



Numerical Solutions of Fractional Integro-Differential Equations with Weakly Singular Kernel by Using a Hybrid of Block-Pulse Functions and Taylor Polynomials

Weerachai Thadee¹, Onuma Ruksachol², Walisa Intarapak², Rattiya Rittichuai², and Nattinee Khongnual^{2*}

¹ Faculty of Liberal Arts, Rajamangala University of Technology Srivijaya, Songkhla, 90000, Thailand

² Faculty of Science and Technology, Nakhon Si Thammarat Rajabhat University, Nakhon Si Thammarat, 80280, Thailand

* Correspondence: nattinee_kho@nstru.ac.th

Citation:

Thadee, W.; Ruksachol, O.; Intarapak, W.; Rittichuai, R.; Khongnual, N. Numerical solutions of fractional integro-differential equations with weakly singular kernel by using a hybrid of block-pulse functions and Taylor polynomials. *ASEAN J. Sci. Tech. Report.* **2026**, 29(7), e262844. <https://doi.org/10.55164/ajstr.v29i7.262844>

Article history:

Received: April 3, 2026

Revised: April 11, 2026

Accepted: April 30, 2026

Available online: June 14, 2026

Publisher's Note:

This article is published and distributed under the terms of the Thaksin University.

Abstract: Fractional integro-differential equations (FIDEs) with weakly singular kernels arise in mathematical models of viscoelasticity, anomalous diffusion, and hereditary systems, yet closed-form solutions are rarely attainable. This study proposes a numerical scheme based on a hybrid combination of block-pulse functions and Taylor polynomials for solving both linear and nonlinear FIDEs with weakly singular kernels of the form $(t-s)^{-\beta}$, $0 < \beta < 1$, formulated within the Caputo fractional derivative framework. The fractional problem is transformed into an equivalent system of algebraic equations by constructing a Riemann–Liouville operational matrix and applying Newton–Cotes collocation nodes, yielding a computationally straightforward solver. Three illustrative examples were examined, including two linear equations with exact solutions $u(t) = t^2 + t^3$ and $u(t) = -t + t^2$, and one nonlinear equation with an exact solution $u(t) = t^2$. The proposed method achieved absolute errors below 1.0×10^{-4} across all test points using $N = 1, 2$ block-pulse intervals and $M = 2, 3$ Taylor polynomial terms. Comparison with the Legendre wavelet method (LWM) demonstrates that the present scheme matches or surpasses LWM accuracy at most collocation points while requiring fewer basis functions. These results confirm that the hybrid block-pulse–Taylor polynomial approach offers a reliable and efficient alternative for solving FIDEs encountered in applied science and engineering.

Keywords: Hybrid functions; fractional differential equations; Taylor polynomials; integro-differential equations; weakly singular kernel

1. Introduction

In general calculus, derivatives of order n are considered only for the particular case when n is a positive integer. The next logical direction is to explore generalized definitions of derivatives and integrals that are valid for arbitrary real orders. This area of mathematics is known as fractional calculus, with origins traceable to 1695, when L'Hospital posed the question to Leibniz:

'What could $\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} f(x)$ mean?' For more than two centuries, this subject remained

largely within pure mathematics. Pioneering mathematicians such as Euler, Fourier, Abel, Liouville, Riemann, and Hadamard contributed significantly to

the early development of fractional operators by introducing various definitions and examining their fundamental properties. In recent decades, however, fractional calculus has expanded considerably, demonstrating applicability across a wide range of natural and applied sciences.

The growing significance of fractional differential equations (FDEs) in various fields of science and engineering has led to increasing research interest [1–12]. Fractional calculus provides a powerful framework for modelling memory effects and hereditary properties in complex systems, making it particularly valuable in applications ranging from viscoelasticity to anomalous diffusion processes [13–15]. Distributed-order fractional derivatives refer to fractional derivatives integrated over a continuum of derivative orders within a specified range [6, 16]. Since analytical solutions to many fractional differential problems are often unavailable, considerable research has been devoted to developing efficient numerical and analytical methods [1, 8, 10, 11, 17–19]. Numerous researchers have developed numerical methods for solving fractional integro-differential equations (FIDEs), which combine the complexities of both fractional derivatives and integral operators [20–24]. In cases where exact analytical solutions are unattainable, attention has been directed toward accurate approximate solutions using various computational techniques [7, 25–27]. For instance, Mingxu et al. [1] employed the Legendre wavelet approach to investigate numerical solutions of FIDEs with weakly singular kernels, transforming the original equations into an algebraic system for efficient computation. More recently, several wavelet-based methods have been proposed, including Haar wavelets [7], Bell wavelets [28], and modified hat functions [29].

The challenge of handling weakly singular kernels in FIDEs has attracted significant attention due to their prevalence in modelling real-world phenomena [30–32]. These kernels, typically of the form $(t-s)^{-\beta}$ where $0 < \beta < 1$, introduce computational difficulties requiring specialized numerical treatments [33, 34]. In this research, we develop a numerical method for solving linear and nonlinear FIDEs with a weakly singular kernel as described by Equations (1) and (2), respectively, subject to appropriate initial conditions. The method applies to both linear and nonlinear problems.

$$D^\alpha f(t) = \lambda_1 \int_0^t \frac{f(x)}{(t-x)^\gamma} dx + \lambda_2 \int_0^1 u(t,x) f(x) dx + g(t) \quad (1)$$

and

$$D^\alpha f(t) = \lambda_1 \int_0^t \frac{f(x)^p}{(t-x)^\gamma} dx + \lambda_2 \int_0^1 u(t,x) f(x)^q dx + g(t) \quad (2)$$

2. Materials and Methods

2.1 Fractional Integral and Derivative

In this section, we present the fundamental definitions, notations, and theoretical concepts related to fractional integrals and derivatives used throughout this study.

Definition 1 (Riemann–Liouville Fractional Integral). The integral operator defined by Equation (3) is called the Riemann–Liouville fractional integral operator of order α [5].

$$I^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(x)}{(t-x)^{1-\alpha}} dx, & \alpha > 0 \\ f(t) & \alpha = 0. \end{cases} \quad (3)$$

For any constants λ and μ , the operator is linear, as expressed in Equation (4).

$$I^\alpha (\lambda f(t) \pm \mu g(t)) = \lambda I^\alpha f(t) \pm \mu I^\alpha g(t) \quad (4)$$

Definition 2 (Caputo Fractional Derivative). The derivative defined by Equation (5) is called the Caputo fractional derivative of order α , where n is the smallest integer such that $n-1 < \alpha < n$ [2–4].

$$(D^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(x)}{(t-x)^{\alpha+1-n}} dx, \quad n-1 < \alpha < n, \quad n \in \mathbb{N}, \quad (5)$$

For any constants, the Caputo derivative is a linear operation, as shown in Equation (6). From Definitions 1 and 2, the composition property is given by Equation (7), and additional key properties are provided as Equations (8) and (9).

$$D^\alpha (\lambda f(t) \pm \mu g(t)) = \lambda D^\alpha f(t) \pm \mu D^\alpha g(t). \quad (6)$$

$$I^\alpha (D^\alpha h(t)) = h(t) - \sum_{k=0}^{n-1} h^{(k)}(0) \frac{t^k}{k!} \quad (7)$$

For any real number α such that $n-1 < \alpha \leq n$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} I^{n-\alpha} D^n h(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{D^n(h(x))}{(t-x)^{1-(n-\alpha)}} dx \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{D^n(h(x))}{(t-x)^{\alpha+1-n}} dx \\ &= D^\alpha h(t) \\ I^{n-\alpha} D^n h(t) &= D^\alpha h(t) \end{aligned} \quad (8)$$

$$D^\beta \bar{B}(t, \alpha) = D^\beta I^\beta I^{\alpha-\beta} B(t) = \bar{B}(t, \alpha - \beta). \quad (9)$$

2.2 Hybrid of Block-Pulse Functions and Taylor Polynomials

2.2.1 Definition of Hybrid Functions

The hybrid functions denoted by $b_{nm}(t)$ are defined by Equation (10), where n is the order of block-pulse functions such that $T_m(t) = t^m$, $n=1, 2, \dots, N$ and $m=0, 1, \dots, M-1$, defined on the interval $[0, t_f]$. The hybrid of block-pulse functions and Taylor polynomial functions constitutes a semi-orthogonal set on the interval $[0, t_f]$.

$$b_{nm}(t) = \begin{cases} T_m(Nt - (n-1)t_f) & , t \in \left[\frac{n-1}{N}t_f, \frac{n}{N}t_f \right) \\ 0 & , \text{otherwise.} \end{cases} \quad (10)$$

For the four elements ($N=2, M=2$), the basis functions are explicitly expressed.

2.2.2 Function Approximation

A function $f(t)$ defined over $[0, t_f]$ may be expanded as a series in hybrid functions as Equation (11). Truncating this expansion yields the approximation given in Equation (12).

$$f(t) \cong \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} b_{nm}(t), \quad (11)$$

where $C^T = [c_{10}, c_{20}, \dots, c_{N0}, c_{11}, c_{21}, \dots, c_{N1}, \dots, c_{1M-1}, c_{2M-1}, \dots, c_{NM-1}]$,

and
$$B^T(t) = [b_{10}(t), b_{20}(t), \dots, b_{N0}(t), b_{11}(t), b_{21}(t), \dots, b_{N1}(t), \dots, b_{1M-1}(t), b_{2M-1}(t), \dots, b_{NM-1}(t)]. \quad (12)$$

2.3 Riemann–Liouville Fractional Integral Operator for Hybrid Functions

The operational matrix for the Riemann–Liouville fractional integral of hybrid functions is introduced in Equation (13) and obtained via the Laplace transform applied to each basis function, as shown in Equation (14).

$$I^\alpha B(t) = \bar{B}(t, \alpha), \quad (13)$$

$$b_{nm}(t) = \mu_{\frac{n-1}{N}t_f}(t) T_m(Nt - (n-1)t_f) - \mu_{\frac{n}{N}t_f}(t) T_m(Nt - (n-1)t_f), \quad (14)$$

2.4 Numerical Scheme

Without loss of generality, assume the initial condition $\beta \geq \alpha$. Expanding $D^\beta f(t)$ in hybrid functions as Equation (15) and applying Equations (7) and (13) yields the fractional derivative approximation Equation (16).

$$D^\beta f(t) = C^T B(t). \quad (15)$$

$$f(t) = C^T \bar{B}(t, \beta) + \sum_{k=0}^{\beta-1} \frac{f^{(k)}(0) t^k}{k!}. \quad (16)$$

Substituting into Equation (1) and collocating at Newton–Cotes nodes t_i as defined by Equation (18) converts the integral equation into an algebraic system (Equation 17), from which the coefficient vector C , and hence the numerical solution $f(t)$, is obtained.

$$\begin{aligned} C^T \bar{B}(t, \beta - \alpha) + \sum_{k=0}^{\beta-1} D^\alpha(t^k) \frac{f^{(k)}(0)}{k!} &= \lambda_1 \int_0^1 \frac{C^T \bar{B}(x, \beta) + \sum_{k=0}^{\beta-1} \frac{f^{(k)}(0) x^k}{k!}}{(t-x)^\beta} + \lambda_2 \int_0^1 u(t, x) C^T \bar{B}(t, \beta) \\ &+ \sum_{k=0}^{\beta-1} \frac{f^{(k)}(0) t^k}{k!} dx + g(t). \end{aligned} \quad (17)$$

$$t_i = \frac{2i+1}{2N(M+1)}, \quad i = 0, 1, \dots, N(M+1)-1. \quad (18)$$

3. Results and Discussion

To assess the performance and efficiency of the proposed method, three illustrative examples are presented: two linear FIDEs and one nonlinear FIDE with weakly singular kernels. Results are compared with the Legendre wavelet method (LWM) [1] and exact solutions.

3.1 Example 1 – Linear FIDE (N = 1, M = 3)

Consider the linear fractional integro-differential equation given by Equation (19), subject to initial conditions $f(0) = 0$ and Equation (20). The equation has a unique exact solution $u(t) = t^2 + t^3$.

$$D^{0.25} f(t) = \frac{1}{2} \int_0^t \frac{f(x)}{(t-x)^{\frac{1}{2}}} dx + \frac{1}{3} \int_0^1 (t-x) f(x) dx + g(t) \quad (19)$$

$$g(t) = \frac{\Gamma(3)}{\Gamma(2.75)} t^{1.75} + \frac{\Gamma(4)}{\Gamma(3.75)} t^{2.75} - \frac{\sqrt{\pi} t^{\frac{5}{2}} \Gamma(3)}{2\Gamma(\frac{7}{2})} - \frac{\sqrt{\pi} t^{\frac{7}{2}} \Gamma(4)}{2\Gamma(\frac{9}{2})} - \frac{7t}{36} + \frac{3}{20}. \quad (20)$$

Applying the hybrid scheme with $N=1$ and $M=3$, the coefficient vector C is obtained by collocating Equation (21) at Newton–Cotes nodes $t_0 = \frac{1}{8}, t_1 = \frac{3}{8}, t_2 = \frac{5}{8}$ and $t_3 = \frac{7}{8}$, we obtain

$c_{10} = -2.4978 \times 10^{-17}, c_{11} = 2$ and $c_{12} = 3$. The numerical solution is then reconstructed via Equation (22).

$$C^T \bar{B}(t, 0.75) = \frac{1}{2} \int_0^t \frac{C^T \bar{B}(x, 1)}{(t-x)^{\frac{1}{2}}} dx + \frac{1}{3} \int_0^1 (t-x) C^T \bar{B}(x, 1) dx + g(t). \quad (21)$$

$$f(t) = C^T \bar{B}(t, 1) = t^2 + t^3, \quad (22)$$

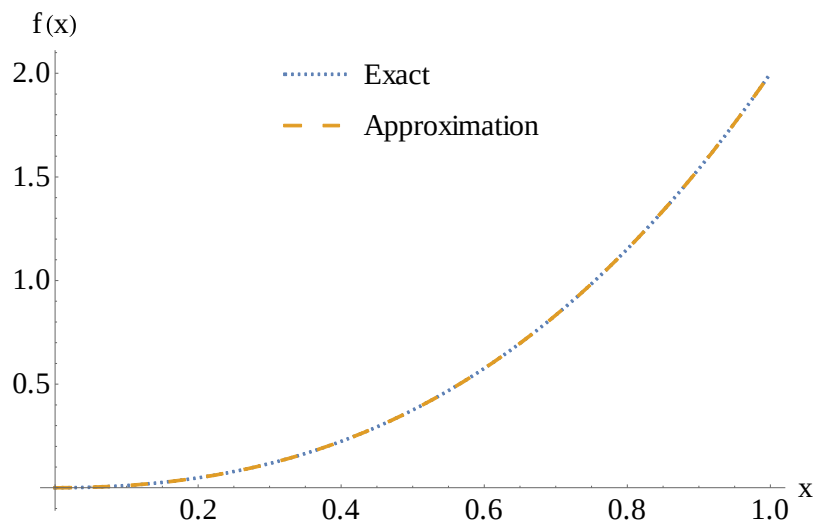


Figure 1. Comparison of exact solution and approximate solution with $N = 1$ and $M = 3$ (Example 1).

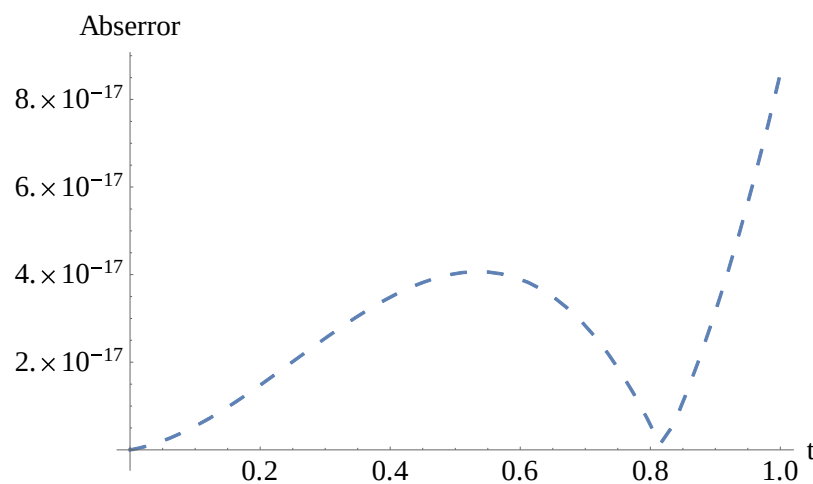


Figure 2. Absolute errors with $N = 1$ and $M = 3$ (Example 1).

3.2 Example 2 – Linear FIDE (N = 1, M = 2)

Consider the linear fractional integro-differential equation given by Equation (23), subject to conditions in Equation (24). The unique exact solution is $u(t) = -t + t^2$. Applying the hybrid scheme with $N = 1$ and $M = 2$ and collocating at Newton–Cotes nodes $t_0 = \frac{1}{6}, t_1 = \frac{3}{6}$ and $t_2 = \frac{5}{6}$, we obtain $c_{10} = -1, c_{11} = 2$. Table 1 presents numerical results at selected collocation points for $N = 1$ and $M = 2$ in Example 2.

$$D^{0.15} f(t) = \frac{1}{4} \int_0^t \frac{f(x)}{(t-x)^{\frac{1}{2}}} dx + \frac{1}{7} \int_0^1 e^{t+x} f(x) dx + g(t) \quad (23)$$

$$g(t) = \frac{\Gamma(3)}{\Gamma(2.85)} t^{1.85} + \frac{\Gamma(2)}{\Gamma(1.85)} t^{0.85} - \frac{\sqrt{\pi} t^{\frac{5}{2}} \Gamma(3)}{4\Gamma\left(\frac{7}{2}\right)} + \frac{\sqrt{\pi} t^{\frac{3}{2}} \Gamma(2)}{4\Gamma\left(\frac{5}{2}\right)} - \frac{e^{t+1} - 3e^t}{7} \quad (24)$$

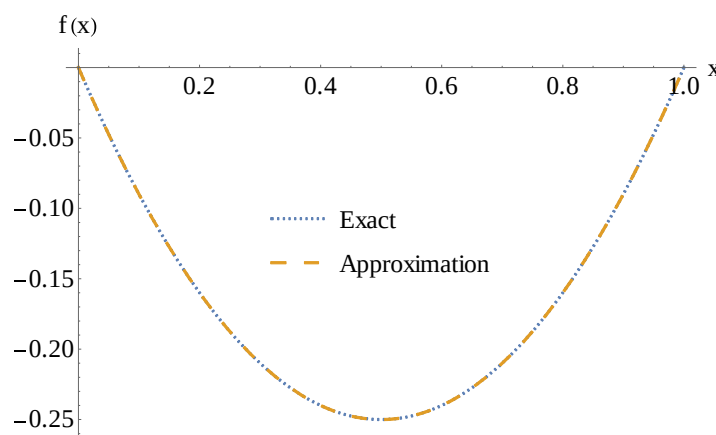


Figure 3. Comparison of exact solution and approximate solution with $N = 1$ and $M = 2$ (Example 2).

Table 1. Approximate solution at selected points for $N = 1$ and $M = 2$ (Example 2).

t	LWM	Present Method	Exact Solution
0	0.0000	0.0000	0.0000
1/8	-0.1090	-0.1094	-0.1094
2/8	-0.1869	-0.1875	-0.1875
3/8	-0.2338	-0.2344	-0.2344
4/8	-0.2498	-0.2500	0.2500
5/8	-0.2337	-0.2344	-0.2344
6/8	-0.1869	-0.1875	-0.1875
7/8	-0.1084	-0.1094	-0.1084

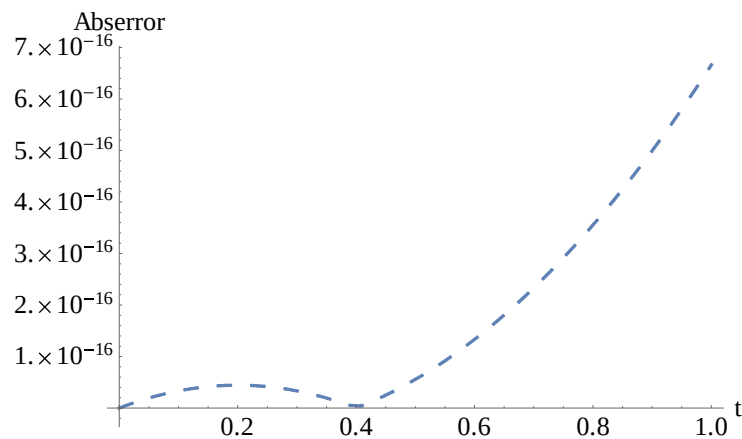


Figure 4. Absolute errors with $N = 1$ and $M = 2$ (Example 2).

3.3 Example 3 – Nonlinear FIDE ($N = 2$, $M = 2$)

Consider the nonlinear fractional integro-differential equation given by Equation (25), subject to conditions in Equation (26). The unique exact solution is $u(t) = t^2$. Applying the hybrid scheme with $N = 2$ and $M = 2$ and collocating at Newton–Cotes nodes produces the following numerical solution.

$$D^{0.5} f(t) = \int_0^x \frac{[f(x)]^2}{(t-x)^{\frac{1}{2}}} dx + \int_0^1 (t + \sin x) [f(x)]^3 dx + g(t) \quad (25)$$

$$g(t) = \frac{\Gamma(3)}{\Gamma(2.5)} t^{1.5} - \frac{\sqrt{\pi} t^{\frac{9}{2}} \Gamma(5)}{\Gamma\left(\frac{11}{2}\right)} - \frac{t}{7} - 389 \cos 1 - 606 \sin 1 + 720. \quad (26)$$

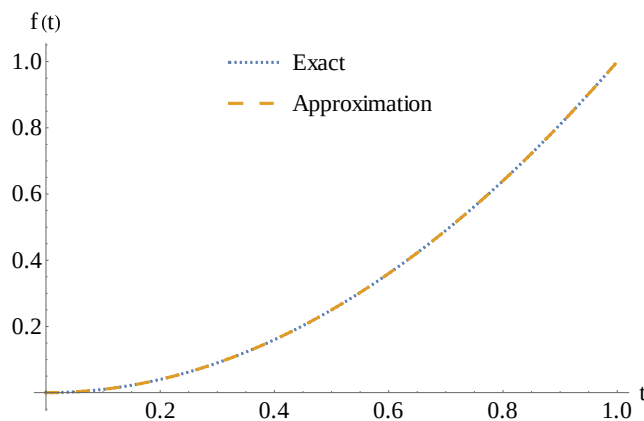


Figure 5. Comparison of exact solution and approximate solution with $N = 2$ and $M = 2$ (Example 3, nonlinear).

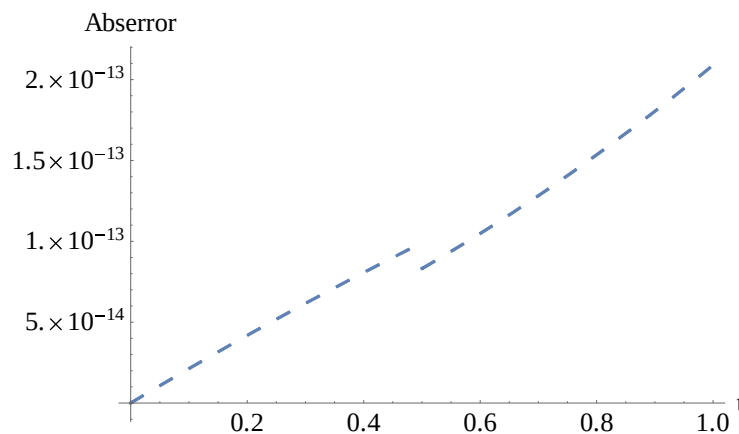


Figure 6. Comparison of absolute errors between the proposed method and exact solution with $N = 2$ and $M = 2$ (Example 3, nonlinear).

4. Conclusions

In this study, a numerical method based on a hybrid combination of block-pulse functions and Taylor polynomials was proposed for solving linear and nonlinear fractional integro-differential equations (FIDEs) with weakly singular kernels, formulated within the Caputo fractional derivative framework. By transforming the fractional problem into an equivalent algebraic system via the Riemann–Liouville operational matrix and Newton–Cotes collocation, three illustrative examples were examined. The results demonstrate that the proposed technique yields solutions in excellent agreement with the exact solutions while achieving lower absolute errors than the Legendre wavelet method (LWM).

5. Acknowledgments

The author would like to express sincere gratitude to the Faculty of Science and Technology, Nakhon Si Thammarat Rajabhat University, for their valuable support and encouragement in conducting this research. Their continuous support has significantly contributed to the successful completion of this study.

Author Contributions: Conceptualization, N.K. and W.T.; methodology, N.K. and W.T.; software, O.R. and W.I.; formal analysis, N.K., O.R., W.I., and R.R.; investigation, all authors; writing — original draft preparation, N.K. and W.T.; writing — review and editing, N.K.; visualization, O.R.; supervision, N.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Conflicts of Interest: The authors declare no conflict of interest.

References

- [1] Mingxu, Y.; Lifeng, W.; Jun, H. Legendre Wavelets Method for the Numerical Solution of Fractional Integro-Differential Equations with Weakly Singular Kernel. *Appl. Math. Modell.* **2016**, *40*, 3422–3437. <https://doi.org/10.1016/j.apm.2015.10.055>
- [2] Garrappa, R. Numerical Solution of Fractional Differential Equations: A Survey and a Software Tutorial. *Mathematics* **2018**, *6*(2), 16. <https://doi.org/10.3390/math6020016>
- [3] Yumei, Z.; Guoping, H. On the Uniqueness of Solutions for a Class of Fractional Differential Equations. *Appl. Math. Lett.* **2017**, *74*, 68–73. <https://doi.org/10.1016/j.aml.2017.05.002>
- [4] Chen, Y.-M.; Wang, Y.-Q.; Liu, D.-Y.; Yang, H. Numerical Solution for a Class of Nonlinear Variable-Order Fractional Differential Equations with Legendre Wavelets. *Appl. Math. Lett.* **2015**, *46*, 83–88. <https://doi.org/10.1016/j.aml.2015.01.012>

- [5] Sun, H.; Chen, A.; Zhang, Y.; Chen, W. A Review on Variable-Order Fractional Differential Equations: Mathematical Foundations, Physical Models, Numerical Methods and Applications. *Fract. Calc. Appl. Anal.* **2019**, *22*(1), 27–59. <https://doi.org/10.1515/fca-2019-0003>
- [6] Nintanavongsa, J.; Boonklurb, Y.; Razzaghi, M. An Efficient Method for Numerical Solutions of Distributed-Order Fractional Differential Equations. *J. Comput. Nonlinear Dyn.* **2018**, *13*(11), 111001. <https://doi.org/10.1115/1.4040495>
- [7] Amin, R.; Shah, K.; Asif, M.; Khan, I.; Ullah, F. An Efficient Algorithm for Numerical Solution of Fractional Integro-Differential Equations via Haar Wavelet. *J. Comput. Appl. Math.* **2021**, *381*, 113028. <https://doi.org/10.1016/j.cam.2020.113028>
- [8] Ravichandran, C.; Logeswari, K.; Jarad, F. New Results on Existence in the Framework of Atangana–Baleanu Derivative for Fractional Integro-Differential Equations. *Chaos Solitons Fractals* **2019**, *125*, 194–200. <https://doi.org/10.1016/j.chaos.2019.05.029>
- [9] Aghajani, A.; Jalilian, Y.; Trujillo, J. J. On the Existence of Solutions of Fractional Integro-Differential Equations. *Fract. Calc. Appl. Anal.* **2012**, *15* (1), 44–69. <https://doi.org/10.2478/s13540-012-0004-8>
- [10] Ma, X.; Huang, C. Numerical Solution of Fractional Integro-Differential Equations by a Hybrid Collocation Method. *Appl. Math. Comput.* **2013**, *219*(12), 6750–6760. <https://doi.org/10.1016/j.amc.2013.01.016>
- [11] Eslahchi, M. R.; Dehghan, M.; Parvizi, M. Application of the Collocation Method for Solving Nonlinear Fractional Integro-Differential Equations. *J. Comput. Appl. Math.* **2014**, *257*, 105–128. <https://doi.org/10.1016/j.cam.2013.08.021>
- [12] Sene, N. Fundamental Results about the Fractional Integro-Differential Equation Described with Caputo Derivative. *J. Funct. Spaces* **2022**, *2022*, 9174488. <https://doi.org/10.1155/2022/9174488>
- [13] Sun, H.; Zhang, Y.; Baleanu, D.; Chen, W.; Chen, Y. A New Collection of Real-World Applications of Fractional Calculus in Science and Engineering. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *64*, 213–231. <https://doi.org/10.1016/j.cnsns.2018.04.010>
- [14] Diethelm, K. *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*; Springer: Berlin, Germany, 2010. <https://doi.org/10.1007/978-3-642-14574-2>
- [15] Baleanu, D.; Diethelm, K.; Scalas, E.; Trujillo, J. J. *Fractional Calculus: Models and Numerical Methods*; World Scientific: Singapore, 2012. <https://doi.org/10.1142/8180>
- [16] Patnaik, S.; Hollkamp, J. P.; Semperlotti, F. Applications of Variable-Order Fractional Operators: A Review. *Proc. R. Soc. A* **2020**, *476*(2234), 20190498. <https://doi.org/10.1098/rspa.2019.0498>
- [17] Li, C.; Zeng, F. *Numerical Methods for Fractional Calculus*; CRC Press: Boca Raton, FL, 2015. <https://doi.org/10.1201/b18588>
- [18] Garrappa, R. Numerical Solution of Fractional Differential Equations: A Survey and a Software Tutorial. *Mathematics* **2018**, *6*(2), 16. <https://doi.org/10.3390/math6020016>
- [19] Stynes, M.; O’Riordan, E.; Gracia, J. L. Error Analysis of a Finite Difference Method on Graded Meshes for a Time-Fractional Diffusion Equation. *SIAM J. Numer. Anal.* **2017**, *55*(2), 1057–1079. <https://doi.org/10.1137/16M1082329>
- [20] Wang, Y.; Zhou, X. An Efficient Numerical Method Based on Bell Wavelets for Solving the Fractional Integro-Differential Equations with Weakly Singular Kernels. *Fractal Fract.* **2024**, *8*(2), 74. <https://doi.org/10.3390/fractalfract8020074>
- [21] Behera, S.; Saha Ray, S. Euler Wavelets Method for Solving Fractional-Order Linear Volterra–Fredholm Integro-Differential Equations with Weakly Singular Kernels. *Comput. Appl. Math.* **2021**, *40*, 192. <https://doi.org/10.1007/s40314-021-01530-5>
- [22] Keshavarz, E.; Ordokhani, Y.; Razzaghi, M. A Fast Numerical Algorithm Based on the Taylor Wavelets for Solving the Fractional Integro-Differential Equations with Weakly Singular Kernels. *Math. Methods Appl. Sci.* **2019**, *42*(13), 4427–4443. <https://doi.org/10.1002/mma.5647>
- [23] Nemati, S.; Lima, P. M. Numerical Solution of Nonlinear Fractional Integro-Differential Equations with Weakly Singular Kernels via a Modification of Hat Functions. *Appl. Math. Comput.* **2018**, *327*, 79–92. <https://doi.org/10.1016/j.amc.2017.12.003>

- [24] Yi, M.; Huang, J. CAS Wavelet Method for Solving the Fractional Integro-Differential Equation with a Weakly Singular Kernel. *Int. J. Comput. Math.* **2015**, 92(8), 1715–1728. <https://doi.org/10.1080/00207160.2014.982674>
- [25] Zhu, L.; Wang, Y. SCW Method for Solving the Fractional Integro-Differential Equations with a Weakly Singular Kernel. *Appl. Math. Comput.* **2016**, 275, 72–80. <https://doi.org/10.1016/j.amc.2015.11.090>
- [26] Mohammed, J. K.; Khudair, A. R.; Pandian, N. R. Numerical Solution of Fractional Integro-Differential Equations via Fourth-Degree Hat Functions. *Iraqi J. Comput. Sci. Math.* **2023**, 4(2), 10–30. <https://doi.org/10.52866/ijcsm.2023.4.2.2>
- [27] Mahdy, A. M. S.; Nagdy, A. S.; Mohamed, D. S. H. Solution of Fractional Integro-Differential Equations Using Least Squares and Shifted Legendre Methods. *J. Appl. Math. Comput. Mech.* **2024**, 23(1), 59–70. <https://doi.org/10.17512/jamcm.2024.1.06>
- [28] Wang, Y.; Zhou, X. An Efficient Numerical Method Based on Bell Wavelets for Solving the Fractional Integro-Differential Equations with Weakly Singular Kernels. *Fractal Fract.* **2024**, 8(2), 74. <https://doi.org/10.3390/fractalfract8020074>
- [29] Nemati, S.; Lima, P. M.; Ordokhani, Y. Numerical Solution of a Class of Two-Dimensional Nonlinear Volterra Integral Equations Using Legendre Polynomials. *J. Comput. Appl. Math.* **2013**, 242, 53–69. <https://doi.org/10.1016/j.cam.2012.10.028>
- [30] Brunner, H. *Volterra Integral Equations: An Introduction to Theory and Applications*; Cambridge University Press: Cambridge, U.K., 2017. <https://doi.org/10.1017/9781108235802>
- [31] Chen, Y.; Tang, T. Convergence Analysis of the Jacobi Spectral-Collocation Methods for Volterra Integral Equations with a Weakly Singular Kernel. *Math. Comput.* **2010**, 79(269), 147–167. <https://doi.org/10.1090/S0025-5718-09-02267-2>
- [32] Alkan, S.; Hatipoglu, V. F. Approximate Solutions of Volterra–Fredholm Integro-Differential Equations of Fractional Order. *Tbilisi Math. J.* **2017**, 10(2), 1–13. <https://doi.org/10.1515/tmj-2017-0001>
- [33] Shu, X. B.; Wang, Q. The Existence and Uniqueness of Mild Solutions for Fractional Differential Equations with Nonlocal Conditions of Order $1 < \alpha < 2$. *Comput. Math. Appl.* **2012**, 64(6), 2100–2110. <https://doi.org/10.1016/j.camwa.2012.02.018>
- [34] Taghipour, M.; Aminikhah, H. A Fast Collocation Method for Solving the Weakly Singular Fractional Integro-Differential Equation. *Comput. Appl. Math.* **2022**, 41, 142. <https://doi.org/10.1007/s40314-022-01834-3>