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Average Run Length of Cumulative Sum Control Charts for SARMA(1,1)_L Models

Suvimol Phanyaem*

Department of Applied Statistics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand.

*Corresponding author; e-mail: suvimol.p@sci.kmutnb.ac.th

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Abstract

The cumulative sum (CUSUM) control chart is widely used in a great variety of practical applications such as finance, medicine, engineering, psychology and in other areas. There are many situations in which the process is serially correlation such as in the manufacturing industry, for example, the dynamics of the process will induce correlations in observations that are closely spaced in time. The average run length (ARL) is a traditional measurement of the performance of control chart. In this paper we derive explicit formula for the ARL of CUSUM control chart when observations are seasonal autoregressive and moving average, SARMA(1,1)_L process with exponential white noise. We use Fredholm integral equation approach to derive an explicit formula for the ARL and use the Gauss-Legendre quadrature rule to approximate the numerical integration which both methods based on the Banach's fixed point theorem which is used to guarantee the existence and uniqueness of the solution. Finally, we compare numerical results obtained from the explicit formula for the ARL of SARMA(1,1)_L processes with results obtained from a numerical solution of an integral equation for the ARL. The results show that the ARL from explicit formula is close to the numerical integration with an absolute percentage difference less than 0.1%. In addition, the explicit formula can reduce in the computational time better than the numerical integration.

Keywords: Seasonal autoregressive and moving average, cumulative sum, average run length, Fredholm integral equation, explicit formula.

1. Introduction

The aim of statistical process control (SPC) consists in detecting deviations within a process over time. The SPC is examined whether the present observations can be considered as realizations of a given target process. The observations are analyzed consecutively. The SPC is desirable to detect a change as quickly as possible after its occurrence. Control charts are one of the efficient tools of SPC for detecting changes in mean or variations of the process. Control charts are prosperously applied in engineering, public health, economics and in other areas of applications.

Shewhart (1931) was introduced the first control chart for statistical process control, namely Shewhart control chart. Recently, the cumulative sum (CUSUM) and exponentially weighted

moving average (EWMA) control charts have been proposed as good alternatives to the Shewhart control chart for detecting small shifts in the process. The CUSUM control chart was initially proposed by Page (1954). Many researchers such as Basseville and Nikiforov (1993), and Brodsky and Darkhovsky (1993) for an introduction to CUSUM control chart and its applications. These control charts are based on the assumption that a process being monitored is independent and identically distributed (Smiley and Keoagile 2005). However the choice of control charts depend on the quality characteristics to be measured in the process.

In applications we are frequently faced with time series data which, for a variety of different reasons have characteristics not compatible with the usual assumptions of linearity or Gaussian errors. Processes with non-Gaussian white noise are useful for modelling a wide range of phenomena that do not allow negative values or have a highly skewed distribution. Many problems such as daily flows of a river, wind speeds, amount of dissolved oxygen in a river, etc. Since 1980 several time series models with non-Gaussian white noise have been suggested. Some references are Gaver and Lewis (1980), Lawrance and Lewis (1985), Bell and Smith (1986), Andel (1988), Davis and McCormick (1989), Andel and Garrido (1991), and McCormick and Mathew (1993).

Usually, traditional control chart methodology is based on the standard assumption that random observations are statistically independent and uniformly distributed. However, for the random data of interest in practical applications the observations are usually serially correlation. In many practical processes such as in chemical processes, the random variables are always serially-correlated. The research related to control charts for serially-correlated processes has been proposed in the work of Wardell et al. (1992), Zhang (1998), Lu and Reynolds (1999), Chen and Elsayed (2002), Rosolowski and Schmid (2006), Vermaat et al. (2008) and Tornng et al. (2009).

The average run length (ARL) is a traditional measurement of control chart's performance. Generally, the ARL is the expected number of observations taken from an in-control process until the control charts falsely signal out-of-control is denoted by ARL_0 . A second common characteristic is the expected number of observations taken from an out-of-control process until the control chart signals that the process is out-of-control is denoted by ARL_1 , ideally it should be small. There are several methods that can be utilized to find the ARL for control charts have been discussed in the literatures. Mastrangelo and Montgomery (1995) have been evaluated the performance of EWMA control charts for serially-correlated process based on Monte Carlo simulation technique. Brook and Evans (1972) proposed the method to approximate the ARL for CUSUM control chart by using the Markov Chain approach (MCA) with finite state. Vanbrackle and Reynold (1997) were estimated the ARL by using an integral equation and Markov chain approach to evaluate EWMA and CUSUM control charts when the observations are first order autoregressive (AR(1)) process with additional random error.

Sukparungsee and Novikov (2008) used the martingale approach to derive close-form formulae for the ARL for EWMA control chart for a variety of light-tailed distributions. Areepong and Novikov (2009) derived an analytical expression for the ARL of EWMA control chart when observations are from an exponential distribution. Later, Mititelu et al. (2010) presented the explicit analytical solutions for the ARL by using the Fredholm integral equation approach for EWMA control chart when observations have a Laplace distribution and CUSUM control chart when observations have a hyperexponential distribution. Petcharat et al. (2011) evaluated the ARL of CUSUM procedure by fitting Pareto and Weibull distributed with hyperexponential distribution. Recently, Busaba et al. (2012) was derived the analytical solutions of ARL for CUSUM control chart, its corresponding in the case of first order stationary autoregressive (AR(1)) process with exponential white noise. Phanyaem et al. (2014) used the integral equation technique to derive the

explicit formula for the ARL of CUSUM control chart for an autoregressive and moving average (ARMA(1,1)) process with exponential white noise. Petcharat et al. (2015) derived an analytical expression for the ARL of CUSUM control chart when the random observations are modeled as a moving average of order q (MA(q)) process.

Consequently, the aim of paper is to derive the explicit formulas of average run length (ARL) of CUSUM control chart for a seasonal autoregressive and moving average, SARMA(1,1)_L process with exponential white noise and compare it with the numerical integration. The organization of this paper is as follows: In Section 2, the characteristic of CUSUM control chart for SARMA(1,1)_L is described. The explicit formula for ARL of CUSUM control chart is proposed in Section 3 and the numerical integration of ARL of CUSUM control chart is presented in Section 4. In Section 5, we compare numerical results obtained from the explicit formula for the ARL of SARMA(1,1)_L processes with results obtained from a numerical solution of an integral equation for the ARL. Conclusions are provided in the final section.

2. The Cumulative Sum Control Chart for SARMA(1,1)_L

In this paper, the CUSUM control chart is considered under the assumption that sequential observation X_1, X_2, \dots of a some process are modeled as a seasonal autoregressive and moving average (SARMA(1,1)_L) with exponential white noise.

The recursive equation of SARMA(1,1)_L process with exponential white noise is defined as:

$$X_t = \mu + \varphi X_{t-L} + \xi_t - \theta \xi_{t-L}; \quad t = 1, 2, \dots \quad (1)$$

where ξ_t is assumed to be a white noise process with exponential distribution. The initial value ξ_{t-L} is usually to be the process mean, an autoregressive coefficient $0 \leq \varphi \leq 1$, a moving average coefficient $0 \leq \theta \leq 1$ and an initial value of SARMA(1,1)_L process $X_{t-L} = 1$.

The CUSUM statistics based on SARMA(1,1)_L process is defined by the following recursion:

$$C_t = \max(C_{t-1} + X_t - a, 0); \quad t = 1, 2, \dots, \quad (2)$$

where X_t is a sequence of SARMA(1,1)_L process, and a is a reference value of CUSUM chart.

Let τ denoted the stopping time of CUSUM control chart and it is given by:

$$\tau_h = \inf \{t > 0; C_t > h\}, \quad h > u, \quad (3)$$

where h is a constant parameter known as the upper control limit.

Let $\mathbb{E}_\infty(\cdot)$ denoted the expectation under density function $f(x, \alpha)$ that the change-point occurs at point θ , where $\theta < \infty$. Thus by definition, the ARL for SARMA(1,1)_L process with an initial value $C_0 = u$ is as follow

$$ARL = H(u) = \mathbb{E}_\infty(\tau_h) < \infty. \quad (4)$$

3. Explicit Formulas for Average Run Length of CUSUM Control Chart for SARMA(1,1)_L

This section is devoted to our analytical derivation as applied to the CUSUM procedure based on SARMA(1,1)_L process with exponential white noise. Crowder (1987) proposed numerical method to evaluate ARL of EWMA chart with Gaussian distribution and showed that the ARL can be presented in the form of Fredholm integral equation of the second kind. Consequently, we used as integral equation approach and solved a Fredholm integral equation of the second kind for finding the ARL.

Supposed that function $H(u)$ is the ARL of CUSUM chart for SARMA(1,1)_L process with the initial value C_0 . We assume that the lower control limit is zero and upper control limit is h . Let \mathbb{P}_c denote the probability measure and \mathbb{E}_c denote the expectation corresponding to initial value $C_0 = u$.

The ARL of CUSUM chart for SARMA(1,1)_L process after it is reset at $u \in [0, h]$ as follows:

$$H(u) = 1 + \mathbb{E}_c [I\{0 < C_1 < h\}H(C_1)] + \mathbb{P}_c \{C_1 = 0\}H(0). \tag{5}$$

Let $C_t = C_{t-1} + X_t - a$; $t = 1, 2, \dots$ where $X_t = \mu + \varphi X_{t-L} + \xi_t - \theta \xi_{t-L}$ and $C_0 = u$.

First, to calculate $\mathbb{E}_c [I\{0 < C_1 < h\}H(C_1)]$ for $t = 1$.

$$\begin{aligned} \mathbb{E}_c [I\{0 < C_1 < h\}H(C_1)] &= \int_{a-u-\mu-\varphi X_{1-L}+\xi_{1-L}}^{h+a-u-\mu-\varphi X_{1-L}+\theta \xi_{1-L}} H(u + \mu + \varphi X_{1-L} + y - \theta \xi_{1-L} - a) \alpha e^{-\alpha y} dy \\ &= \int_0^h H(y) \alpha e^{\alpha(u-a+\mu+\varphi X_{1-L}-\theta \xi_{1-L}-y)} dy \\ &= \alpha e^{\alpha(u-a+\mu+\varphi X_{1-L}-\theta \xi_{1-L})} \int_0^h H(y) e^{-\alpha y} dy \end{aligned}$$

$$\begin{aligned} \mathbb{P}_c \{C_1 = 0\}H(0) &= \mathbb{P}_c \{u + \mu + \varphi X_{1-L} + \xi_1 - \theta \xi_{1-L} - a = 0\}H(0) \\ &= [1 - \mathbb{P}_c \{\xi_1 > a - u - \mu - \varphi X_{1-L} + \theta \xi_{1-L}\}]H(0) \\ &= [1 - e^{-\alpha(a-u-\mu-\varphi X_{1-L}+\theta \xi_{1-L})}]H(0) \end{aligned}$$

In this case the (5) can be written as

$$H(u) = 1 + \alpha e^{\alpha(u-a+\mu+\varphi X_{1-L}-\theta \xi_{1-L})} \int_0^h H(y) e^{-\alpha y} dy + \left(1 - e^{-\alpha(a-u-\mu-\varphi X_{1-L}+\theta \xi_{1-L})}\right)H(0). \tag{6}$$

In this section, we show that the ARL for CUSUM control chart is the unique solution to the integral equation. On the metric space of all continuous functions $(C(I), \|\cdot\|_\infty)$ where I denotes the compact interval and the norm $\|H\|_\infty = \sup_{u \in I} |H(u)|$ and the operator T is named on contraction, if it exists a number of $0 \leq q < 1$ such that

$$\|T(H_1) - T(H_2)\| \leq q \|H_1 - H_2\| \text{ for all } H_1, H_2 \in I.$$

Now, let $C(I_1)$ be the class of all continuous functions defined on a compact interval $I_1 = [0, h]$ and define the operator T by

$$T(H(u)) = 1 + \alpha e^{\alpha(u-a+\mu+\varphi X_{1-L}-\theta \xi_{1-L})} \int_0^h H(y) e^{-\alpha y} dy + \left(1 - e^{-\alpha(a-u-\mu-\varphi X_{1-L}+\theta \xi_{1-L})}\right)H(0). \tag{7}$$

Therefore, the integral equation can be written as $T(H(u)) = H(u)$. According to the Banach's fixed point theorem, if the operator T is a contraction, then fixed point equations $T(H(u)) = H(u)$ have a unique solution.

For any $u \in I$ and $H_1, H_2 \in C(I)$ we have the inequality $\|T(H_1) - T(H_2)\| \leq q \|H_1 - H_2\|$ where $q < 1$. According to (7), we get

$$\begin{aligned} \|T(H_1) - T(H_2)\| &= \sup_{u \in [0, h]} |H_1(0) - H_2(0)| (1 - e^{-\alpha(a-u-\mu-\phi X_{i-L} + \theta \xi_{i-L}^e)}) \\ &\quad + \alpha e^{\alpha(u-a+\mu+\phi X_{i-L} - \theta \xi_{i-L}^e)} \int_0^h (H_1(y) - H_2(y)) e^{-\alpha y} dy \\ &\leq \sup_{u \in [0, h]} \left\| H_1(0) - H_2(0) \right\| (1 - e^{-\alpha(a-u-\mu-\phi X_{i-L} + \theta \xi_{i-L}^e)}) \\ &\quad + \left\| H_1 - H_2 \right\| \alpha e^{\alpha(u-a+\mu+\phi X_{i-L} - \theta \xi_{i-L}^e)} \int_0^h e^{-\alpha y} dy \\ &= \left\| H_1 - H_2 \right\| \sup_{u \in [0, h]} (1 - e^{-\alpha(a-u-\mu-\phi X_{i-L} + \theta \xi_{i-L}^e) - \alpha h}) \\ &\leq q \left\| H_1 - H_2 \right\|, \text{ where } q = (1 - e^{-\alpha(a-u-\mu-\phi X_{i-L} + \theta \xi_{i-L}^e) - \alpha h}) < 1. \end{aligned}$$

We used the Fredholm integral equation to derive the explicit formula of the ARL of CUSUM control chart for SARMA(1,1)_L process.

Let $k = \int_0^h H(y) e^{-\alpha y} dy$, we obtain that

$$H(u) = 1 + \alpha e^{\alpha(u-a+\mu+\phi X_{i-L} - \theta \xi_{i-L}^e)} k + (1 - e^{-\alpha(a-u-\mu-\phi X_{i-L} + \theta \xi_{i-L}^e)}) H(0). \tag{8}$$

Now, we let $u = 0$, then we have

$$\begin{aligned} H(0) &= 1 + \alpha e^{\alpha(-a+\mu+\phi X_{i-L} - \theta \xi_{i-L}^e)} k + (1 - e^{-\alpha(a-\mu-\phi X_{i-L} + \theta \xi_{i-L}^e)}) H(0) \\ &= \frac{1 + (e^{\alpha(-a+\mu+\phi X_{i-L} - \theta \xi_{i-L}^e)})}{(e^{-\alpha(a-\mu-\phi X_{i-L} + \theta \xi_{i-L}^e)})} \alpha k \\ &= e^{\alpha(a-\mu-\phi X_{i-L} + \theta \xi_{i-L}^e)} + \alpha k. \end{aligned} \tag{9}$$

Then, on substituting (9) into (8); we obtain that

$$\begin{aligned} H(u) &= 1 + \alpha e^{\alpha(u-a+\mu+\phi X_{i-L} - \theta \xi_{i-L}^e)} k + (1 - e^{-\alpha(a-u-\mu-\phi X_{i-L} + \theta \xi_{i-L}^e)}) e^{\alpha(a-\mu-\phi X_{i-L} + \theta \xi_{i-L}^e)} + \alpha k \\ &= 1 + \alpha k e^{\alpha(u-a+\mu+\phi X_{i-L} - \theta \xi_{i-L}^e)} + e^{\alpha(a-\mu-\phi X_{i-L} + \theta \xi_{i-L}^e)} + \alpha k \\ &= 1 + \alpha k + e^{\alpha(a-\mu-\phi X_{i-L} + \theta \xi_{i-L}^e)} - e^{\alpha u}. \end{aligned} \tag{10}$$

To find a constant k as following form

$$\begin{aligned} k &= \int_0^h H(y) e^{-\alpha y} dy \\ &= \int_0^h (1 + \alpha k + e^{\alpha(a-\mu-\phi X_{i-L} + \theta \xi_{i-L}^e)} - e^{\alpha y}) e^{-\alpha y} dy \\ &= (1 + \alpha k + e^{\alpha(a-\mu-\phi X_{i-L} + \theta \xi_{i-L}^e)}) \int_0^h e^{-\alpha y} dy - \int_0^h e^{\alpha y - \alpha y} dy \\ &= \frac{e^{\alpha h}}{\alpha} (1 - e^{-\alpha h}) (1 + e^{\alpha(a-\mu-\phi X_{i-L} + \theta \xi_{i-L}^e)}) - h e^{\alpha h}. \end{aligned}$$

Thus, a constant k can be found as follows

$$k = \frac{e^{\alpha h}}{\alpha} (1 - e^{-\alpha h}) (1 + e^{\alpha(a-\mu-\phi X_{i-L} + \theta \xi_{i-L}^e)}) - h e^{\alpha h}.$$

Substituting a constant k into (10) as follows

$$\begin{aligned}
 H(u) &= 1 + \alpha \left(\frac{e^{\alpha h}}{\alpha} (1 - e^{-\alpha h}) (1 + e^{\alpha(a-\mu-\varphi X_{t-L} + \theta \xi_{t-L})}) - h e^{\alpha h} \right) + e^{\alpha(a-\mu-\varphi X_{t-L} + \theta \xi_{t-L})} - e^{\alpha u} \\
 &= 1 + (e^{\alpha h} (1 - e^{-\alpha h}) (1 + e^{\alpha(a-\mu-\varphi X_{t-L} + \theta \xi_{t-L})}) - \alpha h e^{\alpha h}) + e^{\alpha(a-\mu-\varphi X_{t-L} + \theta \xi_{t-L})} - e^{\alpha u} \\
 &= 1 + (e^{\alpha h} - e^{(-\alpha h + \alpha h)}) (1 + e^{\alpha(a-\mu-\varphi X_{t-L} + \theta \xi_{t-L})}) - \alpha h e^{\alpha h} + e^{\alpha(a-\mu-\varphi X_{t-L} + \theta \xi_{t-L})} - e^{\alpha u} \\
 &= 1 + (e^{\alpha h} - 1) (1 + e^{\alpha(a-\mu-\varphi X_{t-L} + \theta \xi_{t-L})}) - \alpha h e^{\alpha h} + e^{\alpha(a-\mu-\varphi X_{t-L} + \theta \xi_{t-L})} - e^{\alpha u} \\
 &= e^{\alpha h} + e^{\alpha h + \alpha(a-\mu-\varphi X_{t-L} + \theta \xi_{t-L})} - e^{\alpha(a-\mu-\varphi X_{t-L} + \theta \xi_{t-L})} - \alpha h e^{\alpha h} + e^{\alpha(a-\mu-\varphi X_{t-L} + \theta \xi_{t-L})} - e^{\alpha u} \\
 &= e^{\alpha h} (1 + e^{\alpha(a-\mu-\varphi X_{t-L} + \theta \xi_{t-L})} - \alpha h) - e^{\alpha u}.
 \end{aligned}$$

Consequently, the explicit formulas of the ARL of CUSUM control chart for SARMA(1,1)_L process for $t = 1, 2, \dots, L$ is given by

$$H(u) = e^{\alpha h} (1 + e^{\alpha(a-\mu-\varphi X_{t-L} + \theta \xi_{t-L})} - \alpha h) - e^{\alpha u}. \tag{11}$$

Suppose that the process produces readings that in-control, have mean $\alpha = \alpha_0$ which is assumed known. The explicit formula of ARL_0 of CUSUM control chart for SARMA(1,1)_L process as follows:

$$ARL_0 = e^{\alpha_0 h} (1 + e^{\alpha_0(a-\mu-\varphi X_{t-L} + \theta \xi_{t-L})} - \alpha_0 h) - e^{\alpha_0 u}. \tag{12}$$

Since an assignable cause event occurs, it will lead to a change in mean of exponential distribution. This situation is called an out-of-control state, with exponential parameter $\alpha = \alpha_1$ where $\alpha_1 = \alpha_0(1 + \delta)$. The explicit formula of ARL_1 of CUSUM control chart for SARMA(1,1)_L process as follows:

$$ARL_1 = e^{\alpha_1 h} (1 + e^{\alpha_1(a-\mu-\varphi X_{t-L} + \theta \xi_{t-L})} - \alpha_1 h) - e^{\alpha_1 u}, \tag{13}$$

where α is a parameter of exponential white noise, h is upper control limit, X_{t-L}, ξ_{t-L} are the initial values, φ is an autoregressive coefficient; $0 \leq \varphi \leq 1$ and θ is a moving average coefficient; $0 \leq \theta \leq 1$.

4. Numerical Integration of Average Run Length of CUSUM Chart for SARMA(1,1)_L

In this section we present the scheme to evaluate numerically the solutions of the integral equation by using Gauss-Legendre quadrature rule.

Since $y \sim Exp(\alpha)$, then $F(u) = 1 - e^{-\alpha u}$ and $f(u) = \frac{dF(u)}{du} = \lambda e^{-\lambda u}$.

Consequently, the integral equation in (6) can be rewritten as follows

$$\tilde{H}(u) = 1 + H(0)F(a - u - \mu - \varphi X_{t-L} + \theta \xi_{t-L}) + \int_0^h H(y)f(y + a - u - \mu - \varphi X_{t-L} + \theta \xi_{t-L})dy. \tag{14}$$

The numerical approximation to integral equation is denoted by $\tilde{H}(a_i)$, which can be found as the solution of linear equations as follows:

$$\begin{aligned}
 \tilde{H}(a_i) &= 1 + \tilde{H}(a_1)F(a - a_i - \mu - \varphi X_{t-L} + \theta \xi_{t-L}) \\
 &\quad + \sum_{j=1}^m w_j \tilde{H}(a_j)f(a_j + a - a_i - \mu - \varphi X_{t-L} + \theta \xi_{t-L}).
 \end{aligned} \tag{15}$$

Thus

$$\begin{aligned} \tilde{H}(a_1) &= 1 + \tilde{H}(a_1)[F(a - a_1 - \mu - \varphi X_{t-L} + \theta \xi_{t-L}) + w_1 f(a - \mu - \varphi X_{t-L} + \theta \xi_{t-L})] \\ &\quad + \sum_{j=2}^m w_j \tilde{H}(a_j) f(a_j + a - a_1 - \mu - \varphi X_{t-L} + \theta \xi_{t-L}) \\ \tilde{H}(a_2) &= 1 + \tilde{H}(a_1)[F(a - a_2 - \mu - \varphi X_{t-L} + \theta \xi_{t-L}) + w_1 f(a_1 + a - a_2 - \mu - \varphi X_{t-L} + \theta \xi_{t-L})] \\ &\quad + \sum_{j=2}^m w_j \tilde{H}(a_j) f(a_j + a - a_2 - \mu - \varphi X_{t-L} + \theta \xi_{t-L}) \\ &\quad \vdots \\ \tilde{H}(a_m) &= 1 + \tilde{H}(a_1)[F(a - a_m - \mu - \varphi X_{t-L} + \theta \xi_{t-L}) + w_1 f(a_1 + a - a_m - \mu - \varphi X_{t-L} + \theta \xi_{t-L})] \\ &\quad + \sum_{j=2}^m w_j \tilde{H}(a_j) f(a_j + a - a_m - \mu - \varphi X_{t-L} + \theta \xi_{t-L}) \end{aligned}$$

or in matrix form as

$$\mathbf{H}_{m \times 1} = \mathbf{1}_{m \times 1} + \mathbf{R}_{m \times m} \mathbf{H}_{m \times 1} \tag{16}$$

where $\mathbf{H}_{m \times 1} = \begin{pmatrix} \tilde{H}(a_1) \\ \tilde{H}(a_2) \\ \vdots \\ \tilde{H}(a_m) \end{pmatrix}$, $\mathbf{1}_{m \times 1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

$$\mathbf{R} = \begin{pmatrix} F(a - a_1 - \mu - \varphi X_{t-L} + \theta \xi_{t-L}) + w_1 f(a - \mu - \varphi X_{t-L} + \theta \xi_{t-L}) & \dots & w_m f(a_m + a - a_1 - \mu - \varphi X_{t-L} + \theta \xi_{t-L}) \\ F(a - a_1 - \mu - \varphi X_{t-L} + \theta \xi_{t-L}) + w_1 f(a_1 + a - a_2 - \mu - \varphi X_{t-L} + \theta \xi_{t-L}) & \dots & w_m f(a_m + a - a_2 - \mu - \varphi X_{t-L} + \theta \xi_{t-L}) \\ \vdots & & \vdots \\ F(a - a_m - \mu - \varphi X_{t-L} + \theta \xi_{t-L}) + w_1 f(a_1 + a - a_m - \mu - \varphi X_{t-L} + \theta \xi_{t-L}) & \dots & w_m f(a_m + a - a_m - \mu - \varphi X_{t-L} + \theta \xi_{t-L}) \end{pmatrix}$$

and $\mathbf{I}_m = \text{diag}(1, 1, \dots, 1)$. If $(\mathbf{I}_m - \mathbf{R}_{m \times m})^{-1}$ there exist

$$\mathbf{H}_{m \times 1} = (\mathbf{I}_m - \mathbf{R}_{m \times m})^{-1} \mathbf{1}_{m \times 1} \tag{17}$$

Here $\tilde{H}(u)$ denotes the numerical integration solution of $H(u)$, then the integral equation in (6) can be approximated by

$$\tilde{H}(u) 1 + \tilde{H}(a_1) F(a - u - \mu - \varphi X_{t-L} + \theta \xi_{t-L}) + \sum_{j=1}^m w_j \tilde{H}(a_j) f(a_j + a - u - \mu - \varphi X_{t-L} + \theta \xi_{t-L}) \tag{18}$$

where $w_j = \frac{h}{m}$ and $a_j = \frac{h}{m} \left(j - \frac{1}{2} \right)$; $j = 1, 2, \dots, m$.

5. Numerical Result

In this section, we present the results obtained from the explicit formulas of ARL for CUSUM control chart when observations are SARMA(1,1)₄ process with exponential white noise and compare to the ARL from numerical integral equation with $m = 500$ nodes. The explicit formula solution is denoted by $H(u)$ and the numerical integration solution is denoted by $\tilde{H}(u)$. We use the absolute percentage difference of ARL to measure of accuracy of comparison defined as

$$\text{Diff}(\%) = \frac{|H(u) - \tilde{H}(u)|}{H(u)} \times 100.$$

In Table 1, the value of parameters a and h for CUSUM control chart were chosen by setting the desired $ARL_0 = 370$ and the value of exponential parameter $\alpha_0 = 1$ in the case of SARMA(1,1)₄ process with parameter $(\varphi, \theta) = (0.10, 0.10), (0.10, 0.20)$ and $(0.10, 0.30)$ respectively.

The results from Table 1 present the value of parameters for CUSUM control chart and show that the ARL_0 from analytical solution is close to the numerical integration with the absolute percentage difference less than 0.1% for the case of division points $m = 500$ nodes. However, the results also show that the computational time for evaluating the purposed explicit formula is much less than the computational time required for numerical integral equation method.

Table 1 Comparison of ARL_0 computed using explicit formulas against numerical integration for SARMA(1,1)₄ process with parameter $\alpha_0 = 1$ and $u = 1$ for $ARL_0 = 370$

SARMA(1,1) ₄ Process with $\varphi = 0.10$ and $\theta = 0.10$				
a	h	Explicit Formulas	Numerical Integration	Diff (%)
2.00	4.585	370.091	369.790 (49.63) ^a	0.08133
2.50	3.669	370.331	370.033 (49.84)	0.08047
3.00	3.028	370.276	369.990 (49.58)	0.07724
SARMA(1,1) ₄ Process with $\varphi = 0.10$ and $\theta = 0.20$				
a	h			
2.00	4.349	370.136	369.833 (49.97) ^a	0.08186
2.50	3.529	370.045	369.814 (49.94)	0.06242
3.00	2.911	370.058	369.783 (49.94)	0.07431
SARMA(1,1) ₄ Process with $\varphi = 0.10$ and $\theta = 0.30$				
a	h			
2.00	4.151	370.276	369.964 (49.94) ^a	0.08426
2.50	3.397	370.195	369.917 (49.94)	0.07510
3.00	2.797	370.040	369.772 (49.85)	0.07242

^a The values in parentheses are CPU times in numerical integration methods (minutes).

In Table 2 to Table 4, we compare the results of $ARL_0 = 370$ and ARL_1 , which obtained by analytical formula with the results by numerical integration method for CUSUM control chart. We assume that process starts with in-control state, the value of the in-control parameter $\alpha_0 = 1$. For example, in Table 2 if we fixed an $ARL_0 = 370$ for SARMA(1,1)₄ process with parameter $\varphi = 0.10$ and $\theta = 0.10$ then the parameter of CUSUM control chart are $a = 2.00$ and $h = 4.585$. In the case of SARMA(1,1)₄ process with parameter $\varphi = 0.10$ and $\theta = 0.20$, we use the parameter of CUSUM chart are $a = 2.50$ and $h = 3.529$. Finally, in the case of SARMA(1,1)₄ process with parameter $\varphi = 0.10$ and $\theta = 0.30$, then the parameter of CUSUM control chart are $a = 3.00$ and $h = 2.797$. Numerical values for ARL_0 are computed for an in-control parameter value and numerical values for ARL_1 are computed for range of out-of-control parameter values $\alpha_1 = \alpha_0(1 + \delta)$ where $\delta = 0.01, 0.03, 0.05, 0.07, 0.09, 0.10, 0.20, 0.30, 0.40, 0.50$ and 1.00 respectively.

Table 2 Comparison of ARL computed using explicit formulas against numerical integration for SARMA(1,1)₄ process with parameter $\varphi = 0.10$, $\theta = 0.10$, $a = 2.00$ and $h = 4.585$

Shift size (δ)	Explicit Formulas	Numerical Integration	Diff (%)
0.00	370.091	369.790 (49.63) ^a	0.08133
0.01	344.256	343.971 (49.44)	0.08279
0.03	299.160	298.925 (49.04)	0.07855
0.05	261.413	261.212 (49.25)	0.07689
0.07	229.633	229.438 (49.24)	0.08492
0.09	202.728	202.571 (49.71)	0.07744
0.10	190.825	190.686 (49.58)	0.07284
0.20	110.602	110.477 (49.52)	0.11302
0.30	70.319	70.217 (49.43)	0.14505
0.40	48.139	48.046 (49.40)	0.19319
0.50	34.975	34.894 (49.67)	0.23159
1.00	12.466	12.397 (49.67)	0.55351

^a The values in parentheses are CPU times in numerical integration methods (minutes).

Table 3 Comparison of ARL computed using explicit formulas against numerical integration for SARMA(1,1)₄ process with parameter $\varphi = 0.10$, $\theta = 0.20$, $a = 2.50$ and $h = 3.529$

Shift size (δ)	Explicit Formulas	Numerical Integration	Diff (%)
0.00	370.045	369.814 (49.94) ^a	0.06242
0.01	347.159	346.933 (49.81)	0.06510
0.03	306.669	306.456 (49.75)	0.06946
0.05	272.172	271.963 (49.63)	0.07679
0.07	242.627	242.448 (49.59)	0.07378
0.09	217.199	217.049 (49.48)	0.06906
0.10	205.812	205.689 (49.39)	0.05976
0.20	126.213	116.094 (49.30)	0.09429
0.30	83.519	83.414 (49.23)	0.12572
0.40	58.710	58.617 (49.17)	0.15841
0.50	43.332	43.280 (49.10)	0.12000
1.00	15.311	15.295 (49.03)	0.10450

^a The values in parentheses are CPU times in numerical integration methods (minutes).

Table 4 Comparison of ARL computed using explicit formulas against numerical integration for SARMA(1,1)_L process with parameter $\varphi = 0.10$, $\theta = 0.30$, $a = 3.00$ and $h = 2.797$

Shift size (δ)	Explicit Formulas	Numerical Integration	Diff (%)
0.00	370.040	369.772 (49.85) ^a	0.07242
0.01	348.126	347.869 (49.72)	0.07382
0.03	309.192	308.943 (49.68)	0.08053
0.05	275.834	275.602 (49.52)	0.08411
0.07	247.109	246.900 (49.40)	0.08458
0.09	222.257	222.079 (49.32)	0.08009
0.10	211.085	210.929 (49.29)	0.07390
0.20	132.052	131.911 (49.17)	0.10678
0.30	88.735	88.609 (49.03)	0.14200
0.40	63.099	59.601 (49.12)	0.13788
0.50	46.961	44.070 (49.04)	0.08731
1.00	16.793	15.576 (49.01)	0.10719

^a The values in parentheses are CPU times in numerical integration methods (minutes).

The results from Table 2 to Table 4, show that the absolute percentage difference up to 0.1% for shift size less than 0.20 by the numerical integration for the case of division points $m = 500$ nodes, and the computational times of approximately 49-50 minutes. However, the computational times from the proposed explicit formulas are less than 1 second.

6. Conclusions

In this paper, we have proposed the explicit formula for the average run length (ARL) of cumulative sum (CUSUM) control chart for the seasonal autoregressive and moving average, SARMA(1,1)_L process with exponential distribution white noise. We derived the explicit formulas by using Fredholm integral equation technique and used the Banach's Fixed Point theorem to guarantee the existence and uniqueness of solution. In addition, we developed numerical integration for evaluating the ARL of CUSUM control chart for SARMA(1,1)_L process with exponential distribution white noise by using Gauss-Legendre quadrature rule. We checked the accuracy of the proposed explicit formulas in term of absolute percentage difference between the explicit formula solution and the numerical integration solution.

The results show that the ARL from the proposed explicit formulas is close to the numerical integration with an absolute percentage difference less than 0.1% for shift size (δ) < 0.20. In another word, it give an absolute percentage difference greater than 0.1% when the process has a large shift size (δ) \geq 0.20. However, for all magnitude of shift size give an absolute percentage difference less than 1.0%. Consequently, the proposed analytical formula is sufficiently high accuracy and easy to calculate in comparison with numerical integration technique. In addition, the computational times for evaluating the proposed explicit formula takes less than 1 second while the numerical integration method takes approximately 49-50 minutes in the case of SARMA(1,1)_L process. Therefore, the explicit formulas can reduce in the computational times much better than the numerical integration.

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