fit is appropriate at 5% level of significance.

Similarly, for the mixture of three exponential, the calculated value of the chi-square statistics is 0.706226. The tabulated value of the chi-square statistics with 20-1-5=14 df at 5% level of significance is 23.684790. Since the calculated value of chi-square is less than the tabulated value of chi-square with 14 df at 5% level of significance, we have reasons to believe that the even the mixture of three exponential is providing excellent fit to the observed data. Table 3 displays the observed and expected observations for our data under the fitted mixture of two exponential and mixture of three exponential.

Hence, from the goodness of fit tests, we conclude that both the mixture of two exponential and the mixture of three exponential are providing very good fit to the data and hence these two models can be appropriate for explaining the underlying claim severity distribution. However, compared to the mixture of two exponential, the mixture of three exponential, in a way, is providing better fit to the data as concluded from the higher value of the log-likelihood function of the sample under the mixture of three exponential compared to the mixture of two exponential.

In finding the exact value for the probability of ultimate ruin in case of mixture of two exponential distribution, the values of R_1, R_2, C_1 and C_2 are listed in Table 4 whereas the Table 5 displays the values of R_1, R_2, R_3, C_1, C_2 and C_3 for finding the exact value of the probability of ultimate ruin for the mixture of three exponential. It needs mentioning that in case of mixture of two exponential, R_1 and R_2 are the roots of the quadratic equation $(6.050136e+04)r^2 - 1.600092r + (6.446592e-6) = 0$ whereas in case of the mixture of three exponential, R_1, R_2 and R_3 are the roots of the cubic equation $(-2.593811e+04)r^3 + 3.955219r^2 - (1.446887e-04)r + (5.125431e-10) = 0$. The cubic equation was solved using the polyroot function of R Software.

Table 3 Observed and expected frequencies under mixture of two exponential and mixture of three exponential

Class intervals	Observed frequency (O_i)	Expected frequency (E_i) Under mixture of two exponential	Expected frequency (E_i) Under mixture of three exponential
0-10000	75693	31403	85187
10000-20000	45966	25331	37129
20000-30000	16188	20433	16656
30000-40000	7148	16483	7825
40000-50000	4292	13296	3955
50000-60000	2710	10725	2220
60000-70000	1844	8651	1413
70000-80000	1319	6978	1014
80000-90000	978	5629	799
90000-100000	806	4541	669
100000-110000	588	3663	579
110000-120000	506	2954	511
120000-130000	411	2383	454
130000-140000	352	1922	407
140000-150000	350	1551	365
150000-160000	257	1251	327
160000-170000	218	1009	294
170000-180000	207	814	264
180000-190000	167	657	237
>190000	2413	2739	2108
Total	162413	162413	162413

Table 4 The values of R_1, R_2, C_1 and C_2 appearing in equation (24) for computing the exact value of the probability of ultimate ruin in case of the fitted mixture of two exponential

Term	Value
R_1	4.958566e-06
R_2	2.148864e-05
C_1	7.692308e-01
C_2	6.202348e-11

Hence the exact expression for the probability of ultimate ruin in case of mixture of two exponential is given by $\psi(u) = (7.692308e-01)e^{-(4.958566e-06)u} + (3.202348e-01)e^{-(2.148864e-05)u}$, $u > 0$

Table 5 The values of R_1, R_2, R_3, C_1, C_2 and C_3 appearing in equation (23) for computing the exact value of the probability of ultimate ruin in case of the fitted mixture of three exponentials

Term	Value
R_1	3.959906e-06
R_2	5.135094e-05
R_3	9.717594e-05
C_1	6.485202e-01
C_2	1.178686e-01
C_3	2.842031e-03

Hence the exact expression for the probability of ultimate ruin in case of mixture of three exponential is given by $\psi(u) = (6.485202e-01)e^{-(3.959906e-06)u} + (1.178686e-01)e^{-(5.135094e-05)u}$ $+(2.842031e-03)e^{-(9.717594e-05)u}$.

Table 6 Exact values for the ultimate ruin probabilities for the mixture of two exponential distribution with $\hat{\lambda}_1 = 2.148864e-05$, $\hat{\lambda}_2 = 2.148712e-05$, $\hat{w}_1 = 3.800000e-06$ and $\hat{w}_2 = 9.999962e-01$

u (in Rs)	$\psi(u)$ (Ultimate ruin probability)
10	0.7691927
20	0.7691545
30	0.7691164
40	0.7690782
50	0.7690401
60	0.7690020
70	0.7689638
80	0.7689257
90	0.7688876
100	0.7688495
200	0.7684683
500	0.7673260
1000	0.7654260

Table 6 gives the exact values of the probability of ultimate ruin for the mixture of two exponential whereas the Table 7 gives the corresponding values for the mixture of three exponential. Here, an illustrative value of the security loading was taken as $\theta = 0.3$. The values of the initial surplus were limited to a set of smaller values although a set of larger values would have been more realistic. In both the cases, the probability of ultimate ruin was found to be decreasing with an increase in the initial surplus which is as expected, because the induction of larger surpluses should diminish the chance of ruin, if any. Although, these values are claimed to be the exact values for the probability of ultimate ruin, yet a slight element of approximation might have entered through the numerical solution of the quadratic equation and the cubic equation appearing in case of mixture of two and mixture of three exponential respectively. Also, it is observed that the probability of ultimate ruin obtained in case of mixture of two exponential is slightly higher than those for the mixture of three exponential for the same values of the initial surplus.

Table 7 Exact values for the ultimate ruin probabilities for the mixture of three exponential distribution with $\hat{\lambda}_1 = 1.066995e-05$, $\hat{\lambda}_2 = 7.979466e-05$, $\hat{\lambda}_3 = 1.005759e-05$, $\hat{w}_1 = 0.0985499$, $\hat{w}_2 = 6.769671e-01$ and $\hat{w}_3 = 2.244830e-01$

u (in Rs)	$\psi(u)$ (Ultimate ruin probability)
10	0.7691419
20	0.7690530
30	0.7689641
40	0.7688752
50	0.7687864
60	0.7686977
70	0.7686089
80	0.7685202
90	0.7684315
100	0.7683429
200	0.7674584
500	0.7648255
1000	0.7605048

Table 8 shows the probability of ultimate ruin for the fitted mixture of two exponential whereas Table 9 display the corresponding values for the fitted mixture of three exponential obtained through the method of product integration. For accelerating the convergence, as mentioned in (Ramsay and Usabel 1997), we have used the Richardson's extrapolation technique with $\gamma = 20$ and $j = 0,1,2,3,4$ thereby giving $n_4 = 320$. Both the tables reveal that the values of the ultimate ruin probability obtained through the method of product integration are extremely close to the exact values of the probability of ultimate ruin in case of our fitted mixture of two and three exponential. This in a way, leads us to conclude that the method of product integration, though an approximate method, is performing efficiently in terms of yielding almost accurate approximations to the probability of ultimate ruin in case of mixture of exponential distributions.

Table 8 Ultimate ruin probabilities for the mixture of two exponential distribution with $\hat{\lambda}_1 = 2.148864e-05$, $\hat{\lambda}_2 = 2.148712e-05$, $\hat{w}_1 = 3.800000e-06$ and $\hat{w}_2 = 9.999962e-01$ obtained through the method of Product integration

u (in Rs)	$\psi(u)$ (Ultimate ruin probability)
10	0.7691926
20	0.7691545
30	0.7691163
40	0.7690782
50	0.7690401
60	0.7690019
70	0.7689638
80	0.7689257
90	0.7688876
100	0.7688494
200	0.7684683
500	0.7673260
1000	0.7654259

Table 9 Ultimate ruin probabilities for the mixture of three exponential distribution with $\hat{\lambda}_1 = 1.066995e-05$, $\hat{\lambda}_2 = 7.979466e-05$, $\hat{\lambda}_3 = 1.005759e-05$, $\hat{w}_1 = 0.0985499$, $\hat{w}_2 = 6.769671e-01$ and $\hat{w}_3 = 2.244830e-01$ obtained through the method of Product integration

u (in Rs)	$\psi(u)$ (Ultimate ruin probability)
10	0.7691418
20	0.7690529
30	0.7689640
40	0.7688752
50	0.7687863
60	0.7686975
70	0.7686088
80	0.7685201
90	0.7684314
100	0.7683427
200	0.7674579
500	0.7648228
1000	0.7604941

4. Conclusions

The merit of the mixture of exponential lies on the fact it provides both flexibility and smoothness. Since with the mixture of exponential distribution, it is possible to derive an exact expression for the probability of ultimate ruin, it abolishes the risk of taking vital decisions on the basis of approximate values of this very important quantity in assessing the solvency of an insurance company and this stands in contrary to the cases for most of the other claim amount distributions, where exact expressions for the probability of ultimate ruin are not found. The methodology for fitting the mixture of exponential distribution might be beneficial in other domains of risk analysis where a mixture of exponential emerges as a potential model. In fact, any claim amount distribution lying between zero and infinity, can be approximated by a mixture of two exponential by virtue of what is known as Tijm's approximation (Tijm, 1994).

An effort has been made to illustrate the methodology to find the exact value of the probability of ultimate ruin through the moment generating function of the maximal aggregate loss random variable. The actual use of this methodology give rise to a problem of finding the roots of polynomials, for example in case of the mixture of three exponential, it is required to solve a cubic equation, in case of mixture of four exponential, it would require solving a bi quadratic equation and so on. The methodology presented for finding the probability of ultimate ruin in case of mixture of three exponential and two exponential can be readily extended to a higher mixture of exponential although the complexity in terms of solving for the roots of higher order polynomials would increase. Furthermore, the paper also utilized a numerical algorithm, namely, the method of product integration to compute the probability of ultimate ruin for the fitted mixture of exponential. The consistency in the exact values of the probability of ultimate ruin with the numerical approximations obtained through the method of product integration provides vital support in favour of the accuracy and the efficiency of the method of product integration.

Extension of this work can be directed towards the establishment of the actual number of exponentials to be mixed based on the verification of the KKT conditions from the data. Complexities were observed in fitting the mixture of three exponential and two exponential in terms of the choice of initial values for the multi parameter Newton-Raphson and convergence issues. Further probe may be directed towards resolution of these issues. The mixture of exponential, as an insurance loss model, leads to explicit expressions of various actuarial quantities like the aggregate claim models, moments of the time to ruin, deficit at ruin and surplus just prior to ruin, probability function for the number of claims until ruin etc. Those quantities can be derived and also further investigation may be directed towards the influence of an inter arrival time of claims having density other than exponential on the probability of ruin and the influence of interest earnings and tax payments on the surplus accumulated up to a time point.

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Appendix

The Newton-Raphson method: the multiparameter situation

One of the most used methods for optimization in the Multi Parameter situation in Statistics is the Newton-Raphson method which is described briefly as given below:

Assume $\theta = (\theta_1, \theta_2, ..., \theta_p)^T$ is a vector of p (say unknown parameters and the log-likelihood of the distribution involving θ is given by $l(\theta, \underset{\sim}{x})$. Then the MLE for θ are obtained by solving the equations

$$l'(\theta, \underset{\sim}{x}) = 0.$$

Let us now define what is known as the gradient matrix and it is given by

$$S(\theta) = \begin{pmatrix} \frac{\partial l}{\partial \theta_1} \\ \frac{\partial l}{\partial \theta_2} \\ \vdots \\ \frac{\partial l}{\partial \theta_p} \end{pmatrix},$$

and the Hessian matrix is given by

$$J(\theta) = \left(J_{ij}\right)_{i,j=1,2,...,p} \text{ where } J_{ij} = -\frac{\partial^2 l(\theta)}{\partial \theta_i \partial \theta_j}.$$

Then the iterative relationship for the multi parameter Newton-Raphson method is given by

$$\theta^{(S+1)} = \theta^{(S)} + [J(\theta^{(S)})]^{-1} S(\theta^{(S)}),$$

where $\theta^{(S)}$ is the estimated value of θ at the S^{th} iteration. The iteration is carried out until there is no significant difference between $\theta^{(S)}$ and $\theta^{(S+1)}$.