ทฤษฎีบทการลู่เข้าของจุดตรึงร่วมสำหรับการส่งแบบไม่ขยาย G

Convergence Theorem of Common Fixed Points for G-Nonexpansive Mappings

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บทคัดย่อ

วัตถุประสงค์ของงานวิจัยนี้คือ นำเสนอขั้นตอนวิธีใหม่สำหรับการหาจุดตรึงร่วมของการส่งแบบไม่ขยาย G บนปริภูมิบานาค ภายใต้เงื่อนไขที่เหมาะสม พวกเราได้พิสูจน์ทฤษฎีบทการลู่เข้าของลำดับที่ได้จากขั้นตอนวิธีที่ได้นำเสนอ

คำสำคัญ: กราฟระบุทิศทาง ทฤษฎีจุดตรึง การส่งแบบไม่กระจาย G

Abstract

The purpose of this paper is to introduce a new algorithm for finding a common fixed point of G-nonexpansive mappings on a Banach space. Under appropriate conditions, we prove a convergence theorem for the sequence generated by the proposed algorithm.

Keywords: Directed Graph, Fixed Point Theory, G-Nonexpansive Mapping

Introduction

Let $G=(V(G),A(G))$ be a directed graph (digraph) where $V(G)$ is a set of vertices of graph and $A(G)$ is a set of ordered pair of element of $V(G)$. We call the elements of $A(G)$ arcs or directed edges. We assume that G has no multiple edges. We denote by G^{-1} the directed graph obtained from G by reversing the direction of edges. That is, $A(G^{-1})=\{(y,x):(x,y)\in A(G)\}$. A weighted directed graph is a directed graph in which a number (the weight) is assigned to each edge. Let x and y be vertices of G. A path in G from x to y of length $N\in\mathbb{N}\cup\{0\}$ is a sequence $\{x_i\}_{i=0}^{N}$ of $N+1$ vertices for which $x_0=x$ $x_N=y$ and $(x_i,x_{i+1})\in A(G)$ for $i=1,2,...,N-1$. A directed graph G is weakly connected if there is an undirected path between any pair of vertices, and strongly connected if there is a directed path between every pair of vertices. A directed graph G is said to be transitive if, for any $x,y,z\in V(G)$ such that (x,y) and (y,z) are in $A(G)$, we have $(x,z)\in A(G)$.

Let (E,d) be a metric space. A mapping $T:E\to E$ is said to be contractive if there is $0<k<1$ such that $d(Tx,Ty)\le kd(x,y)$ for all $x,y\in E$. A mapping T is said to be nonexpansive if $d(Tx,Ty)\le d(x,y)$ for all $x,y\in E$. Recall that a point $x\in E$ is a fixed point of a mapping T if $Tx=x$. The set of fixed point of T is denoted by $F(T)$.

Theorem 1.1 [1] Let (E,d) be a complete metric space and $T:E\to E$ be a contractive mapping. Then T has a unique fixed point.

Let E be a Banach space with the norm $\|\cdot\|$. A Banach space E is said to be strictly convex if $\left\|\frac{x+y}{2}\right\|<1$ for all $x,y\in E$ with $\|x\|=\|y\|=1$ and $x\ne y$ (see [2] for more details). For a sequence $\{x_n\}$ of a Banach space E and a point $x\in E$, the strong convergence of $\{x_n\}$ to x is denoted by $x_n\to x$.

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Lemma 1.2 [3] Let E be a Banach space and $B_r(0)=\{x\in E:\|x\|\le r, r>1\}$ be a closed ball of E. Then E is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g=[0,\infty)\to[0,\infty)$ with $g(0)=0$ such that

$$\|\lambda x+(1-\lambda)y\|^2\le\lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)g(\|x-y\|),$$

for all $x,y\in B_r(0)$ and $\lambda\in[0,1]$.

Lemma 1.3 [4] Let E be a uniformly convex Banach space and $\{\alpha_n\}$ be a sequence in $[\alpha,1-\alpha]$ for some $\alpha\in(0,1)$. Suppose that the sequences $\{x_n\}$ and $\{y_n\}$ in E are such that $\limsup_{n\to\infty}\|\alpha_n x_n+(1-\alpha_n)y_n\|=c$ where $\limsup_{n\to\infty}\|x_n\|\le c$ and $\limsup_{n\to\infty}\|y_n\|\le c$ for some $c\ge 0$. Then $\lim_{n\to\infty}\|x_n-y_n\|=0$.

Let C be a nonempty subset of a Banach space E. A point p in C is said to be a strongly asymptotic fixed point of T [5] if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. The set of strong asymptotic fixed point of T is denoted by $\tilde{F}(T)$. Let C be a nonempty subset of a real Banach space E and Δ denote diagonal of the cartesian product $C\times C$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with C and the set of its edges with $\Delta\subset A(G)$. We assume G has no multiple edges.

A mapping $T:C\to C$ is G-nonexpansive (see [6]), if T satisfies the following conditions.

(i) T preserves edges of G, i.e.,

$$(x,y)\in A(G)\Rightarrow(Tx,Ty)\in A(G),\forall(x,y)\in A(G);$$

(ii) T non-increases weights of edges of G in the following way:

$$(x,y)\in A(G)\Rightarrow\|Tx-Ty\|\le\|x-y\|,\forall(x,y)\in A(G).$$

Example 1.4 [7] Let $E=\mathbb{R}$ and $C=\left[0,\frac{1}{2}\right]$ with norm $\|x-y\|=|x-y|$ and let $G=(V(G),A(G))$ be such that $V(G)=C,\ A(G)=\{(x,y):x,y\in\left[0,\frac{3}{8}\right]\ \textit{such that}\ |x-y|\le\frac{1}{8}\}$. Define $T:C\to C$ by

$$Tx=\begin{cases}\frac{8}{6}x^2 & \text{if } x\in[0,\frac{1}{2}),\\ \frac{25}{64} & \text{if } x=\frac{1}{2}.\end{cases}$$

Note that T is G-nonexpansive.

The study of fixed point theorem on Banach spaces were investigated by many authors (see [8 - 14]). In 2008, Jachymski [15] proved generalizations of the Banach's contraction principle in complete metric spaces endowed with a graph. In 2015, Tiammee et al. [7] proved Browder's convergence theorem for G-nonexpansive mappings in Hilbert spaces with a directed graph. The study of fixed point theorem for G-nonexpansive mappings in Hilbert spaces and Banach spaces were investigated by many authors (see [6 - 7], [13 - 16]). In 2017, Suparatulatorn *et al*. [17] proved a strong convergence theorem for two different hybrid methods by using CQ method for a finite family of G-nonexpansive mappings in a Hilbert space. Recently, Saewan *et al*. [18] proved a strong convergence theorem for three-step iterative scheme for G-nonexpansive mappings under condition (II). In this paper, inspired and motivated by the works mentioned above, we introduce an iterative process for finding a common element of G-nonexpansive mappings in Banach spaces E endowed with a directed graph G and we prove the strong convergence theorem under the appropriate conditions.

Materials and Methodology

1. Studying and investigating on the fixed point problem.
2. Studying and investigating on a directed graph.
3. Studying and investigating on the G-nonexpansive mapping.
4. Establishing a new theorem of the fixed point problem for G-nonexpansive mappings on Banach spaces.

Results

In this section, we prove a fixed point theorem for G-nonexpansive mapping in a Banach space endowed with a directed graph. First, we begin with some well-known results and useful definitions that will be use in results.

Property G: Let C be a nonempty subset of a normed space E and let $G=(V(G),A(G))$, where $V(G)=C$, be a directed graph. Then C is said to have Property G if every sequence $\{x_n\}$ in C converging weakly to $x\in C$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k},x)\in A(G)$ for any $k\in\mathbb{N}$.

Lemma 3.1 ([7]) Let E be a norm space and $G=(V(G),A(G))$ be a directed graph with $V(G)=E$. Suppose $T:E\to E$ is G-nonexpansive mapping. If E has a Property G, then T is continuous.

Theorem 3.2 ([7]) Let E be a norm space and let C be a subset of E having Property G. Let $G=(V(G),A(G))$ be a directed graph such that $V(G)=C$ and $A(G)$ is convex. Suppose $T:C\to C$ is G-nonexpansive mapping and $F(T)\times F(T)\subseteq$ A(G). Then $F(T)$ is closed and convex.

Lemma 3.3 ([19]) Let the sequences $\{a_n\}$ and $\{\delta_n\}$ of real number be satisfied:

$$a_{n+1}\le(1+\delta_n)a_n,\ \ \textit{where}\ \ a_n\ge 0,\ \ \delta_n\ge 0,\ \ \forall n=1,2,3,...\ \text{and}\ \sum_{n=1}^{\infty}\delta_n<\infty.$$

Then (i) $\lim_{n\to\infty}a_n$ exists. (ii) If $\liminf_{n\to\infty}a_n=0$, then $\lim_{n\to\infty}a_n=0$.

Definition 3.4 Let C be a nonempty subset of a real Banach space E. If $F(T)$ is nonempty, then T is call semi-compact if for a bounded sequence $\{x_n\}$ in C with $\lim_{n\to\infty}\|x_n-Tx_n\|=0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}\to p\in C$.

Next, we introduce a new iterative scheme for finding a common fixed point of G-nonexpansive mappings in a real Banach space as follows.

Theorem 3.5

Let E be a real Banach space and let C be a nonempty closed convex subset of E endowed with a directed graph $G=(V(G),A(G))$ such that $V(G)=C$ and $A(G)$ is convex. The mappings $T_i\,(i=1,2,3)$ are G-nonexpansive from C into itself. Assume that $F:=\cap_{i=1}^{3}F(T_i)$ is nonempty and closed subset of C. For an initial point $x_0\in C$, define the sequence $\{x_n\}$ in C by the iterative schemes:

$$\begin{aligned} x_{n+1}&=(1-\alpha_{n,1})x_n+\alpha_{n,1}T_1y_n,\\ y_n&=(1-\alpha_{n,2})x_n+\alpha_{n,2}T_2z_n,\\ z_n&=(1-\alpha_{n,3})x_n+\alpha_{n,3}T_3x_n,\end{aligned}\tag{3.1}$$

Where $\{\alpha_{n,i}\}_{i=1}^{3}$ are real sequences in $[\alpha,1\text{-}\ \alpha]$ for some $\alpha\in(0,\frac{1}{2})$.

If the graph G is transitive and for $u\in F$ be such that $(x_0,u),(y_0,u),(z_0,u),(u,x_0),(u,y_0),(u,z_0)\in A(G)$. Then

(i) $(x_n,u),(y_n,u),(z_n,u),(u,x_n),(u,y_n),(u,z_n),(x_n,y_n),(y_n,z_n)$ and (x_n,z_n) are in $A(G)$.

(ii) $\lim_{n\to\infty}\|x_n-u\|$ exists.

(iii) $\lim_{n\to\infty}\|T_ix_n-x_n\|=0$, for $i=1,2,3$.

(iv) Assume that $\liminf_{n\to\infty}d(x_n,F)=0$, where $d(x,F)=\inf_{p\in F}\|x-p\|$. Then $\{x_n\}$ converges strongly to a common fixed point of F.

(v) Assume that T_i is semi-compact for some $i=1,2,3$. Then $\{x_n\}$ converges strongly to a common fixed point of F.

Proof. For $u\in F$, $(x_0,u),(y_0,u),(z_0,u)\in A(G)$ and $T_i\,(i=1,2,3)$ are edge-preserving. Then we have $(T_1y_0,u),(T_2z_0,u),(T_3x_0,u)\in A(G)$. By the convexity of $A(G)$ and $(T_1y_0,u),(x_0,u)\in A(G)$, we have $(x_1,u)\in A(G)$. By edge-preserving of T_3, then $(T_3x_1,u)\in A(G)$. By the convexity of $A(G)$ and $(T_3x_1,u),(x_1,u)\in A(G)$, we have $(z_1,u)\in A(G)$. By edge-preserving of T_2, then $(T_2z_1,u)\in A(G)$. By the convexity of $A(G)$ and $(T_2z_1,u),(x_1,u)\in A(G)$, we have $(y_1,u)\in A(G)$. For $(x_k,u),(y_k,u),(z_k,u)\in A(G)$. Since $A(G)$ is convex and $T_i,i=1,2,3$ are edge-preserving, then $(T_1y_k,u),(T_2z_k,u),(T_3x_k,u)\in A(G)$. Since (x_k,u) and $(T_1y_k,u)\in A(G)$ and $A(G)$ is convex, then we get

$$(1-\alpha_{k,1})(x_k,u)+\alpha_{k,1}(T_1y_k,u)=((1-\alpha_{k,1})x_k+\alpha_{k,1}T_1y_k,u)=(x_{k+1},u)\in A(G). \tag{3.2}$$

By edge-preserving of T_3, then $(T_3x_{k+1},u)\in A(G)$. Since $(T_3x_{k+1},u),(x_{k+1},u)\in A(G)$ and $A(G)$ is convex, we get

$$(1-\alpha_{k,3})(x_{k+1},u)+\alpha_{k,3}(T_3x_{k+1},u)=((1-\alpha_{k,3})x_{k+1}+\alpha_{k,3}T_3x_{k+1},u)=(z_{k+1},u)\in A(G). \tag{3.3}$$

By edge-preserving of T_2, then $(T_2z_{k+1},u)\in A(G)$. Since $(T_2z_{k+1},u),(x_{n+1},u)\in A(G)$ and $A(G)$ is convex, we get

$$(1-\alpha_{k,2})(x_{k+1},u)+\alpha_{k,2}(T_2z_{k+1},u)=((1-\alpha_{k,2})x_{k+1}+\alpha_{k,2}T_2z_{k+1},u)=(y_{k+1},u)\in A(G). \tag{3.4}$$

By induction, we get $(x_n,u),(y_n,u),(z_n,u)\in A(G)$ for all $n\geq 1$.

For $(u,x_0),(u,y_0),(u,z_0)\in A(G)$ by a similar argument that $(u,x_n),(u,y_n),(u,z_n)\in A(G)$. From the transitivity of G, that $(x_n,y_n),(y_n,z_n),(x_n,z_n)\in A(G)$.

Next, we show that $\lim_{n\to\infty}\|x_n-u\|$ exists.

For $u\in F$ and $(x_0,u),(y_0,u),(z_0,u),(u,x_0),(u,y_0),(u,z_0)\in A(G)$. Notice that

$$\begin{aligned}
\|x_{n+1}-u\| &= \|(1-\alpha_{n,1})x_n+\alpha_{n,1}T_1y_n-u\| \\
&\leq (1-\alpha_{n,1})\|x_n-u\|+\alpha_{n,1}\|T_1y_n-u\| \\
&\leq (1-\alpha_{n,1})\|x_n-u\|+\alpha_{n,1}\|y_n-u\| \\
&= (1-\alpha_{n,1})\|x_n-u\|+\alpha_{n,1}\|(1-\alpha_{n,2})x_n+\alpha_{n,2}T_2z_n-u\| \\
&\leq (1-\alpha_{n,1})\|x_n-u\|+\alpha_{n,1}\left((1-\alpha_{n,2})\|x_n-u\|+\alpha_{n,2}\|T_2z_n-u\|\right) \\
&\leq (1-\alpha_{n,1})\|x_n-u\|+\alpha_{n,1}\left((1-\alpha_{n,2})\|x_n-u\|+\alpha_{n,2}\|z_n-u\|\right) \\
&= (1-\alpha_{n,1})\|x_n-u\|+\alpha_{n,1}(1-\alpha_{n,2})\|x_n-u\|+\alpha_{n,1}\alpha_{n,2}\|z_n-u\| \qquad (3.5) \\
&= (1-\alpha_{n,1})\|x_n-u\|+\alpha_{n,1}(1-\alpha_{n,2})\|x_n-u\|+\alpha_{n,1}\alpha_{n,2}\|(1-\alpha_{n,3})x_n+\alpha_{n,3}T_3x_n-u\| \\
&\leq (1-\alpha_{n,1})\|x_n-u\|+\alpha_{n,1}(1-\alpha_{n,2})\|x_n-u\|+\alpha_{n,1}\alpha_{n,2}\left((1-\alpha_{n,3})\|x_n-u\|+\alpha_{n,3}\|T_3x_n-u\|\right) \\
&\leq (1-\alpha_{n,1})\|x_n-u\|+\alpha_{n,1}(1-\alpha_{n,2})\|x_n-u\|+\alpha_{n,1}\alpha_{n,2}\left((1-\alpha_{n,3})\|x_n-u\|+\alpha_{n,3}\|x_n-u\|\right) \\
&= (1-\alpha_{n,1})\|x_n-u\|+\alpha_{n,1}(1-\alpha_{n,2})\|x_n-u\|+\alpha_{n,1}\alpha_{n,2}\left(\|x_n-u\|\right) \\
&= (1-\alpha_{n,1})\|x_n-u\|+\alpha_{n,1}\|x_n-u\| \\
&= \|x_n-u\|.
\end{aligned}$$

It mean that, $\lim_{n\to\infty}\|x_n-u\|$ exists. Then $\{x_n\}$ is bounded.

Next, we show that $\lim_{n\to\infty}\|T_ix_n-x_n\|=0$, for $i=1,2,3$.

Let $u\in F$, there exists $r>0$ such that $\|x_n-u\|\leq r$, $\|y_n-u\|\leq r$ and $\|z_n-u\|\leq r$ for all $n\geq 1$.

Put $c=\lim_{n\to\infty}\|x_n-u\|$.

If $c=0$, then by G-nonexpansiveness of T_i, $(i=1,2,3)$ and $\lim_{n\to\infty}\|x_n-u\|$ exists, we have

$\|x_n-T_ix_n\|\leq\|x_n-u\|+\|u-T_ix_n\|\leq\|x_n-u\|+\|u-x_n\|$. Then $\lim_{n\to\infty}\|T_ix_n-x_n\|=0$.

If $c>0$, by Lemma 1.1 and T_3 is G-nonexpansive, we have

$$\begin{aligned}\|z_n-u\|^2 &= \|(1-\alpha_{n,3})x_n+\alpha_{n,3}T_3x_n-u\|^2\\ &= \|(1-\alpha_{n,3})(x_n-u)+\alpha_{n,3}(T_3x_n-u)\|^2\\ &\le (1-\alpha_{n,3})\|x_n-u\|^2+\alpha_{n,3}\|T_3x_n-u\|^2-\alpha_{n,3}(1-\alpha_{n,3})g(\|T_3x_n-x_n\|)\\ &\le (1-\alpha_{n,3})\|x_n-u\|^2+\alpha_{n,3}\|T_3x_n-u\|^2\\ &\le (1-\alpha_{n,3})\|x_n-u\|^2+\alpha_{n,3}\|x_n-u\|^2\\ &= \|x_n-u\|^2.\end{aligned}$$

We have

$$\limsup_{n\to\infty}\|z_n-u\|\le c. \tag{3.6}$$

Since T_2 is G-nonexpansive and $\|z_n-u\|^2\le\|x_n-u\|^2$, we have

$$\begin{aligned}\|y_n-u\|^2 &= \|(1-\alpha_{n,2})x_n+\alpha_{n,2}T_2z_n-u\|^2\\ &= \|(1-\alpha_{n,2})(x_n-u)+\alpha_{n,2}(T_2z_n-u)\|^2\\ &\le (1-\alpha_{n,2})\|x_n-u\|^2+\alpha_{n,2}\|T_2z_n-u\|^2-\alpha_{n,2}(1-\alpha_{n,2})g(\|T_2x_n-x_n\|)\\ &\le (1-\alpha_{n,2})\|x_n-u\|^2+\alpha_{n,2}\|T_2z_n-u\|^2\\ &\le (1-\alpha_{n,2})\|x_n-u\|^2+\alpha_{n,2}\|z_n-u\|^2\\ &\le (1-\alpha_{n,2})\|x_n-u\|^2+\alpha_{n,2}\|x_n-u\|^2\\ &= \|x_n-u\|^2.\end{aligned}$$

Thus

$$\limsup_{n\to\infty}\|y_n-u\|\le c. \tag{3.7}$$

Since T_1 is G-nonexpansive and $\|y_n-u\|^2\le\|x_n-u\|^2$, we have

$$\begin{aligned}\|x_{n+1}-u\|^2 &= \|(1-\alpha_{n,1})x_n+\alpha_{n,1}T_1y_n-u\|^2\\ &= \|(1-\alpha_{n,1})(x_n-u)+\alpha_{n,1}(T_1y_n-u)\|^2\\ &\le (1-\alpha_{n,1})\|x_n-u\|^2+\alpha_{n,1}\|T_1y_n-u\|^2-\alpha_{n,1}(1-\alpha_{n,1})g(\|T_1y_n-x_n\|)\\ &\le (1-\alpha_{n,1})\|x_n-u\|^2+\alpha_{n,1}\|y_n-u\|^2-\alpha_{n,1}(1-\alpha_{n,1})g(\|T_1y_n-x_n\|)\\ &\le (1-\alpha_{n,1})\|x_n-u\|^2+\alpha_{n,1}\|x_n-u\|^2-\alpha_{n,1}(1-\alpha_{n,1})g(\|T_1y_n-x_n\|)\\ &= \|x_n-u\|^2-\alpha_{n,1}(1-\alpha_{n,1})g(\|T_1y_n-x_n\|).\end{aligned}$$

Then we have that

$$\alpha_{n,1}(1-\alpha_{n,1})g(\|T_1y_n-x_n\|)\le\|(x_n-u)\|^2-\|x_{n+1}-u\|^2. \tag{3.8}$$

Since $\lim_{n\to\infty}\|x_n-u\|$ exists, we have that

$$\lim_{n\to\infty} g(\|T_1y_n-u\|)=0.$$

Thus

$$\lim_{n\to\infty}\|T_1y_n-x_n\|=0. \tag{3.9}$$

Since T_1 is G-nonexpansive, we have

$$\|x_n-u\|\le\|x_n-T_1y_n\|+\|T_1y_n-T_1u\|\le\|x_n-T_1y_n\|+\|y_n-u\|.$$

From (3.9), we have that

$$\lim_{n\to\infty}\|x_n-u\|\le\lim_{n\to\infty}((\|x_n-T_1y_n\|)+\|y_n-u\|)=\lim_{n\to\infty}\|y_n-u\|.$$

Then

$$c \le \liminf_{n\to\infty} \|y_n - u\|. \tag{3.10}$$

From (3.7) and (3.10), we have

$$\lim_{n\to\infty} \|y_n - u\| = c. \tag{3.11}$$

Since

$$\begin{aligned}
\|y_n - u\|^2 &= \|\alpha_{n,2} T_2 z_n + (1-\alpha_{n,2}) x_n - u\|^2 \\
&= \|\alpha_{n,2}(T_2 z_n - u) + (1-\alpha_{n,2}) x_n - u\|^2 \\
&\le \alpha_{n,2}\|T_2 z_n - u\|^2 + (1-\alpha_{n,2})\|x_n - u\|^2 - \alpha_{n,2}(1-\alpha_{n,2}) g\|T_2 z_n - x_n\| \\
&\le \alpha_{n,2}\|z_n - u\|^2 + (1-\alpha_{n,2})\|x_n - u\|^2 - \alpha_{n,2}(1-\alpha_{n,2}) g\|T_2 z_n - x_n\| \\
&\le \alpha_{n,2}\|x_n - u\|^2 + (1-\alpha_{n,2})\|x_n - u\|^2 - \alpha_{n,2}(1-\alpha_{n,2}) g\|T_2 z_n - x_n\| \\
&= \|x_n - u\|^2 - \alpha_{n,2}(1-\alpha_{n,2}) g\|T_2 z_n - x_n\|.
\end{aligned}$$

It imply that $\alpha_{n,2}(1-\alpha_{n,2}) g\|T_2 z_n - x_n\| \le \|x_n - u\|^2 - \|y_n - u\|^2$.

Since $\lim_{n\to\infty}\|x_n - u\| = c$ and from (3.11), then we have $\lim_{n\to\infty} g\|T_2 z_n - x_n\| = 0$, we get

$$\lim_{n\to\infty} \|T_2 z_n - x_n\| = 0. \tag{3.12}$$

Since $\|x_n - u\| \le \|x_n - T_2 z_n\| + \|T_2 z_n - T_2 u\| \le \|x_n - T_2 z_n\| + \|z_n - u\|$.

From (3.12), we get $\lim_{n\to\infty}\|z_n - u\| \ge \lim_{n\to\infty}\|x_n - u\|$, thus

$$\liminf_{n\to\infty} \|z_n - u\| \ge c. \tag{3.13}$$

From (3.6) and (3.13), that

$$\lim_{n\to\infty} \|z_n - u\| = c. \tag{3.14}$$

Since

$$\begin{aligned}
\|z_n - u\|^2 &= \|(1-\alpha_{n,3}) x_n + \alpha_{n,3} T_3 x_n - u\|^2 \\
&= \|(1-\alpha_{n,3})(x_n - u) + \alpha_{n,3}(T_3 x_n - u)\|^2 \\
&\le (1-\alpha_{n,3})\|x_n - u\|^2 + \alpha_{n,3}\|T_3 x_n - u\|^2 - \alpha_{n,3}(1-\alpha_{n,3}) g(\|T_3 x_n - x_n\|) \\
&\le (1-\alpha_{n,3})\|x_n - u\|^2 + \alpha_{n,3}\|x_n - u\|^2 - \alpha_{n,3}(1-\alpha_{n,3}) g(\|T_3 x_n - x_n\|) \\
&= \|x_n - u\|^2 - \alpha_{n,3}(1-\alpha_{n,3}) g(\|T_3 x_n - x_n\|).
\end{aligned}$$

Then $\alpha_{n,3}(1-\alpha_{n,3}) g(\|T_3 x_n - x_n\|) \le \|x_n - u\|^2 - \|z_n - u\|^2$.

Since $\lim_{n\to\infty}\|x_n - u\| = c$ and $\lim_{n\to\infty}\|z_n - u\| = c$, we get $\lim_{n\to\infty} g(\|T_3 x_n - x_n\|) = 0$. From Lemma 1.2, we get

$$\lim_{n\to\infty} \|T_3 x_n - x_n\| = 0. \tag{3.15}$$

Since T_2 is G-nonexpansive,

$$\begin{aligned}
\|T_2 x_n - x_n\| &\le \|T_2 x_n - T_2 z_n\| + \|T_2 z_n - x_n\| \le \|x_n - z_n\| + \|T_2 z_n - x_n\| \\
&= \alpha_{n,3}\|T_3 x_n - x_n\| + \|T_2 z_n - x_n\| \le \|T_3 x_n - x_n\| + \|T_2 z_n - x_n\|.
\end{aligned}$$

From $\lim_{n\to\infty}\|T_2 z_n - x_n\| = 0$ and $\lim_{n\to\infty}\|T_3 x_n - x_n\| = 0$, we have

$$\lim_{n\to\infty} \|T_2 x_n - x_n\| = 0. \tag{3.16}$$

Since T_1 is G-nonexpansive,

$$\begin{aligned}
\|T_1 x_n - x_n\| &\le \|T_1 x_n - T_1 y_n\| + \|T_1 y_n - x_n\| \le \|x_n - y_n\| + \|T_1 y_n - x_n\| \\
&= \alpha_{n,2}\|T_2 x_n - x_n\| + \|T_2 y_n - x_n\| \le \|T_2 x_n - x_n\| + \|T_2 y_n - x_n\|.
\end{aligned}$$

From $\lim_{n\to\infty}\|T_2x_n - x_n\| = 0$ and $\lim_{n\to\infty}\|T_2y_n - x_n\| = 0$, we have

$$\lim_{n\to\infty}\|T_1x_n - x_n\| = 0.$$

From the assumption $\liminf_{n\to\infty} d(x_n, F) = 0$ and from (3.5), we get $d(x_{n+1}, F) \le d(x_n, F)$. By using Lemma 3.3 (ii), we get $\lim_{n\to\infty} d(x_n, F) = 0.$ Since $\lim_{n\to\infty}\|x_n - u\|$ exists and $\{x_n\}$ is bounded, we get that, for any $\varepsilon > 0,$ there exists a positive integer n_0 such that, $\|x_n - u\| < \frac{\varepsilon}{2}$ for $n \ge n_0$. For any positive integer $m,$ we obtain

$$\|x_{n+m} - x_n\| \le \|x_{n+m} - u\| + \|u - x_n\| = \|x_{n+m} - u\| + \|x_n - u\| \le \|x_n - u\| + \|x_n - u\| \le 2\|x_n - u\| \le \varepsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence in E. Since E is complete, then there exists an element $q \in E$ such that $\{x_n\} \to q$. Because $\{x_n\} \subset C$ and C is closed subset of $E,$ then $q \in C.$ Since $F := \cap_{i=1}^{3} F(T_i)$ and $F(T_i)$ is a closed subset in C for all $i = 1,2,3$ and from the continuity of $d(x,\mathrm{F})$ with $d(x_nF) \to 0$ and $x_n \to q$ as $n \to \infty$, we have $d(q, F) = 0$. It follow that $q \in F$. From the assumption T_i is semi-compact for some $i = 1,2,3$ and from $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to q \in C$ as $j \to \infty.$ By continuity of $T_i,$ we get $\lim_{n\to\infty}\|T_ix_{n_j} - x_{n_j}\| = \|q - T_iq\|$ for all $i = 1,2,3$ then $q \in F.$ The proof is complete.

Remark If $\alpha_{n,3} \equiv 0,$ then Theorem 3.5 reduces to the results of Tripak [15].

Discussion

The result of this paper holds under the assumptions that the some mapping is semi-compact and $\liminf_{n\to\infty} d(x_n, F) = 0.$ We construct and prove a new convergence theorem for G-nonexpansive mappings in a real Banach space endowed with a graph.

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References

[1] Banach, S. (1922). Sur les Operations Dans Les Ensembles Abstraits Et Leur Application Aux Equationa Integrals. *Fundamenta Mathematicae*, *3*(1), 133-181.

[2] Takahashi, W. (2000). *Nonlinear Functional Analysis: Fixed Point Theory and Its Application*. Yokohama: Yokohama - Publishers, Inc.

[3] Xu, H.K. (1991). Inequalities in Banach Spaces with Applications. *Nonlinear Analysis*, *16*(12), 1127-1138.

[4] Schu, J. (1991). Weak and Strong Convergence to Fixed Points of Asymptotically Nonexpansive Mappings. *Bulletin of the Australian Mathematical Society*, *43*(1), 153-159.

[5] Reich, S. (1996). A Weak Convergence Theorem for the Alternating Method with Bregman Distance, In A.G. Kartsatos (Ed.). *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*. 313-318. Marcel Dekker, New York.

[6] Alfuraidan, M.R. (2015). Fixed Points of Monotone Nonexpansive Mappings with a Graph. *Fixed Point Theory and Applications*, *2015*(2015), 49.

[7] Tiammee, J., Kaewkhao, A. & Suantai, S. (2015). On Browder's Convergence Theorem and Halpern Iteration Process for G-Nonexpansive Mappings in Hilbert Spaces Endowed with Graphs. *Fixed Point Theory and Applications*, *2015* (2015), 187.

[8] XU, B.L. & Noor, M.A. (2002). Fixed Point Iterations for Asymptotically Nonexpansive Mappings in Banach Spaces. *Journal of Mathematical Analysis and Applications*, *267*(2002), 444-453.

[9] Shahzad, N. & Al-Dubiban, R. (2006). Approximating Common Fixed Points of Nonexpansive Mappings in Banach Spaces. *Georgian Mathematical Journal*, *13*(3), 529-537.

[10] Nakajo, K. & Takahashi, W. (2003). Strongly Convergence Theorems for Nonexpansive Mappings and Nonexpansive Semigroups. *Journal of Mathematical Analysis and Applications*, *279*(2003), 372–379.

[11] Chairat, S., Noisri, C., Kanjanasamranwong, P. & Saewan, S. (2016). A Convergence Theorem of Variational Inequality Problems and Fixed Point Problems of Multi-Valued Mappings. *Thaksin University Journal*, *19*(1), 1-11.

[12] Khongtham, Y. (2017). Fixed Point Theorems for Compatible Maps and Commuting Maps in Rectangular B-Metric Spaces, In *27th Thaksin University National Academic Conference*, 155. May 3-6, 2017. BP Samila Beach Hotel & Resort. Songkhla: Thaksin University.

[13] Jachymski, J. (2008). The Contraction Principle for Mappings on a Metric Space with a Graph. *Proceedings of the American Mathematical Society*, *136*(4), 1359–1373.

[14] Jhade, P.K. & Saluja, A.S. (2014). On The Convergence of Three Step Iterative Process for Three Asymptotically Nonexpansive Multi-Maps in Banach Spaces. *Acta Universitatis Matthiae Belii, series Mathematics*, *22*(2014), 23–34.

[15] Tripak, O. (2016). Common Fixed Points of G-Nonexpansive Mmppings on Banach Spaces with a Graph. *Fixed Point Theory and Application*, 87. *2016*(2016), 87.

[16] Alfuraidan, M. R. (2015). Fixed Points of Monotone Nonexpansive Mappings with a Graph. *Fixed Point Theory and Application*, *2015*(2015), 49.

[17] Suparatulatorna, R., Suantai, S. & Cholamjiak, W. (2017). Hybrid Methods for A Finite Family of G-Nonexpansive Mappings in Hilbert Spaces Endowed with Graphs. *AKCE International Journal of Graphs and Combinatorics*, *14* (2017), 101–111.

[18] Saewan. S., Noisri, C. & Kanjanasamranwong, P. (2018). Fixed Points of G-Nonexpansive Mappings in Banach Spaces. In *28th Thaksin University National Academic Conference*. 1097. May 8-9, 2018. BP Samila Beach Hotel & Resort. Songkhla: Thaksin University.

[19] Sun, Z.H. (2003). Strong Convergence of an Implicit Iteration Process for a Finite Family of Asymptotically Quasi -Nonexpansive Mappings. *Journal of Mathematical Analysis and Applications*, *286* (2003), 351–358.