บทความวิจัย

ทฤษฎีบทการลู่เข้าสำหรับปัญหาอสมการการแปรผันและปัญหาจุดตรึงของการส่งหลายค่า
A Convergence Theorem for Variational Inequality Problems and Fixed Points Problems of Multi-Valued Mapping

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บทคัดย่อ

จุดประสงค์ของงานวิจัยชิ้นนี้คือ สร้างกระบวนการทำซ้ำผสมใหม่เพื่อประมาณค่าคำตอบร่วมของปัญหาอสมการการแปรผันและเซตของจุดตรึงของการส่งค่อนข้างไม่ขยายหลายค่า นอกจากนั้น ภายใต้เงื่อนไขที่เหมาะสม ได้พิสูจน์ว่ากระบวนการทำซ้ำที่สร้างขึ้นลู่เข้าอย่างเข้มสู่คำตอบร่วมของปัญหาอสมการการแปรผันและเซตของจุดตรึงของการส่งค่อนข้างไม่ขยายหลายค่าในปริภูมิบานาค

คำสำคัญ: ปัญหาอสมการการแปรผัน การส่งหลายค่า การส่งค่อนข้างไม่ขยาย

Abstract

The purpose of this paper is to introduce a new hybrid iterative scheme for finding a common element of the solution set of variational inequality problem and the set of fixed point of relatively nonexpansive multi-valued mapping. Under suitable conditions, we prove some strong convergence theorems for the proposed schemes in Banach spaces. The results presented in this paper were to improve and extend some recent works written by other authors.

Keywords: Variational Inequality, Multi-Valued Mapping, Relatively Nonexpansive Mapping.

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Introduction and Preliminaries

Let E be a real Banach space with dual E^* and $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. E is said to be *smooth* if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U$. The norm of E is said to be *Fréchet differentiable* if, for each $x \in U$, the limit is attained uniformly for $y \in U$. E is said to be *uniformly smooth* if the limit exists uniformly in $x, y \in U$. The *modulus of convexity* of E is the function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta(\varepsilon) = \inf\Big\{1 - \Big\|\frac{x+y}{2}\Big\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\Big\}.$$

A Banach space E is *uniformly convex* if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

Let C be a nonempty closed and convex subset of a real Banach space E. Let $A : C \to E^*$ be a mapping. Then A is said to be:

(1) *monotone* if

$$\langle x - y, Ax - Ay\rangle \geq 0, \ \forall x, y \in C;$$

(2) *α-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \ \forall x, y \in C.$$

Clearly, the class of monotone mappings include the class of α-inverse-strongly monotone mappings. The class of inverse-strongly monotone have been studied by many authors to approximate a common fixed point (see [1-2] for more details).

Let $A : C \to E^*$ be an operator. The *variational inequality problem* for an operator A is as follows: find $\hat{z} \in C$ such that

$$\langle y - \hat{z}, A\hat{z}\rangle \geq 0, \tag{1.1}$$

for all $y \in C$. The set of solution of (1.1) is denote by $VI(A, C)$.

Let E be a Banach space with the dual space E^*. We denote by J the *normalized duality mapping* from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2, \|f^*\| = \|x\|\}.$$

The functional $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \tag{1.2}$$

for all $x, y \in E$, where J is the normalized duality mapping. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2, \ \forall x, y \in E, \tag{1.3}$$

and

$$\phi(x,y) = \phi(z,y) + \phi(x,z) + 2\langle z-x, Jy - jz\rangle,\ \forall x,y,z \in E. \tag{1.4}$$

Remark 1.1. If E is a reflexive, strictly convex and smooth Banach space, then, for any $x, y \in E$, $\phi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that, if $\phi(x,y) = 0$, then $x = y$. From (1.2), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$ (see [3-4] for more details).

A mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in C$. An element $p \in C$ is called a *fixed point* of T if $Tp = p$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. A point p in C is called an *asymptotic fixed point* of T [5] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The asymptotic fixed point set of T is denoted by $\widehat{F}(T)$.

A mapping $T : C \to C$ is called *relatively nonexpansive* [6-8] if
(R1) $F(T)$ is nonempty;
(R2) $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$;
(R3) $\widehat{F}(T) = F(T)$.
A mapping $T : C \to C$ is called *relatively quasi-nonexpansive* if the conditions (R1) and (R2) hold.

On the author hand, the *generalized projection* $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem:

$$\phi(\bar{x}, x) = \inf_{y\in C} \phi(y, x). \tag{1.5}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y,x)$ and the strict monotonicity of the mapping J (see, for example, [3-5, 9-10]). If E is a smooth, strictly convex and reflexive Banach space, then Π_C is a closed relatively quasi-nonexpansive mapping from E onto C with $F(\Pi_C) = C$ [11].

Let $N(C)$ and $CB(C)$ denoted the family of nonempty subsets and nonempty closed bounded subsets of C, respectively. The *Hausdorff metric* on $CB(C)$ is defined by

$$H(A_1, A_2) = \max\{\sup_{x\in A_2} d(x, A_1), \sup_{y\in A_1} d(y, A_2)\}$$

for all $A_1, A_2 \in CB(C)$, where $d(x, A_1) = \inf\{\|x - y\|; y \in A_1\}$.

An element $p \in C$ is called a *fixed point* of $T : C \to N(C)$ if $p \in F(T)$, where $F(T) := \{p \in C : p \in T(p)\}$. A point $p \in C$ is call an *asymptotic fixed point* of a multi-valued mapping $T : C \to N(C)$ if there exists a sequence $\{x_n\}$ in C which converges

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weakly to p and $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$. The asymptotic fixed point set of T is denoted by $\widehat{F}(T)$.

A multi-valued mapping $T : C \to N(C)$ is said to be *relatively nonexpansive* if
($\acute{R}1$) $F(T)$ is nonempty;
($\acute{R}2$) $\phi(p, z) \leq \phi(p, x)$ for all $x \in C$, $z \in T(x)$ and, $p \in F(T)$;
($\acute{R}3$) $\widehat{F}(T) = F(T)$.

Sastry and Babu [12] prove that the Mann and Ishikawa iteration schemes for a multi-valued mapping T with a fixed point p converge to a fixed point q of T. Panyanak [13] extended the result of Sastry and Babu to uniformly convex Banach spaces. In 2009, Shahzad and Zegeye [14] proved strong convergence theorems for the Ishikawa iteration scheme involving quasi-nonexpansive multi-valued mappings in uniformly convex Banach spaces. Iterative methods for approximating fixed points of multi-valued mappings in Banach spaces have been studied by many authors, see [15-18] for more detail. Recently, Homaeipour and Razani [19] introduced an iterative sequence for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of relatively nonexpansive multi-valued mappings. Motivated by other recent works, in this paper, we introduce an iterative process for the approximation of a common element of the set of fixed point of relatively nonexpansive multi-valued mapping and the solution set of variational inequality problem. We prove that an iterative sequence converges strongly to a common element of the set of fixed point of relatively nonexpansive multi-valued mapping and the solution set of variational inequality problem in Banach spaces.

We also need the following lemmas for the proof of our main results.

Lemma 1.2. *Let E be a strictly convex and smooth Banach space. Then, for all $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$* [20].

Lemma 1.3. *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0\rangle \geq 0$ for all $y \in C$* [5].

Lemma 1.4. *Let E be a reflexive, strictly convex and smooth Banach space, C be a nonempty closed convex subset of E and $x \in E$. Then $\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x)$ for all $y \in C$* [5].

Lemma 1.5. *Let E be a smooth and strictly convex Banach space and C be a nonempty closed convex subset of E. Suppose $T : C \to N(C)$ is a relatively nonexpansive multi-valued mapping. Then $F(T)$ is a closed convex subset of C* [19].

Lemma 1.6. *Let E be a uniformly convex and smooth Banach space and $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty] \to [0, \infty]$ with $g(0) = 0$ such that*

$$g(\|y - z\|) \leq \phi(y, z),\ \forall y, z \in B_r(0) = \{\|x\| \leq r\}.$$

See [21] for more detail.

Lemma 1.7. *Let E be a uniformly convex Banach space and $r > 0$. Then, there exists a strictly increasing, continuous and convex function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ such that*

$$\|\alpha x + \beta y\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2 - \alpha\beta h(\|x - y\|)$$

for all $x, y \in B_r := \{z \in X : \|z\| \leq r$ and all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1.$

See [22] for more detail.

Lemma 1.8. *Let C be a nonempty closed convex subset of a uniformly smooth, strictly convex real Banach space E and $A : C \to E^*$ be a continuous monotone mapping. For any $r > 0$, define a mapping $F_r : E \to C$ as follows:*

$$F_r x = \{z \in C : \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \ \forall y \in C\}$$

for all $x \in C$. Then the following hold:

(1) *F_r is a single-valued mapping;*
(2) *$F(F_r) = VI(A, C)$;*
(3) *$VI(A, C)$ is a closed and convex subset of C;*
(4) *$\phi(q, F_r x) + \phi(F_r x, x) \leq \phi(q, x)$ for all $q \in F(F_r)$.*

See [23] for more detail.

Remark 1.9. We remark that if E is a Banach space. Then the following are well known (see [3] for more details):

(1) If E is an arbitrary Banach space, then J is monotone and bounded;
(2) If E is a strictly convex, then J is strictly monotone;
(3) If E is a smooth, then J is single valued and semi-continuous;
(4) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E;
(5) If E is reflexive, smooth and strictly convex, then the normalized duality mapping $J = J_2$ is single valued, one-to-one and onto;
(6) If E is reflexive, smooth and strictly convex, then J^{-1} is also single valued, one-to-one, onto and it is the duality mapping from E^* into E;
(7) If E is uniformly smooth, then E is smooth and reflexive;
(8) E is uniformly smooth if and only if E^* is uniformly convex.

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Methodology

1. Studying and investigating on the variation inequality problems.
2. Studying and investigating on the relatively nonexpansive multi-valued mapping.
3. Establishing the new theorem for the variation inequality problems and fixed point problems in Banach spaces.

Main result

In this section, we introduce an iterative scheme which converges strongly to a common solution of the variational inequality problems and a fixed point of a relatively nonexpansive multi-valued mapping in a real uniformly smooth and uniformly convex Banach space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let $T : C \to N(C)$ be a relatively nonexpansive multi-valued mapping and let A be a continuous monotone mapping of C into E^*. Define a mapping $F_{r_n} : E \to C$ by*

$$F_{r_n}u = \{x \in C : \langle y - x, Ax\rangle + \frac{1}{r_n}\langle y - x, Jx - Ju\rangle \geq 0,\ \forall y \in C\}.$$

Assume that $\Theta := F(T) \cap VI(A, C) \neq \emptyset$. For an initial point $x_1 \in C$ let $\{x_n\}$ be the sequence generated by

$$\begin{cases} u_n = F_{r_n}x_n, \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n Ju_n + (1-\alpha_n)Jz_n), \end{cases} \tag{3.1}$$

where $z_n \in T(x_n)$, for all $n \geq 1$. Assume that $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and $\{r_n\} \subset [d, \infty)$ for some $d > 0$. Then $\{x_n\}$ converges strongly to some point of Θ.*

Proof. Let T be a relatively nonexpansive multi-value mapping. Since $\Theta := F(T) \cap VI(A, C)$ is nonempty, closed and convex, for any $p \in \Theta$, we have

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi(p, \Pi_C J^{-1}(\alpha_n Ju_n + (1-\alpha_n)Jz_n)) \\
&\leq \phi(p, J^{-1}(\alpha_n Ju_n + (1-\alpha_n)Jz_n)) \\
&= \|p\|^2 - 2\langle p, \alpha_n Ju_n + (1-\alpha_n)Jz_n\rangle + \|\alpha_n Ju_n + (1-\alpha_n)Jz_n\|^2 \\
&\leq \|p\|^2 - 2\alpha_n\langle p, Ju_n\rangle - 2(1-\alpha_n)\langle p, Jz_n\rangle + \|\alpha_n Ju_n\|^2 + \|(1-\alpha_n)Jz_n\|^2 \\
&= \alpha_n\phi(p, u_n) + (1-\alpha_n)\phi(p, z_n) \\
&\leq \alpha_n\phi(p, u_n) + (1-\alpha_n)\phi(p, x_n) \\
&= \phi(p, F_{r_n}x_n) + (1-\alpha_n)\phi(p, x_n) \\
&\leq \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, x_n) \\
&= \phi(p, x_n).
\end{aligned}$$

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(3.2)

That is $\{\phi(p, x_n)\}$ is non-increasing. Hence $\{\phi(p, x_n)\}$ is a convergence sequence. So $\lim_{n\to\infty} \phi(p, x_n)$ exists. Follow by (1.3), $\{x_n\}$ is bounded and so are $\{z_n\}$ and $\{u_n\}$. So, there exists $r_1 = \sup_{n\geq 1}\{\|x_n\|, \|z_n\|, \|u_n\|\}$ such that $x_n, z_n \in B_r(0)$ for all $n \geq 1$. Since E is a uniformly smooth Banach space, E^* is a uniformly convex Banach space. By Lemma (1.7), there exists a continuous, strictly increasing and convex function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ such that

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi(p, \Pi_C J^{-1}(\alpha_n J u_n + (1-\alpha_n) J z_n)) \\
&\leq \phi(p, J^{-1}(\alpha_n J u_n + (1-\alpha_n) J z_n)) \\
&= \|p\|^2 - 2\alpha_n \langle p, J u_n\rangle - 2(1-\alpha_n)\langle p, J z_n\rangle + \|\alpha_n J z_n + (1-\alpha_n) J z_n\|^2 \\
&\leq \|p\|^2 - 2\alpha_n \langle p, J u_n\rangle - 2(1-\alpha_n)\langle p, J z_n\rangle + \alpha_n \|J u_n\|^2 + (1-\alpha_n)\|J z_n\|^2 \\
&\quad -\alpha_n(1-\alpha_n) h(\|J u_n - J z_n\|) \\
&= \alpha_n \phi(p, u_n) + (1-\alpha_n)\phi(p, z_n) - \alpha_n(1-\alpha_n) h(\|J u_n - J z_n\|) \\
&\leq \alpha_n \phi(p, u_n) + (1-\alpha_n)\phi(p, u_n) - \alpha_n(1-\alpha_n) h(\|J u_n - J z_n\|) \\
&\leq \phi(p, u_n) - \alpha_n(1-\alpha_n) h(\|J u_n - J z_n\|)
\end{aligned} \tag{3.3}$$

and so

$$\alpha_n(1-\alpha_n) h(\|J u_n - J z_n\|) \leq \phi(p, x_n) - \phi(p, x_{n+1}).$$

Since $\lim_{n\to\infty} \phi(p, u_n)$ exists and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we have

$$\lim_{n\to\infty} h(\|J u_n - J z_n\|) = 0 \tag{3.4}$$

hence

$$\lim_{n\to\infty} \|J u_n - J z_n\| = 0. \tag{3.5}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{3.6}$$

Let $r_2 = \sup_{n\geq 1}\{\|x_n\|, \|u_n\|\}$, from lemma(1.6) and (3.3) , we have

$$g(\|x_n - u_n\|) \leq \phi(x_n, u_n). \tag{3.7}$$

Since $u_n = F_{r_n} x_n$ and $p \in \Theta$, by lemma (1.8), we get

$$g(\|x_n - u_n\|) \leq \phi(x_n, u_n) \leq \phi(p, x_n) - \phi(p, u_n). \tag{3.8}$$

Since $\lim_{n\to\infty} \phi(p, x_n)$ exists and following from (3.2), we have

$$\lim_{n\to\infty} g(\|x_n - u_n\|) = 0. \tag{3.9}$$

From property of g, we have that

$$\lim_{n\to\infty} \|x_n - u_n\|) = 0. \tag{3.10}$$

By using the triangle inequality, we obtain

$$\|x_n - z_n\| \leq \|x_n - u_n\| + \leq \|u_n - z_n\|. \tag{3.11}$$

From (3.6) and (3.10), we get

$$\lim_{n\to\infty} \|x_n - z_n\|) = 0. \tag{3.12}$$

Since $d(x_n, Tx_n) \leq \|x_n - z_n\|$, we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.13}$$

Let $p \in \Theta$ and $r > 0$. Then there exists $p + rk \in \Theta$, whenever $\|k\| \leq 1$. Thus, by (1.4), for any $u \in \Theta$, we have

$$\phi(u, x_n) = \phi(x_{n+1}, x_n) + \phi(u, x_{n+1}) + 2\langle x_{n+1} - u, Jx_n - Jx_{n+1}\rangle, \tag{3.14}$$

which implies

$$\frac{1}{2}(\phi(u, x_n) - \phi(u, x_{n+1}) = \frac{1}{2}\phi(x_{n+1}, x_n) + \langle x_{n+1} - u, Jx_n - Jx_{n+1}\rangle. \tag{3.15}$$

Also, we have

$$\begin{aligned} \langle x_{n+1} - p, Jx_n - Jx_{n+1}\rangle &= \langle x_{n+1} - (p + rk) + rk, Jx_n - Jx_{n+1}\rangle \\ &= \langle x_{n+1} - (p + rk), Jx_n - Jx_{n+1}\rangle \\ &\quad + r\langle k, Jx_n - Jx_{n+1}\rangle. \end{aligned} \tag{3.16}$$

Thus it follows from (3.2), (3.15) and (3.16), we have

$$0 \leq \langle x_{n+1} - (p + rk), Jx_n - Jx_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n).$$

From (3.15),

$$\begin{aligned} r\langle k, Jx_n - Jx_{n+1}\rangle &\leq \langle x_{n+1} - p, Jx_n - Jx_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) \\ &= \frac{1}{2}(\phi(p, x_n) - \phi(p, x_{n+1})) \end{aligned}$$

hence

$$\langle k, Jx_n - Jx_{n+1}\rangle \leq \frac{1}{2r}(\phi(p, x_n) - \phi(p, x_{n+1})).$$

Since $\|k\| \leq 1$, we obtain

$$\|Jx_n - Jx_{n+1}\| \leq \frac{1}{2r}(\phi(p, x_n) - \phi(p, x_{n+1})). \tag{3.17}$$

For all $m, n \geq 1$ with $n > m$, we have

$$\begin{aligned} \|Jx_m - Jx_n\| &\leq \textstyle\sum_{i=m}^{n-1} \|Jx_i - Jx_{i+1}\| \\ &\leq \frac{1}{2r} \textstyle\sum_{i=m}^{n-1} (\phi(p, x_i) - \phi(p, x_{i+1})) \\ &= \frac{1}{2r}(\phi(p, x_m) - \phi(p, x_n)). \end{aligned} \tag{3.18}$$

Since $\{\phi(p, x_n)\}$ converges, $\{Jx_n\}$ is a Cauchy sequence. Since E^* is complete, $\{Jx_n\}$ converge strongly to some point in E^*. Since E^* has a Fréchet differentiable, J^{-1} is norm-to-norm continuous on E^*. Then $\{x_n\}$ converges strongly to some point q in C. Thus, from (3.13), we have $q \in F(T)$. From (3.10) and J is norm-to-norm continuous on E, it follows that

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{3.19}$$

Thus, from (3.19), for all $r_n > 0$, we obtain

$$\lim_{n\to\infty} \frac{\|Jx_n - Ju_n\|}{r_n} = 0. \tag{3.20}$$

Thus, from $F_{r_n} x_n = u_n \in C$, we have that

$$\langle v - u_n, Au_n\rangle + \frac{1}{r_n}\langle v - u_n, Ju_n - Jx_n\rangle \geq 0 \tag{3.21}$$

for all $v \in C$, that is,

$$\langle v - u_n, Au_n\rangle + \langle v - u_n, \frac{Ju_n - Jx_n}{r_n}\rangle \geq 0$$

for all $v \in C$. For all $t \in (0, 1)$, define $v_t = tv + (1-t)q$. Then $v_t \in C$ and it follows from (3.21) that

$$\langle v_t - u_n, Au_n\rangle + \langle v_t - u_n, \frac{Ju_n - Jx_n}{r_n}\rangle \geq 0$$

for all $v \in C$ and

$$\langle v_t - u_n, Av_t\rangle \geq \langle v_t - u_n, Av_t\rangle - \langle v_t - u_n, Au_n\rangle - \langle v_t - u_n, \frac{Ju_n - Jx_n}{r_n}\rangle \geq 0 \tag{3.22}$$

for all $v \in C$. By (3.10) and (3.22), we have $\frac{Ju_n - xu_n}{r_n} = 0$. Since A is monotone, we have

$$\langle v_t - u_n, Av_t\rangle \geq \langle v_t - u_n, Av_t - Au_n\rangle \geq 0$$

and

$$\lim_{n\to\infty} \langle v_t - u_n, Av_t\rangle = \langle v_t - q, Av_t\rangle \geq 0.$$

Taking $t \to 0$, it follows that

$$\langle v - q, Aq\rangle \geq 0$$

for all $v \in C$ and so $q \in VI(A, C)$. Therefore, $q \in F(T) \cap VI(A, C)$. This completes the proof. □

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